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q-DONOHO-STARK'S UNCERTAINTY PRINCIPLE AND q-TIKHONOV REGULARIZATION PROBLEM

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This paper deals with the extension of the Donoho-Stark's uncertainty principle for the class of Fourier multiplier operators $T_m := \mathcal{F}_q^{-1}(m\mathcal{F}_q)$ to time scale. Furthermore, the Bochner-Riesz mean operator, the Weierstrass transform and the Poisson integral are given using Fourier multiplier operators. Finally, the exact expression and some properties of the extremal functions of the so-called Tikhonov regularization problem are also determined; using reproducing kernel methods.

Keywords: Time scale, Fourier multiplier operators, Donoho-Stark's uncertainty principle, Tikhonov regularization, approximate formulas.

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1. Introduction

A time scale, denoted by $\mathbb{T}$, is a nonempty closed subset of the real numbers. We assume throughout that $\mathbb{T}$ has the topology that it inherits from the standard topology on the real numbers $\mathbb{R}$. The calculus on time scales is a relatively new area that unifies the difference and differential calculus, which are obtained by choosing the time scale $\mathbb{T}$ for example the real numbers $\mathbb{R}$, the integers $\mathbb{Z}$, the natural numbers $\mathbb{N}$, the nonnegative integers $\mathbb{N}_0$, the h-numbers $h\,\mathbb{Z} := \{h\,k, k \in \mathbb{Z}\}$ with fixed $h > 0$, and the q-numbers $q^{\mathbb{Z}} := \{q^k, k \in \mathbb{Z}\}$ with fixed $q \in (0,1)$. A strong current research has been developed in many different fields in which dynamic processes can be described with discrete, continuous, or hybrid models. The time scale theory was found promising because it demonstrates the interplay between the theories of continuous time and discrete-time systems. It leads to a new understanding and analyzing of dynamical systems on any nonuniform time domains that are closed subsets of $\mathbb{R}$. In particular, calculus on the time scale $\mathbb{T} := \mathbb{R}_{q,+} = \{q^k, k \in \mathbb{Z}\}$, $q \in (0,1)$ is called "Quantum calculus" or "q-calculus" and much recent research activity and applications has focused on this theory. This branch of mathematics continues to find new and useful applications. Quantum calculus is the modern name for the investigation of calculus without limits. It's appeared as a connection between mathematics and physics. Quantum calculus has its own definition of derivative, integral, exponential, sine, cosine etc. All these notions depend on an a priori given number q. Many q-notions approach their classical analogs as $q \uparrow 1$. The real line $\mathbb{R}$ is replaced, essentially, by the set $\mathbb{R}_q$ of points accumulating at 0.

In this paper, we present a unification proof of many inequalities and approximations for the classical and discrete case; by means of the theory of time scales. The idea is to extend the cosine Fourier transformation $\mathcal{F}_q$ to time scale, which sends a function f on $\mathbb{R}_{q,+}$ to a function $\mathcal{F}_q(f)$ on the same set; and has many properties analogous to those of the classical Fourier transform. In particular, it acts as an isometry of the space $L^2(\mathbb{R}_{q,+})$ of

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functions f on $\mathbb{R}_{q,+}$ with finite norm $\|.\|_{L^2(\mathbb{R}_{q,+})}$, defined by

$$\|f\|_{L^2(\mathbb{R}_{q,+})} := (1-q)\Big(\sum_{n\in\mathbb{Z}} q^n \mid f(q^n) \mid^2 \Big)^{1/2}.$$

Moreover, it satisfies $\mathcal{F}_q(\mathcal{F}_q(f)) = f$.

In this paper we study the Fourier multiplier operators T_m defined for $f \in L^2(\mathbb{R}_{q,+})$ by

$$T_m f := \mathcal{F}_q(m\mathcal{F}_q(f)),$$

where $m : \mathbb{T} \to \mathbb{R}$ is a bounded function. Thus the Fourier transform of T_m reduces to the multiplication by m. Such multiplier operators play a decisive role in the classical Fourier analysis.

Central to this work is the best approximation problem in quantum calculus,

$$\inf_{f\in\mathcal{H}^s_{*,q}(\mathbb{R}_q)} \Big\{\eta\|f\|^2_{\mathcal{H}^s_{*,q}(\mathbb{R}_q)} + \|g - T_m f\|^2_{L^2(\mathbb{R}_{q,+})}\Big\} \tag{1.1}$$

for an unknown function f, where $g \in L^2(\mathbb{R}_{q,+})$ is a given function and $\eta > 0$, $s > 1/2$ are parameters with s fixed throughout and η approaching eventually 0. Here $\mathcal{H}^s_{*,q}(\mathbb{R}_q)$ is the q-analog of the Sobolev space of fractional order s,

$$\mathcal{H}^s_{*,q}(\mathbb{R}_q) := \Big\{f \in L^2(\mathbb{R}_{q,+}) : (1+z^2)^{s/2}\mathcal{F}_q(f)(z) \in L^2(\mathbb{R}_{q,+})\Big\}.$$

We provide some analysis of the minimizer $f^*_{\eta,g}$ of the problem (1.1). Especially we use the theory of Fourier transform in quantum calculus, to give integral representations of $f^*_{\eta,g}$; and to examine the convergence rates of these type of representations.

In the limit case $\eta \uparrow 0$, the problem (1.1) reduces to the Tikhonov regularization problem

$$\inf_{f\in\mathcal{H}^s_{*,q}(\mathbb{R}_q)} \Big\{\|g - T_m f\|^2_{L^2(\mathbb{R}_{q,+})}\Big\}.$$

The paper is divided into five sections and is organized as follows. This paper is organized as follows. In Section 2, we present preliminaries, definitions and concepts concerning time scale calculus and basic notions that will be needed in the proofs of the main result. In Section 3, we define and study the Fourier multiplier operators T_m on $\mathbb{R}_{q,+}$ and we give three examples: the q-Bochner-Riesz mean operator, the q-Weierstrass transform and the q-Poisson integral. In Section 4, we present extensions of an Donoho-Stark's uncertainty principle for the class of Fourier multiplier operators T_m on time scale. In the last section, we give an application of the theory of Fourier transform on time scale, to examine the minimizer $f^*_{\eta,g}$ of the problem (1.1).

2. **Preliminaries**

Throughout the paper assume that $0 < q < 1$. For $a \in \mathbb{C}$, the q-shifted factorial $(a;q)_k$ is defined as a product of k factors

$$(a;q)_0 = 1,\ (a;q)_k = (1-a)(1-aq)...(1-aq^{k-1}), \quad k = 1,2,...$$

This definition remains meaningful for $k = \infty$ as a convergent infinite product

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$

The q-derivative of a function f given on a subset of $\mathbb{R}$ or $\mathbb{C}$ is defined by

$$D_{q,x}f(x) := \frac{f(x)-f(qx)}{(1-q)x}, \quad x, q \neq 0,$$

where x and qx should be in the domain of f. By continuity we set $(D_q f)(0) = f^{'}(0)$ provided $f^{'}(0)$ exists.

For $a > 0$ and a function f given on $(0, a]$ we define the q-integral by

$$\int_0^a f(x) \mathrm{d}_q x := (1-q) a \sum_{n=0}^{\infty} f(aq^n) q^n.$$

The improper integral is defined in the following way

$$\int_0^\infty f(x) \mathrm{d}_q x := (1-q) \sum_{k=-\infty}^{+\infty} f\left(q^k\right) q^k.$$

We begin by putting

$$\mathbb{R}_q = \{\pm q^k, k \in \mathbb{Z}\}, \quad \mathbb{R}_{q,+} = \{q^k, k \in \mathbb{Z}\}, \quad \widetilde{\mathbb{R}}_{q,+} = \{q^k, k \in \mathbb{Z}\} \cup \{0\}.$$

Let $E := [0, a]_q := \{q^k, k \in \mathbb{Z}, k \geq n\}$, $a = q^n$ be subset of $\mathbb{R}_{q,+}$.
We denote by μ the measure on $\mathbb{R}_{q,+}$ given by $\mathrm{d}_q\mu(y) := c_q \mathrm{d}_q y$; and by $L^p(\mathbb{R}_{q,+})$, $1 \leq p \leq \infty$, the space of functions f on $\mathbb{R}_{q,+}$, such that

$$\|f\|_{L^p(\mathbb{R}_{q,+})} := \Big(\int_0^\infty |f(y)|^p \mathrm{d}_q\mu(y) \Big)^{1/p} < \infty, \quad 1 \leq p < \infty,$$

$$\|f\|_{L^\infty(\mathbb{R}_{q,+})} := \operatorname{ess} \sup_{y \in \mathbb{R}_{q,+}} |f(y)| < \infty,$$

where

$$c_q = \Big(\frac{1+q}{1-q} \Big)^{-1/2} \Gamma_{q^2}^{-1}(1/2), \tag{2.1}$$

and the q-gamma function (see [7, 8], Section 1.3) is defined by

$$\Gamma_q(z) := \frac{(q;q)_\infty}{(q^z;q)_\infty} (1-q)^{1-z}, \quad 0 < q < 1, \quad z \neq 0, -1, -2, ...$$

We take the definition of q-trigonometric function [9] and we write q-cosine as a series of functions

$$\cos(x;q^2) := \sum_{n=0}^{\infty} (-1)^n b_n(x;q^2) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n+1)}}{(q;q)_{2n}} x^{2n}.$$

On $\pm q^{\mathbb{Z}}$ this function is bounded and there it satisfy

$$|\cos(x;q^2)| \leq \frac{1}{(q;q^2)^2_\infty}.$$

Let f be a function in $L^1(\mathbb{R}_{q,+})$, the q-even translation operators $T_{q,x}$ are defined by

$$T_{q,x} f(y) := \int_0^\infty f(z) D_q(x,y,z) \mathrm{d}_q z,$$

where c_q is given by (2.1) and $D_q(x,y,z)$ is defined for x and y in $\mathbb{R}_{q,+}$ by

$$D_q(x,y,z) := c_q^2 \int_0^\infty \cos(xt;q^2) \cos(yt;q^2) \cos(zt;q^2) \mathrm{d}_q t.$$

In particular the following product formula holds

$$T_{q,y} \cos(tx;q^2) = \cos(tx;q^2) \cos(ty;q^2).$$

Specially, we need the positivity of the q-even translation operator [6] for proving the following inequality for $f \in L^1(\mathbb{R}_{q,+})$,

$$\| T_{q,x} f \|_{L^1(\mathbb{R}_{q,+})} \leq \| f \|_{L^1(\mathbb{R}_{q,+})} .$$

This positivity property holds if $q \in (0, q_0]$, where q_0 is first zero of the q-hypergeometric function [7] $q \mapsto {}_1\varphi_1(0; q; q, q)$.

For $f \in L^1(\mathbb{R}_{q,+})$ the q-cosine Fourier transform is defined by

$$\mathcal{F}_q(f)(x) := \int_0^\infty \cos(xy; q^2) f(y) \mathrm{d}_q\mu(y), \quad x \in \mathbb{R}_{q,+}.$$

The q-analogue of the elementary exponential functions are crucial. They are defined by

$$E(x;q) := (-(1-q)x;q)_\infty = \sum_0^\infty q^{\frac{n(n-1)}{2}} \frac{(1-q)^n}{(q;q)_n} x^n, \quad x \in \mathbb{R}, \tag{2.2}$$

and

$$e(x;q) := \frac{1}{((1-q)x;q^2)_\infty} = \sum_0^\infty \frac{(1-q)^n}{(q;q)_n} x^n, \quad |x| < \frac{1}{1-q}. \tag{2.3}$$

The q-cosine Fourier transform $\mathcal{F}_q$ satisfies the following properties.

(i) $L^1 - L^\infty$*-boundedness.* For all $f \in L^1(\mathbb{R}_{q,+})$, $\mathcal{F}_q(f) \in L^\infty(\mathbb{R}_{q,+})$ and

$$\|\mathcal{F}_q(f)\|_{L^\infty(\mathbb{R}_{q,+})} \le \frac{1}{(q;q^2)^2_\infty} \|f\|_{L^1(\mathbb{R}_{q,+})}. \tag{2.4}$$

(ii) *Inversion theorem* ([4], Theorem 3.2). Let $f \in L^1(\mathbb{R}_{q,+})$, such that $\mathcal{F}_q(f) \in L^1(\mathbb{R}_{q,+})$. Then

$$f(x) = \mathcal{F}_q(\mathcal{F}_q(f))(x), \qquad x \in \mathbb{R}_{q,+}. \tag{2.5}$$

(iii) *Plancherel theorem* ([4], Theorem 7.7). The q-cosine Fourier transform $\mathcal{F}_q$ extends uniquely to an isometric isomorphism of $L^2(\mathbb{R}_{q,+})$ onto itself. In particular,

$$\|\mathcal{F}_q(f)\|_{L^2(\mathbb{R}_{q,+})} = \|f\|_{L^2(\mathbb{R}_{q,+})}. \tag{2.6}$$

3. q-Fourier multiplier operators on $\mathbb{R}_{q,+}$

Let m be a function in $L^\infty(\mathbb{R}_{q,+})$. The q-cosine Fourier multiplier operators T_m [13], are defined for $f \in L^2(\mathbb{R}_{q,+})$ by

$$T_m f := \mathcal{F}_q(m\mathcal{F}_q(f)). \tag{3.1}$$

Then, for $f \in L^2(\mathbb{R}_{q,+})$, we have

$$\|T_m f\|_{L^2(\mathbb{R}_{q,+})} \le \|m\|_{L^\infty(\mathbb{R}_{q,+})} \|f\|_{L^2(\mathbb{R}_{q,+})}. \tag{3.2}$$

As applications, we give the following examples.

Example 3.1. Let m be the function defined for $t > 0$ and $\beta > 0$ by

$$m(z) := W_{\beta+\frac{1}{2}}(z/t; q^2)\, \chi_{[0,t]_q}(z), \quad z \in \mathbb{R}_{q,+},$$

where W_α is the q-binomial function [5] given by $W_\alpha(x;q^2) = \dfrac{(x^2q^2;q^2)_\infty}{(x^2q^{2\alpha+1};q^2)_\infty}$, which tend to $(1-x^2)^{\alpha-1/2}$ as $q \uparrow 1^-$. Then $T_m f = \sigma_{\beta,t;q}(f)$, where $\sigma_{\beta,t;q}(f)$ is the q-analogue of the Bochner-Riesz mean operator of f (see [2, 3]) given by

$$\sigma_{\beta,t;q}(f)(x) := \left(\frac{1-q}{1+q}\right)^{1/2} \frac{\Gamma_{q^2}(\beta+1)\, t}{(1+q)\Gamma_{q^2}(\beta+3/2)} \int_0^\infty f(z)\, T_{q,x} j_{\beta+1/2}(tz;q^2) \mathrm{d}_q\mu(y).$$

Here, $j_\alpha(z;q)$ is The normalized third Jackson's q-Bessel function of order α $j_\alpha(z;q) := \dfrac{(q;q)_\infty}{(q^{\alpha+1};q)_\infty} z^{-\alpha} J_\alpha^{(3)}(z;q)$, where $J_\alpha^{(3)}(.;q)$ is the third Jackson's q-Bessel function (this is called the Hahn-Exton q-Bessel function $J_\alpha(.;q)$ (see [9], (3.3)).

Example 3.2. Let m be the function defined for $t > 0$ by $m(z) := e_{q^2}\left(-tz^2\right)$, $z \in \mathbb{R}_{q,+}$, where $e_{q^2}(x)$ is given by (2.3). Then $T_m f = W_{t,q}(f)$, where $W_{t,q}(f)$ is the q-analogue of the Weierstrass transform of f (see [10, 14]) given by

$$W_{t,q}(f)(x) := A^{-1}(t;q^2)\int_0^\infty f(z)T_{q,x}e_{q^2}\left(-\frac{z^2}{qt(1+q)^2}\right)\mathrm{d}_q\mu(z),$$

where $A(t;q^2) = q^{-\frac{1}{2}}(1-q)^{\frac{1}{2}}\dfrac{(-\frac{1-q}{1+q}\frac{1}{t}, -\frac{1+q}{1-q}q^2t;q^2)_\infty}{(-\frac{1-q}{1+q}\frac{1}{qt}, -\frac{1+q}{1-q}q^3t;q^2)_\infty}$.

Example 3.3. Let m be the function defined for $t > 0$ by $m(z) := \mathcal{E}_q^{(1/2)}(-tz)$, $z \in \mathbb{R}_{q,+}$, where the function $\mathcal{E}_q^{(1/2)}$ is defined [1] by $\mathcal{E}_q^{(\alpha)}(z) := \sum_{k=0}^{\infty}\dfrac{q^{\alpha k^2/2}}{(q;q)_k}\, z^k$ $,\alpha \in \mathbb{C}$.

Then $T_m f = P_{t,q}(f)$, where $P_t(f)$ is the q-Poisson integral of f (see [11]) given by

$$P_{t,q}(f)(x) := \left(\frac{1-q}{1+q}\right)^{1/2} A^{-1}(\frac{1}{q(1+q)^2};q^2)\int_0^\infty f(z)\,T_{q,x}\left[\frac{t}{z^2+t^2}\right]\mathrm{d}_q\mu(z).$$

4. q-Donoho-Stark's uncertainty principle for T_m

In this section we establish a q-Donoho-Stark's uncertainty principle for the operators T_m. Let $E := [0,a]_q := \{q^k, k \in \mathbb{Z}, k \geq n\}$, $a = q^n$ be a subset of $\mathbb{R}_{q,+}$. We say that a function $f \in L^2(\mathbb{R}_{q,+})$, is ε-concentrated on E, if

$$\|f - \chi_E f\|_{L^2(\mathbb{R}_{q,+})} \leq \varepsilon\|f\|_{L^2(\mathbb{R}_{q,+})}, \tag{4.1}$$

where χ_E is the indicator function of the set E.

Let S be a subset of $\mathbb{R}_{q,+}$ and let $f \in L^2(\mathbb{R}_{q,+})$. We say that that $T_m f$ is ν-concentrated on S, if

$$\|T_m f - \chi_S T_m f\|_{L^2(\mathbb{R}_{q,+})} \leq \nu\|T_m f\|_{L^2(\mathbb{R}_{q,+})}. \tag{4.2}$$

Theorem 4.1. *Let $f \in L^2(\mathbb{R}_{q,+})$ and let $m \in L^1 \cap L^\infty(\mathbb{R}_{q,+})$. If f is ε-concentrated on E and $T_m f$ is ν-concentrated on S, then*

$$(\mu(E))^{1/2}(\mu(S))^{1/2} \geq (q;q^2)^4_\infty \frac{\|m\mathcal{F}_q(f)\|_{L^2(\mathbb{R}_{q,+})} - (\nu+\varepsilon)\|m\|_{L^\infty(\mathbb{R}_{q,+})}\|f\|_{L^2(\mathbb{R}_{q,+})}}{\|m\|_{L^1(\mathbb{R}_{q,+})}\|f\|_{L^2(\mathbb{R}_{q,+})}}.$$

Proof. Let $f \in L^2(\mathbb{R}_{q,+})$ and let $m \in L^1 \cap L^\infty(\mathbb{R}_{q,+})$. Assume that $\mu(E) < \infty$ and $\mu(S) < \infty$. From (3.2), (4.1) and (4.2) it follows that

$$\begin{aligned}
\|T_m f - \chi_S T_m(\chi_E f)\|_{L^2(\mathbb{R}_{q,+})} &\leq \|T_m f - \chi_S T_m f\|_{L^2(\mathbb{R}_{q,+})} + \|\chi_S T_m(f - \chi_E f)\|_{L^2(\mathbb{R}_{q,+})}\\
&\leq \nu\|T_m f\|_{L^2(\mathbb{R}_{q,+})} + \|T_m(f - \chi_E f)\|_{L^2(\mathbb{R}_{q,+})}\\
&\leq \|m\|_{L^\infty(\mathbb{R}_{q,+})}\Big(\nu\|f\|_{L^2(\mathbb{R}_{q,+})} + \|f - \chi_E f\|_{L^2(\mathbb{R}_{q,+})}\Big)\\
&\leq (\nu+\varepsilon)\|m\|_{L^\infty(\mathbb{R}_{q,+})}\|f\|_{L^2(\mathbb{R}_{q,+})}.
\end{aligned}$$

Then the triangle inequality shows that

$$\begin{aligned}
\|T_m f\|_{L^2(\mathbb{R}_{q,+})} &\leq \|\chi_S T_m(\chi_E f)\|_{L^2(\mathbb{R}_{q,+})} + \|T_m f - \chi_S T_m(\chi_E f)\|_{L^2(\mathbb{R}_{q,+})}\\
&\leq \|\chi_S T_m(\chi_E f)\|_{L^2(\mathbb{R}_{q,+})} + (\nu+\varepsilon)\|m\|_{L^\infty(\mathbb{R}_{q,+})}\|f\|_{L^2(\mathbb{R}_{q,+})}.
\end{aligned}$$

But

$$\|\chi_S T_m(\chi_E f)\|_{L^2(\mathbb{R}_{q,+})} = \left(\int_S |T_m(\chi_E f)(x)|^2\mathrm{d}_q\mu(x)\right)^{1/2},$$

and

$$\begin{aligned} |T_m(\chi_E f)(x)| &\le \frac{1}{(q;q^2)^2_\infty}\|m\mathcal{F}_q(\chi_E f)\|_{L^1(\mathbb{R}_{q,+})} \\ &\le \frac{1}{(q;q^2)^2_\infty}\|m\|_{L^1(\mathbb{R}_{q,+})}\|\mathcal{F}_q(\chi_E f)\|_{L^\infty(\mathbb{R}_{q,+})} \\ &\le \frac{1}{(q;q^2)^4_\infty}\|m\|_{L^1(\mathbb{R}_{q,+})}\|\chi_E f\|_{L^1(\mathbb{R}_{q,+})} \\ &\le \frac{1}{(q;q^2)^4_\infty}\|m\|_{L^1(\mathbb{R}_{q,+})}\|f\|_{L^2(\mathbb{R}_{q,+})}(\mu(E))^{1/2}. \end{aligned}$$

Thus,

$$\|\chi_S T_m(\chi_E f)\|_{L^2(\mathbb{R}_{q,+})} \le \frac{1}{(q;q^2)^4_\infty}\,\|m\|_{L^1(\mathbb{R}_{q,+})}\|f\|_{L^2(\mathbb{R}_{q,+})}(\mu(E))^{1/2}(\mu(S))^{1/2}$$

and

$$\begin{aligned} \|T_m f\|_{L^2(\mathbb{R}_{q,+})} &\le \frac{1}{(q;q^2)^4_\infty}\,\|m\|_{L^1(\mathbb{R}_{q,+})}\|f\|_{L^2(\mathbb{R}_{q,+})}(\mu(E))^{1/2}(\mu(S))^{1/2} \\ &+ (\nu+\varepsilon)\|m\|_{L^\infty(\mathbb{R}_{q,+})}\|f\|_{L^2(\mathbb{R}_{q,+})}. \end{aligned}$$

By applying (2.6), we obtain

$$(\mu(E))^{1/2}(\mu(S))^{1/2} \ge (q;q^2)^4_\infty\,\frac{\|m\mathcal{F}_q(f)\|_{L^2(\mathbb{R}_{q,+})} - (\nu+\varepsilon)\|m\|_{L^\infty(\mathbb{R}_{q,+})}\|f\|_{L^2(\mathbb{R}_{q,+})}}{\|m\|_{L^1(\mathbb{R}_{q,+})}\|f\|_{L^2(\mathbb{R}_{q,+})}},$$

which gives the desired result. □

Example 4.2. Let m be the function defined for $t>0$ and $\beta>0$ by

$$m(z) := W_{\beta+\frac{1}{2}}(z/t;q^2)\,\chi_{[0,t]_q}(z), \quad z\in\mathbb{R}_{q,+},$$

Then $\|m\|_{L^\infty(\mathbb{R}_{q,+})}=1$, $\|m\|_{L^1(\mathbb{R}_{q,+})} = D_q(t) = \frac{(1-q)^{1/2}\Gamma_{q^2}(\beta+1)\,t}{(1+q)^{3/2}\Gamma_{q^2}(\beta+3/2)}$ and

$$(\mu(E))^{1/2}(\mu(S))^{1/2} \ge D_q^{-1}(t)(q;q^2)^4_\infty\,\frac{\|\sigma_{\beta,t;q}(f)\|_{L^2(\mathbb{R}_{q,+})} - (\nu+\varepsilon)\|f\|_{L^2(\mathbb{R}_{q,+})}}{\|f\|_{L^2(\mathbb{R}_{q,+})}}.$$

Example 4.3. Let m be the function defined for $t>0$ by $m(z) := e_{q^2}\left(-tz^2\right)$, $z\in\mathbb{R}_{q,+}$.

Then $\|m\|_{L^\infty(\mathbb{R}_{q,+})}=1$, $\|m\|_{L^1(\mathbb{R}_{q,+})} = A^{-1}(t;q^2)$ and

$$(\mu(E))^{1/2}(\mu(S))^{1/2} \ge A(t;q^2)\,(q;q^2)^4_\infty\,\frac{\|W_{t,q}(f)\|_{L^2(\mathbb{R}_{q,+})} - (\nu+\varepsilon)\|f\|_{L^2(\mathbb{R}_{q,+})}}{\|f\|_{L^2(\mathbb{R}_{q,+})}}.$$

Example 4.4. Let m be the function defined for $t>0$ by $m(z) := \mathcal{E}_q^{(1/2)}(-tz)$, $z\in\mathbb{R}_{q,+}$.

Then $\|m\|_{L^\infty(\mathbb{R}_{q,+})}=1$, $\|m\|_{L^1(\mathbb{R}_{q,+})} = K_q = \left(\frac{1-q}{1+q}\right)^{1/2} A^{-1}(\frac{1}{q(1+q)^2};q^2)$ and

$$(\mu(E))^{1/2}(\mu(S))^{1/2} \ge K_q^{-1}(q;q^2)^4_\infty\,\frac{\|P_{t,q}(f)\|_{L^2(\mathbb{R}_{q,+})} - (\nu+\varepsilon)\|f\|_{L^2(\mathbb{R}_{q,+})}}{\|f\|_{L^2(\mathbb{R}_{q,+})}}.$$

5. Extremal functions and q-Tikhonov regularization for the operators T_m

We define the q-Sobolev space [12] of order $s \geq 0$, that will be denoted $\mathcal{H}^s_{*,q}(\mathbb{R}_q)$, as the set of all $f \in L^2(\mathbb{R}_{q,+})$ such that $(1+z^2)^{s/2}\mathcal{F}_q(f) \in L^2(\mathbb{R}_{q,+})$. The space $\mathcal{H}^s_{*,q}(\mathbb{R}_q)$ endowed with the inner product

$$\langle f,g\rangle_{\mathcal{H}^s_{*,q}(\mathbb{R}_q)} := \int_0^\infty (1+z^2)^s \mathcal{F}_q(f)(z)\overline{\mathcal{F}_q(g)(z)}\mathrm{d}_q\mu(z),$$

and the norm $\|f\|_{\mathcal{H}^s_{*,q}(\mathbb{R}_q)} = \sqrt{\langle f,g\rangle_{\mathcal{H}^s_{*,q}(\mathbb{R}_q)}}$.

The Hilbert space $\mathcal{H}^s_{*,q}(\mathbb{R}_q)$ satisfies (see [12]) the following properties.

(a) $\mathcal{H}^0_{*,q}(\mathbb{R}_q) = L^2(\mathbb{R}_{q,+})$.

(b) For all $s > 0$, the space $\mathcal{H}^s_{*,q}(\mathbb{R}_q)$ is continuously contained in $L^2(\mathbb{R}_{q,+})$ and $\|f\|_{L^2(\mathbb{R}_{q,+})} \leq \|f\|_{\mathcal{H}^s_{*,q}(\mathbb{R}_q)}$.

(c) The space $\mathcal{H}^s_{*,q}(\mathbb{R}_q)$, $s \geq 0$, endowed with the inner product $\langle .,.\rangle_{\mathcal{H}^s_{*,q}(\mathbb{R}_q)}$ is a Hilbert space.

Remark 5.1. ([12], Example 3.1) For $s > 1/2$, the function $y \to (1+z^2)^{-s/2}$ belongs to $L^2(\mathbb{R}_{q,+})$. Hence for all $f \in \mathcal{H}^s_{*,q}(\mathbb{R}_q)$, we have $\|\mathcal{F}_q(f)\|_{L^2(\mathbb{R}_{q,+})} \leq \|f\|_{\mathcal{H}^s_{*,q}(\mathbb{R}_q)}$, and by Hölder's inequality,

$$\|\mathcal{F}_q(f)\|_{L^1(\mathbb{R}_{q,+})} \leq \left[\int_0^\infty \frac{\mathrm{d}_q\mu(z)}{(1+z^2)^s}\right]^{1/2} \|f\|_{\mathcal{H}^s_{*,q}(\mathbb{R}_q)}.$$

Then the function $\mathcal{F}_q(f)$ belongs to $L^1 \cap L^2(\mathbb{R}_{q,+})$, and therefore

$$f(x) = \int_0^\infty \cos(xz;q^2)\mathcal{F}_q(f)(z)\mathrm{d}_q\mu(z), \qquad x \in \mathbb{R}_{q,+}.$$

Let $\eta > 0$. We denote by $\langle .,.\rangle_{\eta,\mathcal{H}^s_{*,q}(\mathbb{R}_q)}$ the inner product defined on the space $\mathcal{H}^s_{*,q}(\mathbb{R}_q)$ by

$$\langle f,g\rangle_{\eta,\mathcal{H}^s_{*,q}(\mathbb{R}_q)} := \eta\langle f,g\rangle_{\mathcal{H}^s_{*,q}(\mathbb{R}_q)} + \langle T_m f, T_m g\rangle_{L^2(\mathbb{R}_{q,+})}, \tag{5.1}$$

and the norm $\|f\|_{\eta,\mathcal{H}^s_{*,q}(\mathbb{R}_q)} := \sqrt{\langle f,f\rangle_{\eta,\mathcal{H}^s_{*,q}(\mathbb{R}_q)}}$.

On $\mathcal{H}^s_{*,q}(\mathbb{R}_q)$ the two norms $\|.\|_{\mathcal{H}^s_{*,q}(\mathbb{R}_q)}$ and $\|.\|_{\eta,\mathcal{H}^s_{*,q}(\mathbb{R}_q)}$ are equivalent. This $(\mathcal{H}^s_{*,q}(\mathbb{R}_q), \langle .,.\rangle_{\eta,\mathcal{H}^s_{*,q}(\mathbb{R}_q)})$ is a Hilbert space with reproducing kernel given by the following theorem.

Theorem 5.2. *Let $\eta > 0$, $s > 1/2$ and let $m \in L^\infty(\mathbb{R}_{q,+})$. The space $(\mathcal{H}^s_{*,q}(\mathbb{R}_q), \langle .,.\rangle_{\eta,\mathcal{H}^s_{*,q}(\mathbb{R}_q)})$ has the reproducing kernel*

$$k_s(x,y) = \int_0^\infty \frac{T_{q,x}\cos(yz;q^2)}{|m(z)|^2 + \eta(1+z^2)^s}\mathrm{d}_q\mu(z), \tag{5.2}$$

that is

(i) *For all $y \in \mathbb{R}_{q,+}$, the function $x \to k_s(x,y)$ belongs to $\mathcal{H}^s_{*,q}(\mathbb{R}_q)$.*

(ii) *The q-reproducing property: for all $f \in \mathcal{H}^s_{*,q}(\mathbb{R}_q)$ and $y \in \mathbb{R}_{q,+}$,*

$$\langle f, k_s(.,y)\rangle_{\eta,\mathcal{H}^s_{*,q}(\mathbb{R}_q)} = f(y).$$

Proof. (i) Let $y \in \mathbb{R}_{q,+}$ and $s > 1/2$. The function $\Phi_y : z \to \frac{\cos(yz;q^2)}{|m(z)|^2+\eta(1+z^2)^s}$ belongs to $L^1 \cap L^2(\mathbb{R}_{q,+})$. Then, the function k_s is well defined and by (2.5), we have

$$k_s(x,y) = \int_0^\infty \frac{\cos(xz;q^2)\cos(yz;q^2)}{|m(z)|^2 + \eta(1+z^2)^s}\mathrm{d}_q\mu(z) = \mathcal{F}_q(\Phi_y)(x), \quad x \in \mathbb{R}_{q,+}.$$

From (2.6), it follows that $k_s(.,y)$ belongs to $L^2(\mathbb{R}_{q,+})$, and

$$\mathcal{F}_q(k_s(.,y))(z) = \frac{\cos(yz;q^2)}{|m(z)|^2 + \eta(1+z^2)^s}, \quad z \in \mathbb{R}_{q,+}. \tag{5.3}$$

Then by (2.4), we obtain $|\mathcal{F}_q(k_s(.,y))(z)| \leq \frac{1}{\eta(q;q^2)^2_\infty(1+z^2)^s}$, and

$$\|k_s(.,y)\|_{\mathcal{H}^s_{*,q}(\mathbb{R}_q)} \leq \frac{1}{\eta(q;q^2)^2_\infty}\left(\int_0^\infty \frac{d_q\mu(z)}{(1+z^2)^s}\right)^{1/2} < \infty.$$

This proves that for all $y \in \mathbb{R}_{q,+}$ the function $k_s(.,y)$ belongs to $\mathcal{H}^s_{*,q}(\mathbb{R}_q)$.

(ii) Let $f \in \mathcal{H}^s_{*,q}(\mathbb{R}_q)$ and $y \in \mathbb{R}_{q,+}$. From (5.1) and (5.3), we have

$$\begin{aligned}
\langle f, k_s(.,y)\rangle_{\eta,\mathcal{H}^s_{*,q}(\mathbb{R}_q)} &= \eta\int_0^\infty (1+z^2)^s \frac{\mathcal{F}_q(f)(z)\cos(yz;q^2)}{|m(z)|^2+\eta(1+z^2)^s} d_q\mu(z) \\
&+ \int_0^\infty \mathcal{F}_q(m\mathcal{F}_q(f))(z)\mathcal{F}_q(m\mathcal{F}_q(k_s))(z) d_q\mu(z) \\
&= \eta\int_0^\infty \frac{(1+z^2)^s}{|m(z)|^2+\eta(1+z^2)^s}\mathcal{F}_q(f)(z)\cos(yz;q^2) d_q\mu(z) \\
&+ \int_0^\infty \frac{|m(z)|^2}{|m(z)|^2+\eta(1+z^2)^s}\mathcal{F}_q(f)(z)\cos(yz;q^2) d_q\mu(z) \\
&= \int_0^\infty \mathcal{F}_q(f)(z)\cos(yz;q^2) d_q\mu(z),
\end{aligned}$$

and from Remark 4.1, we obtain the reproducing property

$$\langle f, k_s(.,y)\rangle_{\eta,\mathcal{H}^s_{*,q}(\mathbb{R}_q)} = f(y).$$

This completes the proof of the theorem. □

The main result of this section can be stated as follows.

Theorem 5.3. *Let $s > 1/2$ and let $m \in L^\infty(\mathbb{R}_{q,+})$. For any $g \in L^2(\mathbb{R}_{q,+})$ and for any $\eta > 0$, there exists a unique function $f^*_{\eta,g}$, where the infimum*

$$\inf_{f\in\mathcal{H}^s_{*,q}(\mathbb{R}_q)} \left\{\eta\|f\|^2_{\mathcal{H}^s_{*,q}(\mathbb{R}_q)} + \|g - T_m f\|^2_{L^2(\mathbb{R}_{q,+})}\right\} \tag{5.4}$$

is attained. Moreover, the extremal function $f^*_{\eta,g}$ is given by

$$f^*_{\eta,g}(y) = \int_0^\infty g(x)\mathcal{K}(x,y;q^2) d_q\mu(x), \tag{5.5}$$

where

$$\mathcal{K}(x,y;q^2) = \int_0^\infty \frac{\overline{m(z)}\, T_{q,x}\cos(yz;q^2)}{|m(z)|^2+\eta(1+z^2)^s} d_q\mu(z).$$

Proof. The existence and unicity of the extremal function $f^*_{\eta,g}$ satisfying (5.4) is obtained in [10, 15]. Especially, $f^*_{\eta,g}$ is given by the reproducing kernel of $\mathcal{H}^s_{*,q}(\mathbb{R}_q)$ with $\|.\|_{\eta,\mathcal{H}^s_{*,q}(\mathbb{R}_q)}$ norm (Theorem 5.2) as

$$f^*_{\eta,g}(y) = \langle g, T_m(k_s(.,y))\rangle_{L^2(\mathbb{R}_{q,+})}, \tag{5.5}$$

where k_s is the kernel given by (5.2).
But by (2.5) and (5.3), we have

$$\begin{aligned} T_m(k_s(.,y))(x) &= \int_0^\infty m(z)\mathcal{F}_q(k_s(.,y))(z)\cos(xz;q^2)\mathrm{d}_q\mu(z) \\ &= \int_0^\infty \frac{m(z)}{|m(z)|^2+\eta(1+z^2)^s}\cos(yz;q^2)\cos(xz;q^2)\mathrm{d}_q\mu(z) \\ &= \int_0^\infty \frac{m(z)}{|m(z)|^2+\eta(1+z^2)^s}T_{q,x}\cos(yz;q^2)\mathrm{d}_q\mu(z). \end{aligned}$$

This clearly yields the result. □

As application, we give the following examples.

Example 5.5. Let $s > 1/2$, $\eta > 0$ and $g \in L^2(\mathbb{R}_{q,+})$.

(i) If $m(z) := W_{\beta+\frac{1}{2}}(z/t;q^2)\,\chi_{[0,t]_q}(z)$, $t > 0$ and $\beta > 0$, then $f^*_{\eta,g}(y) = \displaystyle\int_0^\infty g(x)\mathcal{K}(x,y;q^2)\mathrm{d}_q\mu(x)$, where

$$\mathcal{K}(x,y;q^2) = \int_0^t \frac{W_{\beta+\frac{1}{2}}(z/t;q^2)T_{q,x}\cos(yz;q^2)}{W^2_{\beta+\frac{1}{2}}(z/t;q^2)+\eta(1+z^2)^s}\mathrm{d}_q\mu(z).$$

(ii) If $m(z) := e_{q^2}\left(-tz^2\right)$, $t > 0$, then $f^*_{\eta,g}(y) = \displaystyle\int_0^\infty g(x)\mathcal{K}(x,y;q^2)\mathrm{d}_q\mu(x)$, where

$$\mathcal{K}(x,y;q^2) = \int_0^\infty \frac{e_{q^2}\left(-tz^2\right)T_{q,x}\cos(yz;q^2)}{e^2_{q^2}\left(-tz^2\right)+\eta(1+z^2)^s}\mathrm{d}_q\mu_q(z).$$

(iii) If $m(z) := \mathcal{E}_q^{(1/2)}(-tz)$, $t > 0$, then $f^*_{\eta,g}(y) = \displaystyle\int_0^\infty g(x)\mathcal{K}(x,y;q^2)\mathrm{d}_q\mu(x)$, where

$$\mathcal{K}(x,y;q^2) = \int_0^\infty \frac{\mathcal{E}_q^{(1/2)}(-tz)T_{q,x}\cos(yz;q^2)}{\left(\mathcal{E}_q^{(1/2)}(-tz)\right)^2+\eta(1+z^2)^s}\mathrm{d}_q\mu_q(z).$$

Corollary 5.6. *Let $s > 1/2$, $\eta > 0$ and $g \in L^2(\mathbb{R}_{q,+})$. The extremal function $f^*_{\eta,g}$ satisfies*

(i) $|f^*_{\eta,g}(y)| \le \dfrac{D_q(s)}{\sqrt{\eta}}\|g\|_{L^2(\mathbb{R}_{q,+})}$,

(ii) $\|f^*_{\eta,g}\|_{L^2(\mathbb{R}_{q,+})} \le \dfrac{D_q(s)}{\sqrt{\eta}}\left(\displaystyle\int_0^\infty |g(x)|^2 E_{q^2}\left(\frac{x^2}{q(1+q)^2}\right)\mathrm{d}_q\mu(x)\right)^{1/2}$,

where

$$D_q(s) = \frac{1}{2(q;q^2)^2_\infty}\left(\int_0^\infty \frac{\mathrm{d}_q\mu(z)}{(1+|z|^2)^s}\right)^{1/2}.$$

Proof. (i) From (2.6) and (5.5), we have

$$\begin{aligned} |f^*_{\eta,g}(y)| &\le \|g\|_{L^2(\mathbb{R}_{q,+})}\|T_m(k_s(.,y))\|_{L^2(\mathbb{R}_{q,+})} \\ &\le \|g\|_{L^2(\mathbb{R}_{q,+})}\|m\mathcal{F}(k_s(.,y))\|_{L^2(\mathbb{R}_{q,+})}. \end{aligned}$$

Then, by (5.3) we deduce

$$|f^*_{\eta,g}(y)| \le \frac{1}{(q;q^2)^2_\infty}\|g\|_{L^2(\mathbb{R}_{q,+})}\left(\int_0^\infty \frac{|m(z)|^2\mathrm{d}_q\mu(z)}{[|m(z)|^2+\eta(1+z^2)^s]^2}\right)^{1/2}.$$

Using the fact that

$$\Big[|m(z)|^2+\eta(1+z^2)^s\Big]^2 \geq 4\eta(1+z^2)^s|m(z)|^2, \tag{5.6}$$

we obtain the result.

(ii) We write

$$f^*_{\eta,g}(y)=\int_0^\infty \sqrt{e_{q^2}\left(-\frac{x^2}{q(1+q)^2}\right)E_{q^2}\left(\frac{x^2}{q(1+q)^2}\right)}g(x)\mathcal{K}(x,y;q^2)\mathrm{d}_q\mu(x),$$

where $E_{q^2}(x)$ is given by (2.2). Applying Hölder's inequality, we obtain

$$|f^*_{\eta,g}(y)|^2 \leq \int_0^\infty |g(x)|^2E_{q^2}\left(\frac{x^2}{q(1+q)^2}\right)|\mathcal{K}(x,y;q^2)|^2\mathrm{d}_q\mu(x).$$

Thus and from Fubini-Tonnelli's theorem, we get

$$\|f^*_{\eta,g}\|^2_{L^2(\mathbb{R}_{q,+})} \leq \int_0^\infty |g(x)|^2E_{q^2}\left(\frac{x^2}{q(1+q)^2}\right)\|\mathcal{K}(x,.;q^2)\|^2_{L^2(\mathbb{R}_{q,+})}\mathrm{d}_q\mu(x).$$

Let $\Psi_x(z)=\frac{\overline{m(z)}\cos(xz;q^2)}{|m(z)|^2+\eta(1+z^2)^s}$. Since $\Psi_x\in L^1\cap L^2(\mathbb{R}_{q,+})$, then

$$\mathcal{K}(x,y;q^2)=\mathcal{F}_q(\Psi_x)(y),$$

and by (2.6) we deduce that $\mathcal{F}_q(\mathcal{K}(x,.;q^2))(z)=\dfrac{\overline{m(z)}\cos(xz;q^2)}{|m(z)|^2+\eta(1+z^2)^s}$.

Thus,

$$\begin{aligned}\|\mathcal{K}(x,.;q^2)\|^2_{L^2(\mathbb{R}_{q,+})} &= \int_0^\infty |\mathcal{F}_q(\mathcal{K}(x,.;q^2))(z)|^2\mathrm{d}_q\mu(z)\\ &\leq \frac{1}{(q;q^2)^2_\infty}\int_0^\infty \frac{|m(z)|^2\mathrm{d}_q\mu(z)}{[|m(z)|^2+\eta(1+z^2)^s]^2}.\end{aligned}$$

Then using the inequality (5.6), we obtain $\|\mathcal{K}(x,.;q^2)\|_{L^2(\mathbb{R}_{q,+})} \leq \dfrac{D_q(s)}{\sqrt{\eta}}$.

From this inequality we deduce the result. □

Corollary 5.7. *Let $s>1/2$ and $\eta>0$. For every $g\in L^2(\mathbb{R}_{q,+})$, we have*

(i) $f^*_{\eta,g}(y)=\displaystyle\int_0^\infty \cos(yz;q^2)\frac{\overline{m(z)}\mathcal{F}_q(g)(z)}{|m(z)|^2+\eta(1+z^2)^s}\mathrm{d}_q\mu(z).$

(ii) $\mathcal{F}_q(f^*_{\eta,g})(z)=\dfrac{\overline{m(z)}\mathcal{F}_q(g)(z)}{|m(z)|^2+\eta(1+z^2)^s}.$

(iii) $\|f^*_{\eta,g}\|_{\mathcal{H}^s_{*,q}(\mathbb{R}_q)} \leq \dfrac{1}{2\sqrt{\eta}}\|g\|_{L^2(\mathbb{R}_{q,+})}.$

Proof. (i) follows from (5.5) by using (2.6) and (5.3).

(ii) The function $z\rightarrow \frac{\overline{m(z)}\mathcal{F}_q(g)(z)}{|m(z)|^2+\eta(1+z^2)^s}$ belongs to $L^1\cap L^2(\mathbb{R}_{q,+})$. Then by (2.5), we have $f^*_{\eta,g}(y)=\mathcal{F}_q(\dfrac{\overline{m(z)}\mathcal{F}_q(g)(z)}{|m(z)|^2+\eta(1+z^2)^s})(y)$.

From (2.6), it follows that $f^*_{\eta,g}$ belongs to $L^2(\mathbb{R}_{q,+})$, and $\mathcal{F}_q(f^*_{\eta,g})(z)=\dfrac{\overline{m(z)}\mathcal{F}_q(g)(z)}{|m(z)|^2+\eta(1+z^2)^s}$.

(iii) By relation (ii) we have

$$\|f^*_{\eta,g}\|^2_{\mathcal{H}^s_{*,q}(\mathbb{R}_q)}=\int_0^\infty (1+z^2)^s|\mathcal{F}_q(f^*_{\eta,g})(z)|^2\mathrm{d}_q\mu(z)=\int_0^\infty \frac{(1+z^2)^s|m(z)|^2|\mathcal{F}_q(g)(z)|^2}{[|m(z)|^2+\eta(1+z^2)^s]^2}\mathrm{d}_q\mu(z).$$

Using the inequality (5.6), we obtain

$$\|f^*_{\eta,g}\|^2_{\mathcal{H}^s_{*,q}(\mathbb{R}_q)} \le \frac{1}{4\eta}\int_0^\infty |\mathcal{F}_q(g)(z)|^2 \mathrm{d}_q\mu(z) = \frac{1}{4\eta}\|g\|^2_{L^2(\mathbb{R}_{q,+})},$$

which ends the proof. □

Theorem 5.8. *Let $s > 1/2$ and $\eta > 0$. For every $g \in L^2(\mathbb{R}_{q,+})$, we have*

(i) $T_m f^*_{\eta,g}(y) = \displaystyle\int_0^\infty \cos(yz; q^2)\frac{|m(z)|^2\mathcal{F}_q(g)(z)}{|m(z)|^2 + \eta(1+z^2)^s}\mathrm{d}_q\mu(z).$

(ii) $\mathcal{F}_q(T_m f^*_{\eta,g})(z) = \dfrac{|m(z)|^2\mathcal{F}_q(g)(z)}{|m(z)|^2 + \eta(1+z^2)^s}.$

(iii) $T_m f^*_{\eta,g}(y) = f^*_{\eta,T_m g}(y).$

(iv) $\lim\limits_{\eta\to 0^+} \|T_m f^*_{\eta,g} - g\|_{L^2(\mathbb{R}_{q,+})} = 0.$

Proof. From (3.1) and Corollary 4.7 (ii), we have $T_m f^*_{\eta,g}(y) = \mathcal{F}_q\Big(\dfrac{|m(z)|^2\mathcal{F}_q(g)(z)}{|m(z)|^2 + \eta(1+z^2)^s}\Big)(y).$ The function $z \to \frac{|m(z)|^2\mathcal{F}_q(g)(z)}{|m(z)|^2+\eta(1+z^2)^s}$ belongs to $L^1 \cap L^2(\mathbb{R}_{q,+})$. Then by (2.5), we obtain (i), and by (2.6) we obtain (ii).

(iii) follows from (i) and Corollary 4.7 (i).

(iv) From (ii) we have $\mathcal{F}_q(T_m f^*_{\eta,g} - g)(z) = \dfrac{-\eta(1+z^2)^s}{|m(z)|^2 + \eta(1+z^2)^s}\mathcal{F}_q(g)(z).$

Thus, $\|T_m f^*_{\eta,g} - g\|^2_{L^2(\mathbb{R}_{q.+})} = \displaystyle\int_0^\infty \frac{\eta^2(1+z^2)^{2s}|\mathcal{F}_q(g)(z)|^2}{[|m(z)|^2 + \eta(1+z^2)^s]^2}\mathrm{d}_q\mu(z).$

Using the dominated convergence theorem and $\dfrac{\eta^2(1+z^2)^{2s}|\mathcal{F}_q(g)(z)|^2}{[|m(z)|^2+\eta(1+z^2)^s]^2} \le |\mathcal{F}_q(g)(z)|^2$, we deduce that $\lim\limits_{\eta\to 0^+} \|T_m f^*_{\eta,g} - g\|^2_{L^2(\mathbb{R}_{q,+})} = 0,$

which ends the proof. □

Theorem 5.9. *Let $s > 1/2$ and $\eta > 0$. For every $f \in \mathcal{H}^s_{*,q}(\mathbb{R}_q)$ and $g = T_m f$, we have*

(i) $f^*_{\eta,T_m f}(y) = \displaystyle\int_0^\infty \cos(yz; q^2)\frac{|m(z)|^2\mathcal{F}_q(f)(z)}{|m(z)|^2 + \eta(1+z^2)^s}\mathrm{d}_q\mu(z).$

(ii) $\mathcal{F}_q(f^*_{\eta,T_m f})(z) = \dfrac{|m(z)|^2\mathcal{F}_q(f)(z)}{|m(z)|^2 + \eta(1+z^2)^s}.$

(iii) $\lim\limits_{\eta\to 0^+} \|f^*_{\eta,T_m f} - f\|_{L^\infty(\mathbb{R}_{q,+})} = 0.$

(iv) $\lim\limits_{\eta\to 0^+} \|f^*_{\eta,T_m f} - f\|_{\mathcal{H}^s_{*,q}(\mathbb{R}_q)} = 0.$

Proof. (i) and (ii) follow directly from Corollary 4.6 (i) and (ii).

(iii) From Remark 4.1, the function $\mathcal{F}(f) \in L^1 \cap L^2(\mathbb{R}_{q,+})$. Then by (i) and (2.5),

$$f^*_{\eta,T_m f}(y) - f(y) = \int_0^\infty \frac{-\eta(1+z^2)^s\mathcal{F}_q(f)(z)}{|m(z)|^2 + \eta(1+z^2)^s}\cos(yz; q^2)\mathrm{d}_q\mu(z).$$

So

$$\|f^*_{\eta,T_m f} - f\|_{L^\infty(\mathbb{R}_{q,+})} \le \frac{1}{(q;q^2)^2_\infty}\int_0^\infty \frac{\eta(1+z^2)^s|\mathcal{F}_q(f)(z)|}{|m(z)|^2 + \eta(1+z^2)^s}\mathrm{d}_q\mu(z).$$

Again, by dominated convergence theorem and $\dfrac{\eta(1+z^2)^s|\mathcal{F}_q(f)(z)|}{|m(z)|^2+\eta(1+z^2)^s} \le |\mathcal{F}_q(f)(z)|$, we deduce that $\lim\limits_{\eta\to 0^+} \|f^*_{\eta,T_m f} - f\|_{L^\infty(\mathbb{R}_{q,+})} = 0.$

(iv) From (ii) we have $\mathcal{F}_q(f^*_{\eta,T_m f} - f)(z) = \dfrac{-\eta(1+z^2)^s}{|m(z)|^2 + \eta(1+z^2)^s}\mathcal{F}_q(f)(z)$.

Consequently, $\|f^*_{\eta,g} - f\|^2_{\mathcal{H}^s_{*,q}(\mathbb{R}_q)} = \displaystyle\int_0^\infty \frac{\eta^2(1+z^2)^{3s}|\mathcal{F}_q(f)(z)|^2}{[|m(z)|^2 + \eta(1+z^2)^s]^2} \mathrm{d}_q\mu(z)$.

Using the fact that $\dfrac{\eta^2(1+z^2)^{3s}|\mathcal{F}_q(f)(z)|^2}{[|m(z)|^2 + \eta(1+z^2)^s]^2} \leq (1+z^2)^s|\mathcal{F}_q(f)(z)|^2$,

we deduce that $\lim\limits_{\eta\to 0^+} \|f^*_{\eta,T_m f} - f\|^2_{\mathcal{H}^s_{*,q}(\mathbb{R}_q)} = 0$,

which ends the proof. □

6. **Conclusions**

In this paper, an unification proof of many inequalities and approximations for the classical and discrete case by means of the q-theory. An extensions of the q-Donoho-Stark's uncertainty principle for the class of Fourier multiplier operators T_m. Finally, an exact expression and some properties of the extremal functions of the so-called Tikhonov regularization problem are obtained, using reproducing kernel methods.

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