ON APPROXIMATION OF COMMON SOLUTION OF FINITE FAMILY OF MIXED EQUILIBRIUM PROBLEMS INVOLVING $\mu - \alpha$ RELAXED MONOTONE MAPPING IN A BANACH SPACE

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In this paper, we introduce a U-mapping for finite family of mixed equilibrium problems involving $\mu - \alpha$ relaxed monotone operator. We prove a strong convergence theorem for finding the common solution of finite family of these equilibrium problems in a uniformly smooth and strictly convex Banach space which also enjoys Kadec-Klee property. Furthermore, we give some applications of our result and numerical example to show its relevance. Our results improve and generalize many other recent results in literature.

Keywords: Mixed equlibrium problem; U-mapping; fixed point; strong convergence; Banach space; Duality pairing.

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1. INTRODUCTION

Let X be a real Banach space with the dual space X^* and C be a nonempty, closed and convex subset of X. A nonlinear mapping $T : X \to X$ is said to be a contraction, if there exists a constant $\theta \in (0,1)$ such that

$$||Tx - Ty|| \leq \theta||x - y||, \ \forall \ x, y \in X.$$

If $\theta = 1$, then T is said to be nonexpansive. We denote the set of fixed points of T by $F(T)$. Let $F : C \times C \to \mathbb{R}$ be a bifunction, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem with respect to F and C in the sense of Blum and Oettli (1994) is to find $x \in C$ such that

$$F(x, y) \geq 0, \ \forall y \in C. \tag{1}$$

In this paper, we assume that the bifuncton F satisfies the following conditions:
(F1) $F(x, x) = 0$, for all $x \in C$; (F2) F is monotone; i.e. $F(x, y) + F(y, x) \leq 0$, for all $x, y \in C$; (F3) for all $x, y \in C$, $\lim\limits_{t\to 0} F(tz + (1-t)x, y) \leq F(x, y)$; (F4) for all $x \in C$, $F(x, .)$ is convex and lower semicontinuous.
Fang and Huang [5] introduced the concept of relaxed $\mu - \alpha$ monotone mapping for solving a mixed equilibrium problem. A mapping $A : C \to X^*$ is said to be relaxed $\mu - \alpha$ monotone [17], if there exists a mapping $\mu : C \times C \to X$ and a function $\alpha : X \to \mathbb{R}$ with $\alpha(tz) = t^p\alpha(z)$ for all $t > 0$ and $z \in X$, where $p > 1$ such that

$$\langle Ax - Ay, \mu(x, y)\rangle \geq \alpha(x - y).$$

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In particular, if $\mu(x,y) = x - y,\ \forall x,y \in C$ and $\alpha(z) = k||z||^p$, where $p > 1$ and $k > 1$ are constants, then A is called p monotone [7, 20]. Fang and Huang [5] proved that under some appropriate conditions, the following variational inequality is solvable; find $x \in C$ such that

$$\langle Ax, \mu(y,x)\rangle + \phi(y) - \phi(x) \geq 0, \quad \forall y \in C, \tag{1}$$

where $\phi : C \to \mathbb{R} \cup \{\infty\}$ is a nonlinear mapping. They also proved that the following inequality is equivalent to the variational inequality (1) : find $x \in C$ such that

$$\langle Ay, \mu(y,x)\rangle + \phi(y) - \phi(x) \geq \alpha(y-x), \quad \forall y \in C. \tag{2}$$

The mixed equilibrium problem (see e.g [21]) is to find $x \in C$ such that

$$F(x,y) + \langle Ax, \mu(y,x)\rangle + \phi(y) - \phi(x) \geq 0,\ \forall y \in C. \tag{3}$$

We denote the solution set of mixed equilibrium problem (3) by $EP(F,A)$. It is easily observed that if $F(x,y) = 0,\ \forall x,y \in C$, then, the mixed equilibrium problem (3) reduces to the variational inequality problem (1). Also if $A = 0$ and $\phi = 0$, then $EP(F,A)$ coincides with $EP(F)$. Equilibrium and mixed equilibrium problems have been widely used to for solve variational inequalities, fixed point and optimization problems. (see, [25]). There are several iterative methods in literature proposed for finding solutions of fixed point and mixed equilibrium problems with relaxed monotone mappings in various settings, see ([1, 3, 4, 6, 8, 9, 10, 11, 12, 13, 16, 24, 25]).
In this paper, motivated by the research going on in this direction, we study a strong convergence theorem for finding the common solution of finite family of mixed equilibrium problems with $\mu - \alpha$ relaxed monotone mapping in the frame work of a uniformly smooth and strictly convex Banach space which also enjoys the Kadec-Klee property. First, we introduce the following mapping: Let C be a nonempty closed convex subset of a smooth and strictly convex Banach space X. For $i = 1,2,\ldots,N$, let $F_i : C \times C \to \mathbb{R}$ be a finite family of bifunctions, $A_i : C \to X^*$ be a finite family of μ hemicontinuous and relaxed $\mu - \alpha$ monotone mappings and $\phi_i : C \to \mathbb{R} \cup \{+\infty\}$ be a finite family of proper, convex and semicontinuous functions. For $i = 1,2,\ldots,N$ and $\{r_n\} \subset (0,\infty)$, the resolvent operator on F_i is defined in [3] as
$K^i_{r_n}(x) := \{z \in X : F_i(z,y) + \langle A_i z, \mu(y,z)\rangle + \phi_i(y) - \phi_i(z) + \frac{1}{r_n}\langle y - z, Jz - Jx\rangle \geq 0.\ \forall\, y \in X\}$.
However it has been proved in [3] that $K^i_{r_n}$ is single valued for each $i = 1,2\ldots,N$. (See Lemma 2.5). We define the mapping $U_n : C \to C$ as

$$\begin{cases} S_{n,1} &= \lambda_{n,1}K^1_{r_n} + (1-\lambda_{n,1})I, \\ S_{n,2} &= \lambda_{n,2}K^2_{r_n}S_{n,1} + (1-\lambda_{n,2})S_{n,1}, \\ \vdots & \\ S_{n,N-1} &= \lambda_{n,N-1}K^{N-1}_{r_n}S_{n,N-2} + (1-\lambda_{n,N-1})S_{n,N-2}, \\ U_n &= S_{n,N} = \lambda_{n,N}K^N_{r_n}S_{n,N-1} + (1-\lambda_{n,N})S_{n,N-1}, \end{cases} \tag{4}$$

where $0 \leq \lambda_{n,i} \leq 1$, for $i = 1,2,\ldots,N$. In addition, we present the following algorithm for finding a common solution of finite family of mixed equilibrium problems involving a relaxed monotone operator: For arbitrary $x_1 \in C$, let $\{x_n\}$ be generated by

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n U_n x_n,\ \forall n \geq 1, \tag{5}$$

where U_n is as defined in (4) and f a contraction mapping from C to C. Furthermore, we obtain a strong convergence theorem under some appropriate conditions of the proposed iterative algorithm in a uniformly smooth and strictly convex Banach space which also enjoys Kadec-Klee property. Our results improve the results of [3, 21] and many other results in literature.

2. PRELIMINARIES

In this section, we give some basic definitions and results which will be used in the sequel. Let X be a real Banach space and $B = \{x \in X : ||x|| = 1\}$. X is said to be strictly convex, if for any $x, y \in B$,

$$x \neq y \text{ implies } \left|\left|\frac{x+y}{2}\right|\right| < 1.$$

Define a function $\delta : [0,2] \to [0,1]$ called the *modulus of convexity* of X as follows:

$$\delta(\epsilon) = \inf\left\{1 - \left|\left|\frac{x+y}{2}\right|\right| : x, y \in X, ||x|| = ||y|| = 1, ||x-y|| \geq \epsilon\right\}.$$

Then X is uniformly convex if and only if $\delta(\epsilon) > 0$ for all $\epsilon \in (0,2]$. It is known that a uniformly convex Banach space is reflexive and strictly convex. Recall that X has the Kadec-Klee property if for any sequence $\{x_n\} \subset X$ and $x \in X$ with $x_n \rightharpoonup x$ and $||x_n|| \to ||x||$, then $||x_n - x|| \to 0$ as $n \to \infty$. It is well known that if X is uniformly convex, then X enjoys the Kadec-Klee property. For more on the Kadec-Klee property (see [2, 19]). X is said to be smooth if the limit $\lim\limits_{t \to 0} \frac{||x+ty||-||x||}{t}$ exists for every $x, y \in B$. It is also said to be uniformly smooth if the limit is attained uniformly for $x, y \in B$. The generalized duality mapping $J_p : X \to 2^{X^*}$ is defined by

$$J_p(x) = \{x^* \in X^* : \langle x, x^* \rangle = ||x||^p, ||x^*|| = ||x||^{p-1}, x \in X\}.$$

For $p = 2$, we have the normalized duality pairing denoted by J. It is well known that if X is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of X (see [19]).

In order to prove our main result, we will need the following lemmas.

Lemma 2.1. [26] *Let X be a real Banach space. Then for all $x, y \in X$ and $j(x+y) \in J(x+y)$, the following inequality holds:*

$$||x+y||^2 \leq ||x||^2 + 2\langle y, j(x+y)\rangle. \tag{6}$$

Lemma 2.2. [18] *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X such that*

$$x_{n+1} = \beta_n x_n + (1-\beta_n) y_n, \ n \geq 0,$$

where $\{\beta_n\}$ is a sequence in $(0,1)$ such that $0 < \liminf\limits_{n\to\infty} \beta_n \leq \liminf\limits_{n\to\infty} \beta_n < 1$. Assume that

$$\limsup_{n\to\infty}(||y_{n+1} - y_n|| - ||x_{n+1} - x_n||) \leq 0. \textit{ Then } \lim_{n\to\infty} ||y_n - x_n|| = 0.$$

Lemma 2.3. [22] *Let $\{\alpha_n\}$ be a sequence of nonnegative real numbers satisfying the condition*

$$\alpha_{n+1} \leq (1-\gamma_n)\alpha_n + \gamma_n \sigma_n, n \geq 0,$$

where $\{\gamma_n\} \subset (0,1)$ and $\{\sigma_n\}$ is a sequence in $\mathbb{R}$ such that
i. $\lim\limits_{n\to\infty} \gamma_n = 0$ and $\sum_{n=0}^{\infty} \gamma_n = \infty$, ii. either $\lim\limits_{n\to\infty}$ sup $\sigma_n \leq 0$ or
$\sum_{n=0}^{\infty} |\gamma_n \sigma_n| < \infty$. Then, $\lim\limits_{n\to\infty} \alpha_n = 0$.

Lemma 2.4. [23] *Let X be a uniformly smooth Banach space and C be a nonepmty, closed and convex subset of X. Let $U : C \to C$ be a nonexpansive mapping such that $F(U) \neq \emptyset$ and $f : C \to C$ be a contraction mapping. For each t in $(0,1)$, define $z_t = tf(z_t) + (1-t)Uz_t$, then $\{z_t\}$ converges strongly to the unique fixed point q of U as $t \to 0$, where $q = P_{F(U)}f(q)$ and $P_{F(U)} : C \to F(U)$ is the sunny nonexpansive retraction from C to $F(U)$.*

Lemma 2.5. [3] *Let X be a uniformly smooth, strictly convex Banach space with the dual space X^* and let C be a nonempty, closed and convex bounded subset of X. Let $A : C \to X^*$ be a μ-hemicontinuous and relaxed $\mu - \alpha$ monotone mapping, let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying $(F1) - (F4)$ and $\phi : C \to \mathbb{R} \cup \{+\infty\}$. Let $r > 0$ and define a mapping $K_r : X \to C$ as follows:*
$K_r(x) = \{z \in C : F(z, y) + \langle Az, \mu(y, z)\rangle + \phi(y) - \phi(z) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \quad \forall\ y \in C\}$, *for all $x \in X$. Assume that*
(i) $\mu(x, y)+\mu(y, x) = 0$, $\forall\ x, y \in C$; (ii) for any fixed $u, v \in C$, the mapping $x \mapsto \langle Av, \mu(x, u)\rangle$ is convex and lower semicontinuous; (iii) $\alpha : X \to \mathbb{R}$ is weakly lower semicontinuous; (iv) for any $x, y \in C$, $\alpha(x - y) + \alpha(y - x) \geq 0$;
(v) $\langle A(tz_1 + (1-t)z_2), \mu(y, tz_1 + (1-t)z_2)\rangle \geq t\langle Az_1, \mu(y, z_1)\rangle + (1-t)\langle Az_2, \mu(y, z_2)\rangle$ for any $z_1, z_2, y \in C$ and $t \in [0, 1]$. Then the following hold:
(1) K_r is single-valued; (2) K_r is a firmly nonexpansive type mapping;
(3) $F(K_r) = EP(F, A)$; (4) $EP(F, A)$ is closed and convex.

Let C be a nonempty, closed and convex subset of a uniformly smooth and strictly convex Banach space X which also enjoys the Kadec-Klee property. Let $\mu : C \times C \to X$ be a nonlinear mapping. For $i = 1, 2, \ldots, N$, let $F_i : C \times C \to \mathbb{R}$ be a finite family of bifunctions, $A_i : C \to X^*$ be a finite family of μ hemicontinuous relaxed $\mu - \alpha$ monotone mappings and $\phi_i : C \to \mathbb{R} \cup \{+\infty\}$ be a finite family of proper, convex and lower semicontinuous functions. Let $\lambda_1, \lambda_2, \ldots, \lambda_N$ be real numbers such that $0 \leq \lambda_i \leq 1$ for all $i = 1, 2 \ldots, N$. We define a mapping $U : C \to C$ as follows:
$S_1 = \lambda_1 K_r^1 + (1 - \lambda_1)I, \quad S_2 = \lambda_2 K_r^2 S_1 + (1 - \lambda_2)S_1, \quad \cdots,$
$S_{N-1} = \lambda_{N-1} K_r^{N-1} S_{N-2} + (1 - \lambda_{N-1})S_{N-2},$

$$U \quad = \quad S_N = \lambda_N K_r^N S_{N-1} + (1 - \lambda_N)S_{N-1}. \tag{7}$$

The mapping so defined above is called U-mapping generated by $K_r^1, K_r^2, \ldots, K_r^N$ and $\lambda_1, \lambda_2, \ldots, \lambda_N$.

3. MAIN RESULT

In this section, we present our main results.

Lemma 3.1. *Let X be a uniformly smooth, strictly convex Banach space with the dual space X^* and C be a nonempty, closed and convex subset of X. Let $A : C \to X^*$ be a relaxed $\mu-\alpha$ monotone mapping, $F : C \times C \to \mathbb{R}$ be a bifunction satisfying $(F2)$ and $\phi : C \to \mathbb{R} \cup \{+\infty\}$. Assume that*
(i) $\mu(x, y) + \mu(y, x) = 0\ \forall\ x, y \in C$; (ii) for any $x, y \in C$, $\alpha(x - y) + \alpha(y - x) \geq 0$.
For $s > 0$ and $r > 0$, $||K_s x - K_r x|| \leq \left|1 - \frac{r}{s}\right| ||x - K_s x||$.

Proof. Let $z = K_r(x)$ and $w = K_s(x)$, from the definition of K_r, we have

$$F(z, y) + \langle Az, \mu(y, z)\rangle + \phi(y) - \phi(z) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0\ \forall\ y \in C.$$

In particular, we have

$$F(z, w) + \langle Az, \mu(w, z)\rangle + \phi(w) - \phi(z) + \frac{1}{r}\langle w - z, Jz - Jx\rangle \geq 0. \tag{8}$$

Similarly, we obtain

$$F(w, z) + \langle Aw, \mu(z, w)\rangle + \phi(z) - \phi(w) + \frac{1}{s}\langle z - w, Jw - Jx\rangle \geq 0. \tag{9}$$

Adding equation (8) and (9), we obtain from (i) that

$$F(z,w)+F(w,z)+\langle Az-Aw,\mu(w,z)\rangle+\frac{1}{r}\langle w-z,Jz-Jx\rangle$$
$$+\frac{1}{s}\langle z-w,Jw-Jx\rangle\geq 0. \quad (10)$$

Using condition $(F2)$, we have

$$\frac{1}{r}\langle w-z,Jz-Jx\rangle+\frac{1}{s}\langle z-w,Jw-Jx\rangle$$
$$\geq\langle Aw-Az,\mu(w,z)\rangle\geq\alpha(w-z) \quad (11)$$

interchanging the roles of w and z in (11), we obtain

$$\frac{1}{s}\langle z-w,Jw-Jx\rangle+\frac{1}{r}\langle w-z,Jz-Jx\rangle\geq\alpha(z-w). \quad (12)$$

Adding (11) and (12), and using condition (ii), we have $\frac{1}{r}\langle w-z,Jz-Jx\rangle+\frac{1}{s}\langle z-w,Jw-Jx\rangle\geq 0$, which implies that $\langle w-z,Jz-Jx\rangle-\langle w-z,\frac{rJw-rJx}{s}\rangle\geq 0$. That is, $\langle w-z,\frac{rJw-rJx}{s}-(Jz-Jx)\rangle\leq 0$, which implies

$$\langle w-z,\frac{rJw-rJx-sJz+sJw-sJw+sJx}{s}\rangle\leq 0. \quad (13)$$

This further implies that $||w-z||^2\leq\langle w-z,\frac{r-s}{s}(Jx-Jw)\rangle$, from which we obtain that

$$||w-z||\leq\big|1-\frac{r}{s}\big|||x-w||. \quad (14)$$

That is,

$$||K_sx-K_rx||\leq\big|1-\frac{r}{s}\big|||x-K_sx||. \quad (15)$$

□

Proposition 3.1. *Let C be a nonempty closed convex subset of a uniformly smooth and strictly convex Banach space X. Let $\mu: C\times C\to X$ be a nonlinear mapping. For $i=1,2\ldots,N$, let $F_i: C\times C\to\mathbb{R}$ be a finite family of bifunctions, $A_i: C\to X^*$ be a finite family of μ-hemicontinuous relaxed $\mu-\alpha$ monotone mapping and $\phi_i: C\to\mathbb{R}\cup\{+\infty\}$ be a finite family of proper convex lower semicontinuous mapping. Let $\lambda_1,\lambda_n,\ldots,\lambda_N$ be real numbers such that $0\leq\lambda_i\leq 1$ for all $i=1,2,\ldots,N$. Let U be the U-mapping defined in (7). Then $S_1,S_2,\ldots,S_{N-1}$ and U are nonexpansive. Also, $F(U)=\cap_{i=1}^N EP(F_i,A_i)$.*

Proof. By the nonexpansivity of K_r^i, for $i=1,2,\ldots,N$, it follows that $S_1,S_2,\ldots,$ $S_N=U$ are nonexpansive mappings. Since $\cap_{i=1}^N F(K_r^i)=\cap_{i=1}^N EP(F_i,A_i)$, then it suffices to show that $F(U)=\cap_{i=1}^N F(K_r^i)$. To show that $F(U)=\cap_{i=1}^N F(K_r^i)$, we have to show that $\cap_{i=1}^N F(K_r^i)\subseteq F(U)$ and $F(U)\subseteq\cap_{i=1}^N F(K_r^i)$. It is easily observed that the first part is obvious. Next we show that $F(U)\subseteq\cap_{i=1}^N F(K_r^i)$. Let $a\in F(U)$ and $b\in\cap_{i=1}^N F(K_r^i)$. Using the definition of U, we have

$$\begin{aligned}
||a-b|| &= ||Ua-b||=||\lambda_N K_r^N S_{N-1}a+(1-\lambda_N)S_{N-1}a-b||\\
&\leq \lambda_N||K_r^N S_{N-1}a-b||+(1-\lambda_N)||S_{N-1}a-b||\leq||S_{N-1}a-b||\\
&= ||(\lambda_{N-1}K_r^{N-1}S_{N-2}a-b)+(1-\lambda_{N-1}(S_{N-2}a-b||\\
&\leq \lambda_{N-1}||K_r^{N-1}S_{N-2}a-b||+(1-\lambda_{N-1})||S_{N-2}a-b||\\
&\leq ||S_{N-2}a-b||\\
&\vdots\\
&\leq ||S_1a-b||=||\lambda_1K_r^1a+(1-\lambda_1)a-b||\\
&\leq \lambda_1||K_r^1a-b||+(1-\lambda_1)||a-b||\leq||a-b||.
\end{aligned}$$

It follows that $||a-b|| = ||\lambda_1(K_r^1 a - b) + (1-\lambda_1)(a-b)||$ and $||a-b|| = \lambda_1||K_r^1 a - b|| + (1-\lambda_1)||a-b||$, that is $||a-b|| = ||K_r^1 a - b||$. Using the strict convexity of X, we obtain $K_r^1 a = a$, which implies that $a \in F(K_r^1)$. Hence, $S_1 a = a$. Again from (16) and the fact that $S_1 a = a$, we have $||a-b|| = ||\lambda_2(K_r^2 S_1 a - b) + (1-\lambda_2)(a-b)$ and $||a-b|| = \lambda_2||K_r^2 a - b|| + (1-\lambda_2)||a-b||$, that is, $||a-b|| = ||K_r^2 a - b||$. Using the strict convexity of X, we obtain $K_r^2 a = a$, which implies that $a \in F(K_r^2)$. From which we obtain $S_2 a = a$. Proceeding the same way, we obtain $a = K_r^1 a = K_r^2 a = \cdots = K_r^{N-1} a$ and $a = S_1 a = S_2 a = \cdots = S_{N-1} a$. Since $a \in F(U) = F(S_N)$ and $S_{N-1} a = a$, then $a = \lambda_N K_r^N a + (1-\lambda_N)a$. This implies that $a = K_r^N a$. Hence $F(U) \subset F(K_r^i)$ for $i = 1, 2, \ldots, N$ and thus $F(U) \subset \cap_{i=1}^N F(K_r^i)$. Therefore, $F(U) = \cap_{i=1}^N F(K_r^i) = \cap_{i=1}^N EP(F_i, A_i)$. The proof is complete. □

Proposition 3.2. *Let X be a uniformly smooth and strictly convex Banach space. For $i = 1, 2, \ldots, N$ and $n \in \mathbb{N}$, let U_n be a U-mapping defined by (4). Let $\{x_n\}$ be a bounded sequence in X, then the following inequality is satisfied.*

$$||U_{n+1}x_n - U_n x_n|| \leq ||x_{n+1} - x_n|| + M_N \sum_{i=1}^{N} \left|\lambda_{n+1,i} - \lambda_{n,i}\right|. \tag{16}$$

Proof. Using the fact that $K_{r_n}^i$ and $S_{n,i}$ for $i = 1, 2 \ldots, N$ are nonexpansive with Lemma 3.1, we obtain the following estimates:

$$\begin{aligned}
&||U_{n+1}x_n - U_n x_n|| = ||\lambda_{n+1,N} K_{r_{n+1}}^N S_{n+1,N-1} x_n \\
&+ (1-\lambda_{n+1,N}) S_{n+1,N-1} x_n - [\lambda_{n,N} K_{r_n}^N S_{n,N-1} x_n + (1-\lambda_{n,N}) S_{n,N-1} x_n]|| \\
&= ||\lambda_{n+1,N}(K_{r_{n+1}}^N S_{n+1,N-1} x_n - K_{r_{n+1}}^N S_{n,N-1} x_n) + (S_{n+1,N-1} x_n - S_{n,N-1} x_n) \\
&+ \lambda_{n+1,N}(S_{n,N-1} x_n - S_{n+1,N-1} x_n) + (\lambda_{n,N} - \lambda_{n+1,N})(S_{n,N-1} x_n) + \lambda_{n+1,N}(K_{r_{n+1}}^N S_{n,N-1} x_n \\
&- K_{r_n}^N S_{n,N-1} x_n) + (\lambda_{n+1,N} - \lambda_{n,N})(K_{r_n}^N S_{n,N-1} x_n)|| \\
&\leq \lambda_{n+1,N}||K_{r_{n+1}}^N S_{n+1,N-1} x_n - K_{r_{n+1}}^N S_{n,N-1} x_n|| + (1-\lambda_{n+1,N})||S_{n+1,N-1} x_n - S_{n,N-1} x_n|| + \\
&\quad |\lambda_{n+1,N} - \lambda_{n,N}|.||K_{r_n}^N S_{n,N-1} x_n - S_{n,N-1} x_n|| + \lambda_{n+1,N}||K_{r_{n+1}}^N S_{n,N-1} x_n - K_{r_n}^N S_{n,N-1} x_n|| \\
&\leq \lambda_{n+1,N}||S_{n+1,N-1} x_n - S_{n,N-1} x_n|| + (1-\lambda_{n+1,N})||S_{n+1,N-1} x_n - S_{n,N-1} x_n|| + \\
&\quad |\lambda_{n+1,N} - \lambda_{n,N}|.||K_{r_n}^N S_{n,N-1} x_n - S_{n,N-1} x_n|| + \lambda_{n+1,N}||K_{r_{n+1}}^N S_{n,N-1} x_n - K_{r_n}^N S_{n,N-1} x_n|| \\
&\leq ||S_{n+1,N-1} x_n - S_{n,N-1} x_n|| + |\lambda_{n+1,N} - \lambda_{n,N}|.||K_{r_n}^N S_{n,N-1} x_n - S_{n,N-1} x_n|| + \\
&\quad ||K_{r_{n+1}}^N S_{n,N-1} x_n - K_{r_n}^N S_{n,N-1} x_n|| \\
&\leq ||S_{n+1,N-1} x_n - S_{n,N-1} x_n|| + |\lambda_{n+1,N} - \lambda_{n,N}|.||K_{r_n}^N S_{n,N-1} x_n - S_{n,N-1} x_n|| \\
&+ \left|1 - \frac{r_{n+1}}{r_n}\right| ||S_{n,N-1} x_n||,
\end{aligned}$$

which implies that

$$||U_{n+1}x_n - U_n x_n|| \leq ||S_{n+1,N-1} x_n - S_{n,N-1} x_n|| + M_1\left(|\lambda_{n+1,N} - \lambda_{n,N}|, \left|1 - \frac{r_{n+1}}{r_n}\right|\right), \tag{17}$$

where M_1 is a constant such that $M_1 \geq \max\{||K_{r_n}^N S_{n,N-1}x_n - S_{n,N-1}x_n||, ||S_{n,N-1}x_n||\}$. Furthermore,

$$\begin{aligned}||S_{n+1,N-1}x_n - S_{n,N-1}x_n|| &= ||\lambda_{n+1,N-1}K_{r_{n+1}}^{N-1}S_{n+1,N-2}x_n + (1-\lambda_{n+1,N-1})S_{n+1,N-2}x_n \\ &- [\lambda_{n,N-1}K_{r_n}^{N-1}S_{n,N-2}x_n + (1-\lambda_{n,N-1})S_{n,N-2}x_n]|| \\ &= ||\lambda_{n+1,N-1}(K_{r_{n+1}}^{N-1}S_{n+1,N-2}x_n - K_{r_{n+1}}^{N-1}S_{n,N-2}x_n) + (1-\lambda_{n+1,N-1})(S_{n+1,N-2}x_n - S_{n,N-2}x_n) \\ &+ (\lambda_{n+1,N-1} - \lambda_{n,N-1})(K_{r_n}^{N-1}S_{n,N-2}x_n - S_{n,N-2}x_n) \\ &+ \lambda_{n+1,N-1}(K_{r_{n+1}}^{N-1}S_{n+1,N-2}x_n - K_{r_n}^{N-1}S_{n,N-2}x_n)||\end{aligned}$$

Thus

$$\begin{aligned}||S_{n+1,N-1}x_n - S_{n,N-1}x_n|| &\\ \leq ||S_{n+1,N-2}x_n - S_{n,N-2}x_n|| &+ |\lambda_{n+1,N-1} - \lambda_{n,N-1}|.||K_{r_n}^{N-1}S_{n,N-2}x_n - S_{n,N-2}x_n|| \\ &+ \left|1 - \frac{r_{n+1}}{r_n}\right| ||S_{n,N-2}x_n||. \qquad (18)\end{aligned}$$

Substituting (18) into (17), we obtain

$$\begin{aligned}||U_{n+1}x_n - U_nx_n|| &\leq M_1\left(\left|1 - \frac{r_{n+1}}{r_n}\right| + |\lambda_{n+1,N} - \lambda_{n,N}|\right) + ||S_{n+1,N-2}x_n - S_{n,N-2}x_n|| + \\ &|\lambda_{n+1,N-1} - \lambda_{n,N-1}|.||K_{r_n}^{N-1}S_{n,N-2}x_n - S_{n,N-2}x_n|| + \left|1 - \frac{r_{n+1}}{r_n}\right| ||S_{n,N-2}x_n|| \\ &\leq M_2\left(2\left|1 - \frac{r_{n+1}}{r_n}\right| + |\lambda_{n+1,N} - \lambda_{n,N}| + |\lambda_{n+1,N-1} - \lambda_{n,N-1}|\right) + ||S_{n+1,N-2}x_n - S_{n,N-2}x_n||, \qquad (19)\end{aligned}$$

where $M_2 \geq \max\{M_1, ||K_{r_n}^{N-1}S_{n,N-2}x_n - S_{n,N-2}x_n||, ||S_{n,N-2}x_n||\}$.
Proceeding the same way as above, we obtain

$$||U_{n+1}x_n - U_nx_n|| \leq M_{N-1}\left((N-1)\left|1 - \frac{r_{n+1}}{r_n}\right| + \sum_{i=2}^{N-1}|\lambda_{n+1,i} - \lambda_{n,i}|\right) + ||S_{n+1,1}x_n - S_{n,1}x_n||,$$

where $M_{N-1} \geq \max\{M_{N-2}, ||K_{r_n}^2 S_{n,1}x_n - S_{n,1}x_n||, ||S_{n,1}x_n||\}$. Hence,

$$\begin{aligned}||U_{n+1}x_n - U_nx_n|| &\leq M_{N-1}\left((N-1)\left|1 - \frac{r_{n+1}}{r_n}\right| + \sum_{i=2}^{N-1}|\lambda_{n+1,i} - \lambda_{n,i}|\right) + ||S_{n+1,1}x_n - S_{n+1,1}|| \\ &= ||\lambda_{n+1,1}K_r^1 + (1-\lambda_{n+1,1})x_n - \lambda_{n,1}K_r^1 - (1-\lambda_{n,1})x_n|| \\ &+ M_{N-1}\sum_{i=2}^{N-1}|\lambda_{n+1,i} - \lambda_{n,i}| \\ &= |\lambda_{n+1,1} - \lambda_{n,1}.||K_r^1 x_n - x_n|| + M_{N-1}\sum_{i=2}^{N-1}|\lambda_{n+1,i} - \lambda_{n,i}| \\ &= ||\lambda_{n+1,1}K_{r_{n+1}}^1 x_n + (1-\lambda_{n+1,1}x_n) - \lambda_{n,1}K_{r_n}^1 x_n - (1-\lambda_{n,1})x_n|| + \\ &\quad M_{N-1}\left((N-1)\left|1 - \frac{r_{n+1}}{r_n}\right| + \sum_{i=2}^{N-1}|\lambda_{n+1,i} - \lambda_{n,i}|\right) \\ &\leq M_N\left(N\left|1 - \frac{r_{n+1}}{r_n}\right| + \sum_{i=1}^{N}|\lambda_{n+1,i} - \lambda_{n,i}|\right), \qquad (20)\end{aligned}$$

where $M_N > \max\{M_{N-1}, ||K_r^1 x_n - x_n||, ||x_n||\}$. Therefore,

$$\begin{aligned} ||U_{n+1}x_{n+1} - U_n x_n|| &\leq ||U_{n+1}x_{n+1} - U_{n+1}x_n|| + ||U_{n+1}x_n - U_n x_n|| \\ &\leq ||x_{n+1} - x_n|| + M_N\left(N\left|1 - \frac{r_{n+1}}{r_n}\right| + \sum_{i=1}^{N}|\lambda_{n+1,i} - \lambda_{n,i}|\right). \end{aligned}$$

□

Proposition 3.3. *Let C be a nonempty, closed and convex subset of a uniformly smooth and strictly convex Banach space X. Let $\mu : C \times C \to X$ be a nonlinear mapping. For $i = 1, 2 \ldots, N$, let $F_i : C \times C \to \mathbb{R}$ be a finite family of bifunctions, let $A_i : C \to X^*$ be a relaxed $\mu - \alpha$ monotone mappings. Let $\phi_i : C \to \mathbb{R} \cup \{\infty\}$ be a finite family of proper convex lower semicontinuous mapping. For $i = 1, 2 \ldots N$, let $\lambda_{n,i}$ and λ_i be sequences in $[0, 1]$ such that $\lambda_{n,i} \to \lambda_i$ as $n \to \infty$ and $\{r_n\}$ be a sequence in $(0, \infty)$ such that $r_n \to r$ as $n \to \infty$ with $r > 0$. Suppose U is the mapping generated by $K_r^1, K_r^2, \ldots, K_r^N$ and $\lambda_1, \lambda_2, \ldots, \lambda_N$. For $n \in \mathbb{N}$, let U_n be the mapping generated by $K_{r_n}^1, K_{r_n}^2, \ldots, K_{r_n}^N$ and $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N}$. Assuming the conditions of Lemma 3.1 are satisfied, then for each $x \in C$, we have*

$$\lim_{n\to\infty} ||U_n x - Ux|| = 0. \tag{21}$$

Proof. Let $x \in C$, using Lemma 3.1, we have

$$\begin{aligned} ||S_{n,1}x - S_1 x|| &= ||\lambda_{n,1}K_{r_n}^1 x + (1 - \lambda_{n,1})x - \lambda_1 K_r^1 x - (1 - \lambda_1)x|| \\ &= ||\lambda_{n,1}(K_{r_n}^1 - K_r^1 x) + (\lambda_{n,1} - \lambda_1)(K_r^1 x - x)|| \\ &\leq \left|1 - \frac{r_n}{r}\right| ||K_r^1 x - x|| + |\lambda_{n,1} - \lambda_1|.||K_r^1 x - x|| \\ &\leq \left(\left|1 - \frac{r_n}{r}\right| + |\lambda_{n,1} - \lambda_1|\right)||K_r^1 x - x||. \end{aligned}$$

Using the same argument as above, for each $i = 1, 2, \ldots, N$, we obtain

$$\begin{aligned} ||S_{n,N}x - S_N x|| &= ||\lambda_{n,N}K_{r_n}^N S_{n,N-1}x + (1 - \lambda_{n,N})x - \lambda_N K_r^N S_{N-1}x - (1 - \lambda_N)x|| \\ &\leq \lambda_{n,N}||K_{r_n}^N S_{n,N-1}x - K_{r_n}^N S_{N-1}x|| + \lambda_{n,N}||K_{r_n}^N S_{N-1}x - K_r^N S_{N-1}x|| + \\ &\quad |\lambda_{n,1} - \lambda_1|.||K_r^N S_{N-1}x - x|| \\ &\leq ||S_{n,N-1}x - S_{N-1}x|| + \left|1 - \frac{r_n}{r}\right| ||K_r^N S_{N-1}x - S_{N-1}x|| \\ &+ |\lambda_{n,1} - \lambda_1|.||K_r^N S_{N-1}x - x||. \end{aligned}$$

It follows that

$$\begin{aligned} ||U_n x - Ux|| &= ||S_{n,N}x - S_N x|| \leq ||S_{n,1}x - S_1 x|| + \sum_{i=1}^{N}|\lambda_{n,i} - \lambda_i|.||K_r^i S_{i-1}x - S_{i-1}x|| \\ &\leq \left(\left|1 - \frac{r_n}{r}\right| + |\lambda_{n,1} - \lambda_1|\right)||K_r^1 x - x|| + \sum_{i=1}^{N}|\lambda_{n,i} - \lambda_i|.||K_r^i S_{i-1}x - S_{i-1}x||. \end{aligned}$$

Since $r_n \to r$ and $\lambda_{n,i} \to \lambda_i$ as $n \to \infty$, then $\lim_{n\to\infty} ||U_n x - Ux|| = 0$. □

Theorem 3.1. *Let C be a nonempty closed and convex subset of a uniformly smooth and strictly convex Banach space X which also enjoys the Kadec-Klee property. Let $\mu : C \times C \to X$ be a nonlinear mapping. For $i = 1, 2, \ldots, N$, $F_i : C \times C \to \mathbb{R}$ be a finite family of bifunctions satisfying conditions $(F1) - (F4)$, $A_i : C \to X^*$ be a finite family of μ-hemicontinuous relaxed $\mu - \alpha$ monotone mapping and $\phi_i : C \to \mathbb{R} \cup \{\infty\}$ be a finite family of proper convex lower semicontinuous functions. Let $K_r^1, K_r^2, \ldots, K_r^N$ be a finite family*

of resolvent operators for mixed equilibrium problems with relaxed $\mu-\alpha$ mappings on C such that $\cap_{i=1}^{N}F(K_r^i)\neq\emptyset$. Let $f:C\to C$ be a contraction with constant $\theta\in(0,1)$, let $\lambda_{n,1},\lambda_{n,2},\ldots,\lambda_{n,N}$ be real numbers satisfying $0\leq,\lambda_{n,i}\leq 1$ such that $\lim_{n\to\infty}|\lambda_{n,i}-\lambda_i|=0$ for $i=1,2,\ldots,N$ with $0\leq\lambda_i\leq 0$. For $n\in\mathbb{N}$, let U_n be a U-mapping generated by $K_{r_n}^1,K_{r_n}^2,\ldots K_{r_n}^N$ and $\lambda_{n,1},\lambda_{n,2},\ldots,\lambda_{n,N}$. Suppose $\{\alpha_n\},\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$ with $\alpha_n+\beta_n+\gamma_n=1$, r is a positive parameter and $\{r_n\}$ is a sequence in $(0,\infty)$. Assume that the conditions (i)-(v) of Lemma 2.5 and the following conditions are satisfied: (i) $\lim_{n\to\infty}\alpha_n=0$, $\sum_{n=1}^{\infty}\alpha_n=\infty$; (ii) $0<\liminf_{n\to\infty}\beta_n\leq\limsup_{n\to\infty}\beta_n<1$;
(iii) $r_n\to r$, $n\to\infty$; (iv) $\liminf_{n\to\infty} r_n>0$, $\lim_{n\to\infty}\frac{r_{n+1}}{r_n}=1$, $\lim_{n\to\infty}|\lambda_{n+1,i}-\lambda_{n,i}|=0$.
For a given $x_1\in C$, let $\{x_n\}$ be the sequence defined iteratively by

$$x_{n+1}=\alpha_n f(x_n)+\beta_n x_n+\gamma_n U_n x_n,\ \forall n\geq 1. \tag{22}$$

Then, $\{x_n\}$ converges to $P_\Gamma f(q)$, where $\Gamma=\cap_{i=1}^{N}EP(F_i,A_i)$ and P_Γ is the sunny nonexpansive retraction of C onto Γ.

Proof. The proof of this theorem will be divided into several steps.
Step 1: $\{x_n\}$ is bounded. To see this, fix $q\in\Gamma$. We have,

$$\begin{aligned}
||x_{n+1}-q|| &= ||\alpha_n f(x_n)+\beta_n x_n+\gamma_n U_n x_n-q||\\
&= ||\alpha_n(f(x_n)-q)+\beta_n(x_n-q)+\gamma_n(U_nx_n-q)||\\
&= ||\alpha_n(f(x_n)-f(q)+f(q)-q+\beta_n(x_n-q)+\gamma_n(U_nx_n-q)||\\
&\leq \alpha_n||f(x_n)-f(q)||+\alpha_n||f(q)-q||+\beta_n||x_n-q||+\gamma_n||U_nx_n-q||\\
&\leq \theta\alpha_n||x_n-q||+\alpha_n||f(q)-q||+\beta_n||x_n-q||+\gamma_n||x_n-q||\\
&\leq \theta\alpha_n||x_n-q||+(1-\alpha_n)||x_n-q||+\alpha_n||f(q)-q||\\
&\leq [1-\alpha_n(1-\theta)]||x_n-q||+\alpha_n||f(q)-q||\\
&\leq \max\left\{||x_n-q||,\frac{1}{1-\theta}||f(q)-q||\right\}\\
&\ \vdots\\
&\leq \max\left\{||x_1-q||,\frac{1}{1-\theta}||f(q)-q||\right\},\ \forall n\geq 1.
\end{aligned} \tag{23}$$

Therefore, the sequences $\{x_n\}$ and $\{U_nx_n\}$ are bounded.
Step 2: We show that

$$\lim_{n\to\infty}||x_{n+1}-x_n||=0 \tag{24}$$

Putting $y_n=\dfrac{\alpha_n f(x_n)+\gamma_nU_nx_n}{1-\beta_n}$, then (22) becomes $x_{n+1}=\beta_nx_n+(1-\beta_n)y_n$. Since U_n is nonexpansive, $\{x_n\}$ and $\{U_nx_n\}$ are bounded, we get that $\{y_n\}$ is also bounded.
Now,

$$\begin{aligned}
y_{n+1}-y_n &= \frac{\alpha_{n+1}f(x_{n+1})+\gamma_{n+1}U_{n+1}x_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n f(x_n)+\gamma_nU_nx_n}{1-\beta_n}\\
&= \left(\frac{\alpha_{n+1}}{1-\beta_{n+1}}\right)(f(x_{n+1})-f(x_n))+\left(\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\right)(f(x_n)-U_nx_n)\\
&\quad+\left(1-\frac{\alpha_{n+1}}{1-\beta_{n+1}}\right)(U_{n+1}x_{n+1}-U_nx_n),
\end{aligned}$$

hence, using Proposition 3.2 and the fact that $\theta \in (0,1)$, we have

$$\begin{aligned}
||y_{n+1}-y_n|| &\leq \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}}\right| ||f(x_{n+1})-f(x_n)|| + \left|1-\frac{\alpha_{n+1}}{1-\beta_{n+1}}\right| ||U_{n+1}x_{n+1}-U_nx_n|| \\
&\quad + \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right| ||f(x_n)-U_nx_n|| \\
&\leq \frac{\theta\alpha_{n+1}}{1-\beta_{n+1}} ||x_{n+1}-x_n|| + \left|1-\frac{\alpha_{n+1}}{1-\beta_{n+1}}\right| ||x_{n+1}-x_n|| \\
&\quad + M_N\left(N\left|1-\frac{r_{n+1}}{r_n}\right| + \sum_{i=1}^{N}|\lambda_{n+1,i}-\lambda_{n,i}|\right) \\
&\quad + \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right| ||f(x_n)-U_nx_n|| \\
&\leq ||x_{n+1}-x_n|| + \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right| ||f(x_n)-U_nx_n|| \\
&\quad + M_N\left(N\left|1-\frac{r_{n+1}}{r_n}\right| + \sum_{i=1}^{N}|\lambda_{n+1,i}-\lambda_{n,i}|\right).
\end{aligned}$$

This together with $\alpha_n \to 0$, $\frac{r_{n+1}}{r_n} = 1$ and $|\lambda_{n+1,i} - \lambda_{n,i}| \to 0$ as $n \to \infty$ implies that

$$\limsup_{n\to\infty}(||y_{n+1}-y_n|| - ||x_{n+1}-x_n||) \leq 0.$$

Hence, by Lemma 2.2, we obtain $||y_n - x_n|| \to 0$ as $n \to \infty$.
Consequently, $\lim_{n\to\infty} ||x_{n+1} - x_n|| = \lim_{n\to\infty}(1-\beta_n)||y_n - x_n|| = 0$.
Step 3: Next, we show that

$$\lim_{n\to\infty} ||x_n - Ux_n|| = 0. \tag{25}$$

We note that,

$$\begin{aligned}
||x_{n+1}-U_nx_n|| &= ||\alpha_nf(x_n)+\beta_nx_n+\gamma_nU_nx_n-U_nx_n|| \\
&\leq \alpha_n||f(x_n)-U_nx_n|| + \beta_n||x_n-U_nx_n|| \\
&\leq \alpha_n||f(x_n)-U_nx_n|| + \beta_n||x_{n+1}-x_{n+1}+x_n-U_nx_n|| \\
&\leq \frac{\alpha_n}{1-\beta_n}||f(x_n)-U_nx_n|| + \frac{\beta_n}{1-\beta_n}||x_n-x_{n+1}||.
\end{aligned}$$

From conditions (1),(2) and step 2, we have that $\lim_{n\to\infty} ||x_{n+1} - U_nx_n|| = 0$.
Also,

$$||x_n - U_nx_n|| \leq ||x_n - x_{n+1}|| + ||x_{n+1} - U_nx_n|| \to 0, \ as\ n \to \infty. \tag{26}$$

Note also that,

$$\begin{aligned}
||x_n-Ux_n|| &\leq ||x_n-U_nx_n|| + ||U_nx_n-Ux_n|| \\
&\leq ||x_n-U_nx_n|| + \sup_{x\in C}||U_nx-Ux||.
\end{aligned}\tag{27}$$

Therefore from (26) and Proposition 3.3, we have that $\lim_{n\to\infty} ||x_n - Ux_n|| = 0$.
Step 4: We show that

$$\lim_{n\to\infty} \langle f(q)-q, j(q-x_n)\rangle \leq 0. \tag{28}$$

For any $t \in (0,1)$, set $z_t = tf(z_t) + (1-t)Uz_t$. Then we have,

$$\begin{aligned}
||z_t - x_n||^2 &= ||t(f(z_t) - x_n) + (1-t)(Uz_t - x_n)||^2 \\
&\leq (1-t)^2||Uz_t - x_n||^2 + 2t\langle f(z_t) - x_n, j(z_t - x_n)\rangle \\
&\leq (1-t)^2\big[||Uz_t - Ux_n|| + ||Ux_n - x_n||\big]^2 \\
&\quad +2t\langle f(z_t) - z_t, j(z_t - x_n)\rangle + 2t\langle z_t - x_n, j(z_t - x_n)\rangle \\
&\leq (1-t)^2\big[||z_t - x_n|| + ||Ux_n - x_n||\big]^2 + 2t||z_t - x_n||^2 + 2t\langle f(z_t) - z_t, j(z_t - x_n)\rangle, \\
&\leq (1-t)^2||z_t - x_n||^2 + g_n(t) + 2t\langle f(z_t) - z_t, j(z_t - x_n)\rangle + 2t||z_t - x_n||,
\end{aligned}$$

where

$$g_n(t) = (1-t)^2(2||z_t - x_n|| + ||x_n - U_n x_n||)||x_n - U_n x_n|| \to 0 \ as\ n \to \infty. \tag{29}$$

It follows that

$$\langle z_t - f(z_t), j(z_t - x_n) \quad \leq \quad \frac{t}{2}||z_t - x_n||^2 + \frac{1}{2t}g_n(t). \tag{30}$$

Letting $n \to \infty$ in (30) and noting (29), we obtain $\langle z_t - f(z_t), j(z_t - x_n)\rangle \leq \frac{t}{2}M^*$, where $M^* = \limsup_{n\to\infty} ||z_t - x_n||^2$. Clearly $\frac{t}{2}M^* \to 0$ as $t \to 0$ from which we obtain $\limsup_{t\to 0}\limsup_{n\to\infty}\langle z_t - f(z_t), j(z_t - x_n)\rangle \leq 0$. Since j is norm-to-norm continuous on bounded subset of X and by Lemma 2.4, $z_t \to q$, where $q = P_\Gamma f(q)$, we have $||j(z_t - x_n) - j(q - x_n)|| \to 0$.
Observe that

$$\begin{aligned}
&\Big|\langle z_t - f(z_t), j(z_t - x_n)\rangle - \langle q - f(z_t), j(q - x_n)\rangle\Big| \\
&\qquad \leq \Big|\langle z_t - q, j(z_t - x_n)\rangle + \langle q - f(z_t), j(z_t - x_n)\rangle - \langle q - f(z_t), j(q - x_n)\rangle\Big| \\
&\qquad \leq \langle z_t - x_n, j(z_t - x_n)\rangle + \langle q - f(z_t), j(z_t - x_n) - j(q - x_n)\rangle \\
&\qquad \leq ||z_t - q||.||z_t - x_n|| + ||q - f(z_t)||.||j(z_t - x_n) - j(q - x_n)|| \to 0,
\end{aligned}$$

as $n \to \infty$. Therefore,

$$\langle z_t - f(z_t), j(z_t - x_n)\rangle \to \langle q - f(q), j(q - x_n)\rangle. \tag{31}$$

Hence,

$$\lim_{n\to\infty} \langle q - f(q), j(q - x_n)\rangle \leq 0. \tag{32}$$

Step 5: Finally, we show that $x_n \to q$ as $n \to \infty$. From Lemma 2.1 and step 1, we have

$$\begin{aligned}
||x_{n+1} - q||^2 &= ||\alpha_n f(x_n) + \beta_n x_n + \gamma_n U_n x_n - q||^2 \\
&= ||\alpha_n (f(x_n) - q) + \beta_n (x_n - q) + \gamma_n (U_n x_n - q)||^2 \\
&\leq ||\beta_n (x_n - q) + \gamma_n (U_n x_n - q)||^2 + 2\alpha_n \langle f(x_n) - q, j(x_{n+1} - q)\rangle \\
&\leq \{\beta_n ||x_n - q|| + \gamma_n ||x_n - q||\}^2 + 2\alpha_n \langle f(x_n) - f(q), j(x_{n+1} - q)\rangle \\
&\quad +2\alpha_n \langle f(q) - q, j(x_{n+1} - q)\rangle \\
&\leq (1-\alpha)^2 ||x_n - q||^2 + 2\theta\alpha_n (||x_n - q||.||x_{n+1} - q||) + 2\alpha_n \langle f(q) - q, j(x_{n+1} - q)\rangle \\
&\leq (1-\alpha)^2 ||x_n - q||^2 + \theta\alpha_n ||x_n - q|| + \theta\alpha_n ||x_{n+1} - q|| + 2\alpha_n \langle f(q) - q, j(x_{n+1} - q)\rangle \\
&\leq \frac{(1-\alpha_n)^2 + \theta\alpha_n}{1 - \theta\alpha_n} ||x_n - q||^2 + \frac{2\alpha_n}{1-\theta\alpha_n} \langle f(q) - q, j(x_{n+1} - q)\rangle \\
&= \frac{1 - 2\alpha_n + \theta\alpha_n}{1-\theta\alpha_n} ||x_n - q||^2 + \frac{\alpha_n^2}{1-\theta\alpha_n} ||x_n - q||^2 + \frac{2\alpha_n}{1-\theta\alpha_n} \langle f(q) - q, j(x_{n+1} - q)\rangle \\
&\leq \left\{1 - \frac{2(1-\theta)\alpha_n}{1-\theta\alpha_n}\right\} ||x_n - q||^2 + \frac{2(1-\theta)\alpha_n}{1-\theta\alpha_n} \left\{ \frac{M^* * \alpha_n}{2(1-\theta\alpha_n)} \right. \\
&\quad \left. + \frac{1}{1-\theta} \langle f(q) - q, j(x_{n+1} - q)\rangle \right\}.
\end{aligned}$$

Observe that the conditions of Lemma 2.3 are satisfied with $\gamma_n = \frac{2(1-\theta)\alpha_n}{1-\theta\alpha_n}$ and $\sigma_n = \left\{ \frac{M^* * \alpha_n}{2(1-\theta\alpha_n)} + \frac{1}{1-\theta} \langle f(q) - q, j(x_{n+1} - q)\rangle \right\}$. By Lemma 2.3 and (32), it follows that $||x_n - q|| \to 0$ as $n \to 0$. Therefore $\{x_n\}$ converges strongly to $q = P_\Gamma f(q)$. This completes the proof. $\square$

We obtain the following as consequences of Theorem 3.1.
Suppose $A_i = 0$, in Theorem 3.1, the mixed equilibrium problem with $\mu - \alpha$ monotone mapping reduces to the following classical mixed equilibrium problem: $F_i(z, y) + \phi_i(y) - \phi_i(z) \geq 0$. We thus obtain the following result:

Corollary 3.1. *Let C be a nonempty closed and convex subset of a uniformly smooth and strictly convex Banach space X which also enjoys the Kadec-Klee property. Let $\mu : C \times C \to X$ be a nonlinear mapping. For $i = 1, 2, \ldots, N$, let $F_i : C \times C \to \mathbb{R}$ be a finite family of bifunctions satisfying conditions $(F1) - (F4)$, and let $\phi_i : C \to \mathbb{R} \cup \{\infty\}$ be a finite family of proper convex lower semicontinuous functions. Let $K_r^1, K_r^2, \ldots, K_r^N$ be a finite family of resolvent operators for mixed equilibrium problems on C such that $\cap_{i=1}^N F(K_r^i) \neq \emptyset$. Let $f : C \to C$ be a contraction with constant $\theta \in (0,1)$, let $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N}$ be real numbers satisfying $0 \leq, \lambda_{n,i} \leq 1$ such that $\lim_{n\to\infty} |\lambda_{n,i} - \lambda_i| = 0$ for all $i = 1, 2, \ldots, N$. For all $n \in \mathbb{N}$, let U_n be a U-mapping generated by $K_{r_n}^1, K_{r_n}^2, \ldots K_{r_n}^N$ and $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N}$. Suppose $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$ with $\alpha_n + \beta_n + \gamma_n = 1$, r is a positive parameter and $\{r_n\}$ is a sequence in $(0, \infty)$. Assume that the conditions of Lemma 2.5 and the following conditions are satisfied:*
(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$; *(ii)* $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(iii) $r_n \to r$, $n \to \infty$; *(iv)* $\liminf_{n\to\infty} r_n > 0$, $\lim_{n\to\infty} \frac{r_{n+1}}{r_n} = 1$, $\lim_{n\to\infty} |\lambda_{n+1,i} - \lambda_{n,i}| = 0$.
For a given $x_1 \in C$, let $\{x_n\}$ be the sequence defined iteratively by

$$x_{n+1} = \alpha f(x_n) + \beta_n x_n + \gamma_n U_n x_n, \ \forall n \geq 1. \tag{33}$$

Then $\{x_n\}$ converges to $P_\Gamma f(q)$, where $\Gamma = \cap_{i=1}^N EPF_i$, P_Γ is the sunny nonexpansive retraction of C onto Γ.

For $F_i(x,y)=0$, in Theorem 3.1, the mixed equilibrium problem reduces to the following variational inequality

$$\langle A_i z, \mu(y,z)\rangle + \phi_i(y) - \phi_i(z) \geq 0.$$

We obtain a result which solves the finite family of variational inequalities as follows:

Corollary 3.2. *Let C be a nonempty closed and convex subset of a uniformly smooth and strictly convex Banach space X which also enjoys the Kadec-Klee property. Let $\mu : C \times C \to \mathbb{R}$ be a nonlinear mapping. For $i = 1, 2, \ldots, N$, let $A_i : C \to X^*$ be a finite family of μ-hemicontinuous relaxed $\mu - \alpha$ monotone mapping and let $\phi_i : C \to \mathbb{R}$ be a finite family of proper convex lower semicontinuous functions. Let $K_r^1, K_r^2, \ldots, K_r^N$ be a finite family of resolvent operators for variational inequalities with relaxed $\mu - \alpha$ mappings on C such that $\cap_{i=1}^N F(K_r^i) \neq \emptyset$. Let $f : C \to C$ be a contraction with constant $\theta \in (0,1)$, let $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N}$ be real numbers satisfying $0 \leq, \lambda_{n,i} \leq 1$ such that $\lim_{n\to\infty} |\lambda_{n,i} - \lambda_i| = 0$ for all $i = 1, 2, \ldots, N$. For all $n \in \mathbb{N}$, let U_n be a U-mapping generated by $K_{r_n}^1, K_{r_n}^2, \ldots K_{r_n}^N$ and $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N}$. Suppose $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$ with $\alpha_n + \beta_n + \gamma_n = 1$, r is a positive parameter and $\{r_n\}$ is a sequence in $(0,\infty)$. Assume that the conditions of Lemma 2.5 and the following conditions are satisfied:*
(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$; (ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(iii) $r_n \to r$, $n \to \infty$; (iv) $\liminf_{n\to\infty} r_n > 0$, $\lim_{n\to\infty} \frac{r_{n+1}}{r_n} = 1$, $\lim_{n\to\infty} |\lambda_{n+1,i} - \lambda_{n,i}| = 0$.
For a given $x_1 \in C$, let $\{x_n\}$ be the sequence defined iteratively by

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n U_n x_n, \ \forall n \geq 1. \tag{34}$$

Then $\{x_n\}$ converges to $P_\Gamma f(q)$, where $\Gamma = \cap_{i=1}^N VIA_i$, P_Γ is the sunny nonexpansive retraction of C onto Γ.

4. NUERICAL EXAMPLE

Let $X = \mathbb{R} \times \mathbb{R}$ and $C = [-1,1] \times [-1,1]$. Define a mapping $A : C \to \mathbb{R} \times \mathbb{R}$ by $A(x_1, x_2) = (x_1, x_2)$ for all $(x_1, x_2) \in C$, $\alpha : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $\alpha((x_1, x_2)) = 3x_1^2 + 3x_2^2$ for all $(x_1, x_2) \in X$ and $\mu : C \times C \to \mathbb{R} \times \mathbb{R}$ by $\mu((x_1, x_2), (y_1, y_2)) = (2(x_1 - y_1), 2(x_2 - y_2))$ for all $(x_1, x_2) \times (y_1, y_2) \in C \times C$. Then the mapping A is a relaxed $\mu - \alpha$ monotone mapping. Indeed, for all $x = (x_1, x_2), y = (y_1, y_2) \in C$, we have

$$\begin{aligned} \langle Ax - Ay, \mu(x,y)\rangle &= ((x_1 - y_1), (x_2 - y_2)), (2(x_1 - y_1), 2(x_2 - y_2)) \\ &= 4[(x_1 - y_1)^2 + (x_2 - y_2)^2] \\ &\geq 3[(x_1 - y_1)^2 + (x_2 - y_2)^2] = \alpha(x - y). \end{aligned} \tag{35}$$

Hence, A is a relaxed $\mu - \alpha$ monotone mapping. Let $\bar{z} = (z_1, z_2)$, $\bar{y} = (y_1, y_2)$ and $\bar{x} = (x_1, x_2)$. Define $F_i(\bar{z}, \bar{y}) = -3i\bar{z}^2 + 2i\bar{z}\bar{y} + i\bar{y}^2, A_i(\bar{z}) = i\bar{z}$ and $\phi_i(\bar{z}) = i\bar{z}^2$. Lemma 2.5 ensures that there exist $\bar{x} \in \mathbb{R}^2$ such that

$$F_i(\bar{z}, \bar{y}) + \langle A_i\bar{z}, \mu(\bar{y}, \bar{z})\rangle + \phi_i(\bar{y}) - \phi_i(\bar{z}) + \frac{1}{r_n}\langle \bar{y} - \bar{z}, \bar{z} - \bar{x}\rangle \geq 0 \ \forall \ \bar{y} \in \mathbb{R}^2$$

$$\begin{aligned} \iff \quad & -3i\bar{z}^2 + 2i\bar{z}\bar{y} + i\bar{y}^2 + i\bar{z}(2(\bar{y} - \bar{z})) + (i\bar{y}^2) - (i\bar{z}^2) + \frac{1}{r_n}(\bar{y} - \bar{z}) \times (\bar{z} - \bar{x}) \geq 0 \\ \iff \quad & -3i\bar{z}^2 + 2i\bar{z}\bar{y} + i\bar{y}^2 + 2i\bar{y}\bar{z} - 2i\bar{z^2} + (i\bar{y}^2) - (i\bar{z}^2) + \frac{1}{r_n}(\bar{y}\bar{z} - \bar{y}\bar{x} - \bar{z}^2 + \bar{z}\bar{x}) \geq 0 \\ \iff \quad & -3ir_n\bar{z}^2 + 2r_n i\bar{z}\bar{y} + ir_n\bar{y}^2 + 2r_n i\bar{y}\bar{z} - 2r_n i\bar{z^2} + r_n i\bar{y}^2 - r_n i\bar{z}^2 + \bar{y}\bar{z} - \bar{y}\bar{x} - \bar{z}^2 + \bar{z}\bar{x} \geq 0 \\ \iff \quad & 2ir_n\bar{y}^2 + (4ir_n\bar{z} + \bar{z} - \bar{x})\bar{y} + \bar{z}\bar{x} - \bar{z}^2 - 6ir_n\bar{z}^2 \geq 0. \end{aligned}$$

Let $H(\bar{y}) = 2ir_n\bar{y}^2 + (4ir_n\bar{z} + \bar{z} - \bar{x})\bar{y} + \bar{z}\bar{x} - \bar{z}^2 - 6ir_n\bar{z}^2$, then $H(\bar{y})$ is a quadratic equation in $\bar{y}$.

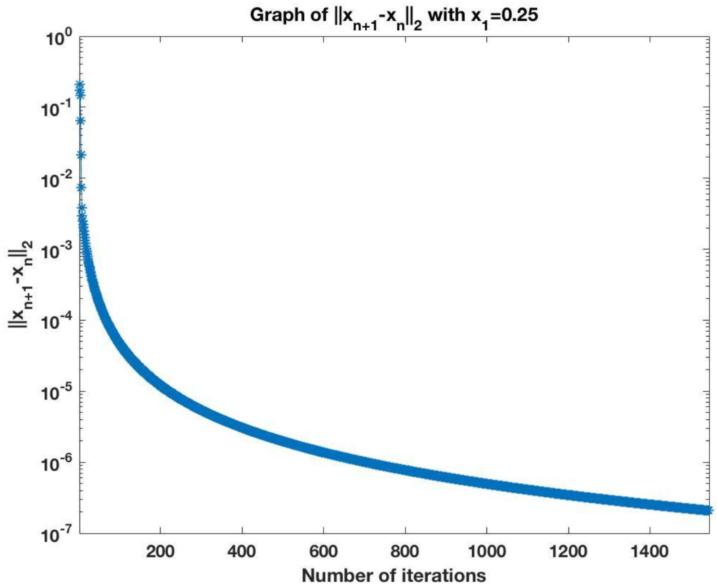

FIGURE 1. Errors vs number of iterations for initial value **1**.

With $a = 2ir_n$, $b = 4ir_n\bar{z} + \bar{z} - \bar{x}$ and $c = -6ir_n\bar{z}^2 - \bar{z}^2 + \bar{z}\bar{x}$. We obtain the discriminant Δ of $H(\bar{y})$ as follows:

$$\begin{aligned}\Delta &= b^2 - 4ac = (4ir_n\bar{z} + \bar{z} - \bar{x})^2 - 4(2ir_n)(-6ir_n\bar{z}^2 - \bar{z}^2 + \bar{z}\bar{x}) \\ &= \bar{x}^2 + 64i^2r_n^2\bar{z}^2 + 16ir_n\bar{z} + \bar{z}^2 - 16ir_n\bar{x}\bar{z} - 2\bar{x}\bar{z} = \bar{x}^2 + (8ir_n\bar{z} + \bar{z})^2 - 2\bar{x}\bar{z} - 16ir_n\bar{x}\bar{z} \\ &= \bar{x}^2 - 2(8ir_n\bar{z} + \bar{z})\bar{x} + (8ir_n\bar{z} + \bar{z})^2 = (\bar{x} - (8r_n\bar{z} + \bar{z})) \geq 0.\end{aligned}$$

Hence, $\bar{z} = \dfrac{\bar{x}}{8ir_n + 1}$. This implies $\bar{z} = \left(\dfrac{x_1}{8ir_n + 1}, \dfrac{x_2}{8ir + n - 1}\right)$ and thus

$$K_{r_n}^i(\bar{x}) = \left(\frac{x_1}{8ir_n + 1}, \frac{x_2}{8ir_n + 1}\right). \tag{36}$$

Assume that $\lambda_{n,i} = \dfrac{1}{in + 2}$ and $S_{n,0}\bar{x} = \bar{x}$. Using (4) and (36), we have

$$S_{n,i}\bar{x} = \frac{1}{in + 2} \times \frac{1}{8ir_n + 1} S_{n,i-1}\bar{x} + \frac{in + 1}{in + 2} S_{n,i-1}\bar{x}, \text{ for } i = 1, 2, \ldots, 100, \tag{37}$$

and $U_n = S_{n,100}$. Choosing $\alpha_n = \frac{1}{n+1}$, $\beta_n = \frac{8}{8n-1}$, $\gamma_n = \frac{16n-7}{8n^2+7n-1}$ and $r_n = \frac{n-1}{2n+1}$. Let $f(\bar{x}) = \frac{1}{10}\bar{x}$, then our iterative algorithm (33) becomes $\bar{x}_{n+1} = \frac{\bar{x}_n}{10(n+1)} + \frac{8\bar{x}_n}{8n-1} + \frac{16n-7}{8n^2+7n-1} U_n\bar{x}_n$, $\forall\, n \geq 1$. We make different choices of our initial value as follow: (1) $\bar{x}_1 = -0.5$, (2) $\bar{x}_1 = 0.05$ and (3) $\bar{x}_1 = 0.25$. We also vary the stopping criterion as: (a) $\dfrac{|\bar{x}_{n+1} - \bar{x}_n|}{|\bar{x}_2 - \bar{x}_1|} < 10^{-6}$ and (b) $\dfrac{|\bar{x}_{n+1} - \bar{x}_n|}{|\bar{x}_2 - \bar{x}_1|} < 10^{-12}$. Matlab version 2014a is used to obtain the graphs of errors against the number of iterations.

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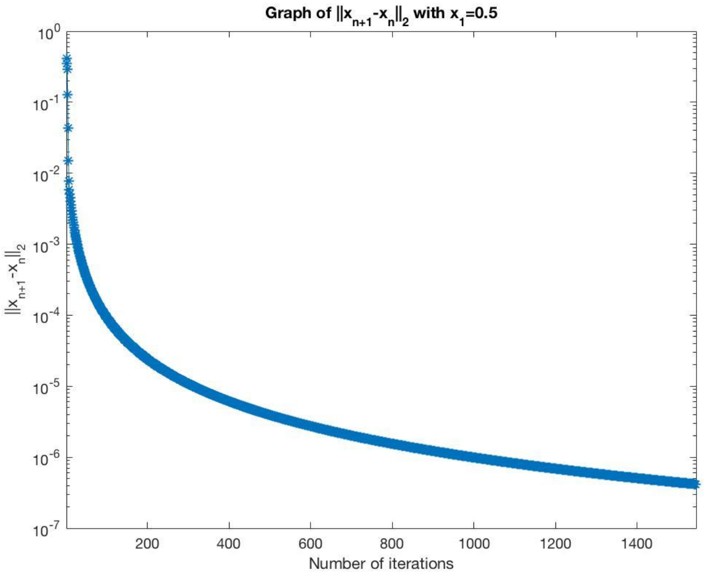

FIGURE 2. Errors vs number of iterations for initial value **2**.

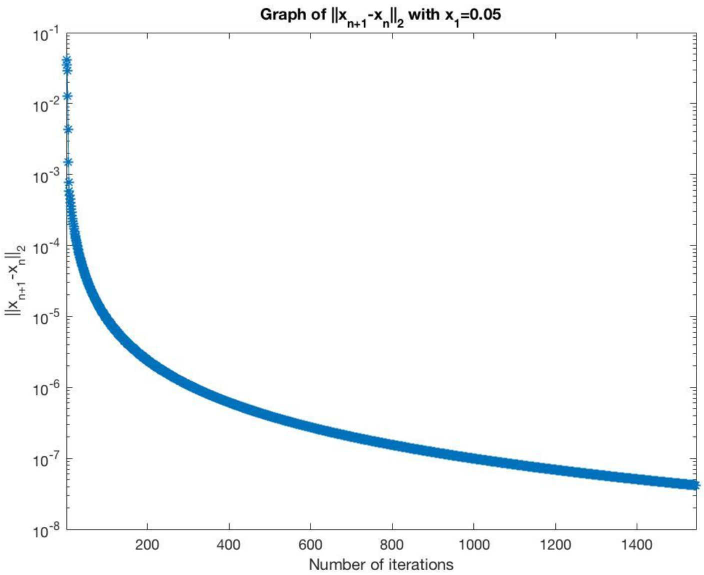

FIGURE 3. Errors vs number of iterations for initial value **3**.

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