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EXISTENCE OF FIXED POINTS OF SET-VALUED MAPS ON MODULAR b-GAUGE SPACES

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In this paper, we shall introduce the notion of modular b-gauge spaces with the help of pseudomodular b-metrics. We shall also prove some fixed point theorems for multivalued mappings on modular b-gauge spaces. Moreover, we shall construct an application of our result in nonlinear integral equations.

Keywords: Modular metric spaces, Modular b-metric spaces, Modular b-gauge spaces, Δ_b-condition, Fatou property.

MSC2020: 47H10 54H25.

1. **Introduction and Preliminaries**

Czerwik [1] introduced the concept of b-metric spaces which appears as a novel generalization of metric spaces. This notion helps us to standardize the measurement of distance between the elements of l_p spaces, specially for $p \in (0,1)$. Czerwik [1] defined this space by modifying the triangular axiom of metric space (X,d) as:

$$d(x,z) \leq s[d(x,y) + d(y,z)] \text{ for each } x,y,z \in X, \text{ where } s \geq 1.$$

After Czerwik [1] many articles published in this direction, see for example, [2, 3] with those cited in these two articles.

Another generalized form of metric spaces is introduced by Chistyakov [4] known as modular metric spaces. In this space the distance between two elements may depends upon a parameter λ. Moreover it is not necessary that the distance between two elements must be finite. Recently Ali [5] extended modular metric spaces by introducing the terminology of modular b-metric spaces, this concept involves the idea of Czerwik [1].

Frigon [6] studied the Banach fixed point result on gauge spaces. After this study many authors worked in the direction of gauge spaces, like, [7, 8, 9, 10, 11, 12, 13]. Recently, Ali *et al.* [14] introduced the notion of modular gauge spaces induced through the family of pseudomodular metrics.

In this paper we define the concept of modular b-gauge spaces induced through the family of pseudomodular b-metrics. We will also prove fixed point results for multivalued mappings in the setting of modular b-gauge spaces induced through the family of pseudomodular b-metrics. An as application of our work we will provide an existence result for nonlinear integral equations. Further, we will provide a particular nonlinear integral equation as an example of our existence theorem.

With the help of bibliography we collect few definitions and results which will be required subsequently.

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Definition 1.1. [4] *A modular metric* $\omega_{ms} : (0,\infty) \times X \times X \to [0,\infty]$ *on* X *is a function with these axioms: for each* $x, y, z \in X$*, we have*

: *(i)* $\omega_{ms}(\beta, x, y) = 0 \ \forall \beta > 0 \Leftrightarrow x = y$;
: *(ii)* $\omega_{ms}(\beta, x, y) = \omega_{ms}(\beta, y, x) \ \forall \beta > 0$;
: *(iii)* $\omega_{ms}(\beta + \gamma, x, z) \leq \omega_{ms}(\beta, x, y) + \omega_{ms}(\gamma, y, z) \ \forall \beta, \gamma > 0$.

For a brief study of modular metric spaces and fixed point results stated on it, we refer the readers to [5, 15, 16, 17, 18, 19]. Ali [5] extended the concept of modular metric space by defining the modular b-metric space as:

Definition 1.2. [5] *A modular b-metric* $\omega_{ms} : (0,\infty) \times X \times X \to [0,\infty]$ *on* X *is a function with these axioms: for each* $x, y, z \in X$*, we have*

: *(i)* $\omega_{ms}(\beta, x, y) = 0 \ \forall \beta > 0 \Leftrightarrow x = y$;
: *(ii)* $\omega_{ms}(\beta, x, y) = \omega_{ms}(\beta, y, x) \ \forall \beta > 0$;
: *(iii)* $\omega_{ms}(\beta + \gamma, x, z) \leq \omega_{ms}(\frac{\beta}{s}, x, y) + \omega_{ms}(\frac{\gamma}{s}, y, z) \ \forall \beta, \gamma > 0$*, here* $s \geq 1$ *is a fixed real number.*

The modular b-metric on X *is called modular b-metric space and denoted by* (X, ω_{ms}, s)

A modular metric space is also a modular b-metric space but converse statement is not true in general.

Example 1.1. [5] *Take* $X = [0,\infty)$ *with* $w_{ms}(\beta, x, y) = \frac{x^2+y^2-2xy}{\beta}$ *is the simplest example of modular b-metric space with* $s = 2$*. Note that it is not a modular metric space.*

Definition 1.3. [5] *A regular modular b-metric* $\omega_{ms} : (0,\infty) \times X \times X \to [0,\infty]$ *on* X *is a function with these axioms: for each* $x, y, z \in X$*, we have*

: *(i)* $x = y \Leftrightarrow \omega_{ms}(\beta, x, y) = 0$ *for some* $\beta > 0$;
: *(ii)* $\omega_{ms}(\beta, x, y) = \omega_{ms}(\beta, y, x) \ \forall \beta > 0$;
: *(iii)* $\omega_{ms}(\beta + \gamma, x, z) \leq \omega_{ms}(\frac{\beta}{s}, x, y) + \omega_{ms}(\frac{\gamma}{s}, y, z) \ \forall \beta, \gamma > 0$*, here* $s \geq 1$ *is a fixed real number.*

A pseudomodular b-metric on X is obtained by replacing axiom (i) of a modular b-metric with (i') for each $x \in X$, $\omega_{ms}(\beta, x, x) = 0 \ \forall \beta > 0$. Note that the function $\beta \to \omega_{ms}(\beta, x, z)$ is nonincreasing on $(0,\infty)$. This fact can be proven by taking $0 < \beta < \gamma$ (with $\beta = \frac{\lambda}{s}$) in triangular axiom, that is,

$$\omega_{ms}(\gamma, x, z) \leq \omega_{ms}(\frac{\gamma - \lambda}{s}, x, x) + \omega_{ms}(\frac{\lambda}{s}, x, z) = \omega_{ms}(\beta, x, z).$$

Further note that when ω_{ms} is a pseudomodular b-metric on X and $x_0 \in X$ is a fixed element, then the sets

$$X_{\omega_{ms}} = X_{\omega_{ms}}(x_0) = \{x \in X : \omega_{ms}(\gamma, x_0, x) \to 0 \text{ as } \gamma \to \infty\}$$

and

$$X^*_{\omega_{ms}} = X^*_{\omega_{ms}}(x_0) = \{x \in X : \exists \gamma = \gamma(x) > 0 \text{ such that } \omega_{ms}(\lambda, x_0, x) < \infty\}$$

are modular spaces (around x_0).

The concepts like ω_{ms}-convergent sequence, ω_{ms}-Cauchy sequence, ω_{ms}-closed sets and ω_{ms}-complete sets in modular b-metric spaces are defined in the similar way as defined for modular metric spaces in [19].

Definition 1.4. *Let* (X, ω_{ms}, s) *be a modular b-metric space, let* $\{x_n\} \subseteq X_\omega$ *and* $x \in X_\omega$*. Then:*

: *(i) the* $\{x_n\}$ *is* ω_{ms}*-convergent sequence in* X *with the limit point* x*, if* $\omega_{ms}(\beta, x_n, x) \to 0$ *for some* $\beta > 0$ *as* $n \to \infty$;

: *(ii) the* $\{x_n\}$ *is* ω_{ms}*-Cauchy sequence if* $\lim_{n,m\to\infty}\omega_{ms}(\beta, x_n, x_m) = 0$ *for some* $\beta > 0$

: *(iii) a subset* M *of* $X_{\omega_{ms}}$ *is* ω_{ms}*-complete if every* ω_{ms}*-Cauchy sequence in* M *is* ω_{ms}*-convergent in* M*;*

: *(iv) a subset* M *of* $X_{\omega_{ms}}$ *is* ω_{ms}*-closed if it contains the limit point of each* ω_{ms}*-convergent sequence contained in* M*.*

: *(v) a subset* M *of* $X_{\omega_{ms}}$ *is* ω_{ms}*-bounded if we have*

$$\delta_{\omega_{ms}}(M) = \sup\{\omega_{ms}(1, x, y) : x, y \in M\} < \infty.$$

In the literature we have seen that the fixed point results on modular metric spaces involve the Δ-condition and Fatou property. Ali [5] extended these conditions for modular b-metric spaces as follows:

Definition 1.5. [5] *Let* (X, ω_{ms}, s) *be a modular b-metric space. Then* ω_{ms} *is satisfies:*

: *(a) the* Δ_b*-condition, if the following axioms hold:*

  : *(i) for each* $\{x_n\}$ *in* X *satisfying* $\omega_{ms}(\beta, x_n, x_{n+1}) \le r^n C$ *for some* $\beta > 0$ *and for each* $n \in \mathbb{N}$*, where* $r \in [0, 1/s)$ *and* $C > 0$ *is some fixed real numbers, then we have* $\omega_{ms}(\gamma, x_n, x_{n+1}) \le r^n C$ *for each* $\gamma > 0$ *and for each* $n \in \mathbb{N}$*;*

  : *(ii) for each* $\{x_n\}$ *in* X *and* $x \in X$ *with* $\lim_{n\to\infty}\omega_{ms}(\beta, x_n, x) = 0$ *for some* $\beta > 0$*, then we have* $\lim_{n\to\infty}\omega_{ms}(\gamma, x_n, x) = 0$ *for all* $\gamma > 0$*.*

: *(b) the Fatou property if for each* $\{x_n\}$ ω_{ms}*-convergent to* x *and* $\{y_n\}$ ω_{ms}*-convergent to* y*, we have* $\omega_{ms}(1, x, y) \le \liminf_{n\to\infty}\omega_{ms}(1, x_n, y_n)$*.*

2. Main Results

This section begins with ω_{ms}-ball with respect to pseudomodular b-metric.

Definition 2.1. *Take a pseudomodular b-metric* ω_{ms} *on* X*. Then the* ω_{ms}*-ball having the radius* $\beta > 0$ *with* $x \in X$ *as a center is the set*

$$B[x, \omega_{ms}, \beta] = \{z \in X : \forall\gamma > 0\ \omega_{ms}(\gamma, x, y) < \beta\}.$$

Example 2.1. *Take* $X = \mathbb{R}$ *with the pseudomodular b-metric* $\omega_{ms}(\beta, x, y) = \frac{x^2+y^2-2xy}{\beta}$ *for each* $x, y \in X$ *and* $\beta > 0$*, where* $s = 2$*. Then*

$$B[x_0, \beta, 1] = \{z \in X : \forall\beta > 0,\ x_0^2 + z^2 - 2zx_0 < \beta\} = \{x_0\}.$$

Example 2.2. *Take* $X = \mathbb{R}$ *with the pseudomodular b-metric* $\omega_{ms}(\beta, x, y) = \frac{x^2+y^2-2xy}{\lceil\beta\rceil}$ *for each* $x, y \in X$ *and* $\beta > 0$*, where* $s = 2$*. Then*

$$\begin{aligned} B[x_0, \beta, 1] &= \{z \in X : \forall\beta > 0,\ x_0^2 + z^2 - 2zx_0 < \lceil\beta\rceil\} \\ &= \{y \in X : x_0^2 + z^2 - 2zx_0 < 1\} = (x_0 - 1, x_0 + 1). \end{aligned}$$

Definition 2.2. *A collection* $\mathfrak{F} = \{\omega_{ms_\eta}$ *with* $s_\eta \ge 1 : \eta \in \mathfrak{A}\}$ *of pseudomodular b-metrics is called separating if for every pair* (x, y) *with* $x \ne y$*, we have atleast one* $\omega_{ms_\eta} \in \mathfrak{F}$ *with* $\omega_{ms_\eta}(\beta, x, y) \ne 0\ \forall\beta > 0$*.*

Definition 2.3. *Take a collection* $\mathfrak{F} = \{\omega_{ms_\eta}$ *with* $s_\eta \ge 1 : \eta \in \mathfrak{A}\}$ *of pseudomodular b-metrics on* $X \ne \emptyset$*. The topology* $\mathfrak{T}(\mathfrak{F})$ *with a collection of subbases*

$$\mathfrak{B}(\mathfrak{F}) = \{B[z, \omega_{ms_\eta}, \gamma] : z \in X, \omega_{ms_\eta} \in \mathfrak{F} \text{ and } \gamma > 0\}$$

of the balls is a modular topology induced by the collection $\mathfrak{F}$ *of pseudomodular b-metrics. The pair* $(X, \mathfrak{T}(\mathfrak{F}))$ *is said to be a modular b-gauge space.*

Before going towards a next definition we define the following notion:

$$X_{\mathfrak{F}} = X_{\mathfrak{F}}(x_0) = \{x \in X : \forall\eta \in \mathfrak{A}\ \omega_{ms_\eta}(\beta, x_0, x) \to 0 \text{ as } \beta \to \infty\}$$

where x_0 is fixed in X.

Definition 2.4. *Take modular b-gauge space $(X, \mathfrak{T}(\mathfrak{F}))$ with respect to the collection $\mathfrak{F} = \{\omega_{ms_\eta}$ with $s_\eta \geq 1 : \eta \in \mathfrak{A}\}$ of pseudomodular b-metrics on X and also take $\{x_n\} \subseteq X_{\mathfrak{F}}$ and $x \in X_{\mathfrak{F}}$. Then:*

- : *(i) $\{x_n\}$ is ω_{ms_η}-convergent to x if for every $\eta \in \mathfrak{A}$ we have $\lim_{n\to\infty} \omega_{ms_\eta}(\beta, x_n, x) = 0$ for some $\beta > 0$. We denote it as $x_n \to^{\mathfrak{F}} x$;*
- : *(ii) $\{x_n\}$ is ω_{ms_η}-Cauchy if for every $\eta \in \mathfrak{A}$ we have $\lim_{n,m\to\infty} \omega_{ms_\eta}(\beta, x_n, x_m) = 0$ for some $\beta > 0$;*
- : *(iii) $X_{\mathfrak{F}}$ is ω_{ms_η}-complete if every ω_{ms_η}-Cauchy sequence in $X_{\mathfrak{F}}$ is ω_{ms_η}-convergent in $X_{\mathfrak{F}}$;*
- : *(iv) a subset W of $X_{\mathfrak{F}}$ is said to be ω_{ms_η}-closed if it contains the limit of each ω_{ms_η}-convergent sequence of its elements.*
- : *(v) a subset W of $X_{\mathfrak{F}}$ is ω_{ms_η}-bounded if we have*

$$\delta_{\mathfrak{F}}(W) = \sup\{\omega_{ms_\eta}(1, x, y) : x, y \in W, \eta \in \mathfrak{A}\} < \infty.$$

Take a separating modular b-gauge space induced through the collection of pseudomodular b-metrics $\mathfrak{F} = \{\omega_{ms_\eta}$ with $s_\eta \geq 1 : \eta \in \mathfrak{A}\}$ on $X \neq \emptyset$ and $\{x_n\}$ is ω_{ms_η}-convergent in $X_{\mathfrak{F}}$, then $\{x_n\}$ ω_{ms_η}-converges to unique limit point.

On contrary we take $x_n \to^{\mathfrak{F}} a$ and $x_n \to^{\mathfrak{F}} b$. Then for every $\eta \in \mathfrak{A}$, there are $\gamma_1, \gamma_2 > 0$ such that $\lim_{n\to\infty} \omega_{ms_\eta}(\gamma_1, x_n, a) = 0$ and $\lim_{n\to\infty} \omega_{ms_\eta}(\gamma_2, x_n, b) = 0$. By the triangular axiom we obtain

$$\omega_{ms_\eta}(s_\eta\gamma_1 + s_\eta\gamma_2, a, b) \leq \omega_{ms_\eta}(\gamma_1, a, x_n) + \omega_{ms_\eta}(\gamma_2, x_n, b) \ \forall n \in \mathbb{N} \text{ and } \eta \in \mathfrak{A}.$$

This yields, $\lim_{n\to\infty} \omega_{ms_\eta}(s_\eta\gamma_1 + s_\eta\gamma_2, a, b) = 0 \ \forall \eta \in \mathfrak{A}$. As $\mathfrak{F} = \{\omega_{ms_\eta}$ with $s_\eta \geq 1 : \eta \in \mathfrak{A}\}$ is separating, hence we get $a = b$.

Subsequently, in the article, $\mathfrak{A}$ is an indexed set and $X \neq \emptyset$ equipped with a modular b-gauge space induced through the collection $\mathfrak{F} = \{\omega_{ms_\eta}$ with $s_\eta \geq 1 : \eta \in \mathfrak{A}\}$ of separating pseudomodular b-metrics which also satisfy the Fatou property and Δ_b-condition. Furthermore, M is ω_{ms_η}-bounded and ω_{ms_η}-complete subset of $X_{\mathfrak{F}}$ under the above considered modular b-gauge space $(X, \mathfrak{T}(\mathfrak{F}))$. Moreover, λ is a mapping from $M \times M$ into $[0, \infty)$. By $CL(M)$, we denote the collection of nonempty ω_{ms_η}-closed subsets of M under the above modular b-gauge space.

Theorem 2.1. *Consider a map $T\colon M \to CL(M)$ such that for each $x, y \in M$ with $\lambda(x, y) \geq 1$ and $u \in Tx$, there exists $v \in Ty$ satisfying the following inequality:*

$$\begin{aligned} \omega_{ms_\eta}(1, u, v) \quad \leq \quad & r_\eta \max\Big\{\omega_{ms_\eta}(1, x, y), \frac{\omega_{ms_\eta}(1, x, u) + \omega_{ms_\eta}(1, y, v)}{2}, \\ & \frac{\omega_{ms_\eta}(2s_\eta, x, v) + \omega_{ms_\eta}(1, y, u)}{2}\Big\} + L_\eta \omega_{ms_\eta}(1, y, u) \end{aligned} \tag{1}$$

for all $\eta \in \mathfrak{A}$, where $r_\eta \in [0, 1/s_\eta)$ and $L_\eta \geq 0 \ \forall \eta \in \mathfrak{A}$. Further, assume the given below conditions hold:

- : *(i) there are two elements $x_0 \in M$ and $x_1 \in Tx_0$ with $\lambda(x_0, x_1) \geq 1$;*
- : *(ii) for $x \in M$ and $y \in Tx$ with $\lambda(x, y) \geq 1$, we have $\lambda(y, z) \geq 1$ for each $z \in Ty$;*
- : *(iii) if $\{x_n\}$ is a sequence in M with $x_n \to^{\mathfrak{F}} x \in M$ and $\lambda(x_n, x_{n+1}) \geq 1 \ \forall n \in \mathbb{N}$, then $\lambda(x_n, x) \geq 1 \ \forall n \in \mathbb{N}$.*

Then atleast one fixed point of T must exists.

Proof. By hypothesis (i), there are two elements $x_0 \in M$ and $x_1 \in Tx_0$ such that $\lambda(x_0, x_1) \geq 1$. From (1), for $\lambda(x_0, x_1) \geq 1$ and $x_1 \in Tx_0$, we have $x_2 \in Tx_1$ such that

$$\begin{aligned}\omega_{ms_\eta}(1, x_1, x_2) &\leq r_\eta \max\Big\{\omega_{ms_\eta}(1, x_0, x_1), \frac{\omega_{ms_\eta}(1, x_0, x_1) + \omega_{ms_\eta}(1, x_1, x_2)}{2}, \\ &\quad \frac{\omega_{ms_\eta}(2s_\eta, x_0, x_2) + \omega_{ms_\eta}(1, x_1, x_1)}{2}\Big\} + L_\eta \omega_{ms_\eta}(1, x_1, x_1) \\ &\leq r_\eta \max\Big\{\omega_{ms_\eta}(1, x_0, x_1), \frac{\omega_{ms_\eta}(1, x_0, x_1) + \omega_{ms_\eta}(1, x_1, x_2)}{2}, \\ &\quad \frac{\omega_{ms_\eta}(1, x_0, x_1) + \omega_{ms_\eta}(1, x_1, x_2)}{2}\Big\} \\ &\leq r_\eta \max\Big\{\omega_{ms_\eta}(1, x_0, x_1), \omega_{ms_\eta}(1, x_1, x_2)\Big\} \ \forall \eta \in \mathfrak{A}. \end{aligned} \tag{2}$$

If we take $\max\{\omega_{ms_\eta}(1, x_0, x_1), \omega_{ms_\eta}(1, x_1, x_2)\} = \omega_{ms_\eta}(1, x_1, x_2)$, then from (2) we get $\omega_{ms_\eta}(1, x_1, x_2) \leq r_\eta \omega_{ms_\eta}(1, x_1, x_2) < \omega_{ms_\eta}(1, x_1, x_2)$, which is impossible. Thus, we have $\max\{\omega_{ms_\eta}(1, x_0, x_1), \omega_{ms_\eta}(1, x_1, x_2)\} = \omega_{ms_\eta}(1, x_0, x_1)$. From (2), we have

$$\omega_{ms_\eta}(1, x_1, x_2) \leq r_\eta \omega_{ms_\eta}(1, x_0, x_1) \ \forall \eta \in \mathfrak{A}. \tag{3}$$

As $x_0 \in M$ and $x_1 \in Tx_0$ with $\lambda(x_0, x_1) \geq 1$, then by hypothesis (ii), for $x_2 \in Tx_1$, we have $\lambda(x_1, x_2) \geq 1$. From (1), for $\lambda(x_1, x_2) \geq 1$ and $x_2 \in Tx_1$, we have $x_3 \in Tx_2$ such that

$$\begin{aligned}\omega_{ms_\eta}(1, x_2, x_3) &\leq r_\eta \max\Big\{\omega_{ms_\eta}(1, x_1, x_2), \frac{\omega_{ms_\eta}(1, x_1, x_2) + \omega_{ms_\eta}(1, x_2, x_3)}{2}, \\ &\quad \frac{\omega_{ms_\eta}(2s_\eta, x_1, x_3) + \omega_{ms_\eta}(1, x_2, x_2)}{2}\Big\} + L_\eta \omega_{ms_\eta}(1, x_2, x_2) \\ &\leq r_\eta \max\Big\{\omega_{ms_\eta}(1, x_1, x_2), \frac{\omega_{ms_\eta}(1, x_1, x_2) + \omega_{ms_\eta}(1, x_2, x_3)}{2}, \\ &\quad \frac{\omega_{ms_\eta}(1, x_1, x_2) + \omega_{ms_\eta}(1, x_2, x_3)}{2}\Big\} \\ &\leq r_\eta \max\Big\{\omega_{ms_\eta}(1, x_1, x_2), \omega_{ms_\eta}(1, x_2, x_3)\Big\} \\ &= r_\eta \omega_{ms_\eta}(1, x_1, x_2) \ \forall \eta \in \mathfrak{A}. \end{aligned} \tag{4}$$

From (3) and (4), we have $\omega_{ms_\eta}(1, x_2, x_3) \leq r_\eta^2 \omega_{ms_\eta}(1, x_0, x_1) \ \forall \eta \in \mathfrak{A}$. Continuing this pattern we get $\{x_n\}$ in M such that $x_n \in Tx_{n-1}$, $\lambda(x_{n-1}, x_n) \geq 1$ and $\omega_{ms_\eta}(1, x_n, x_{n+1}) \leq r_\eta^n \omega(1, x_0, x_1) \leq r_\eta^n \delta_{\mathfrak{F}}(M)$ for each $n \in \mathbb{N}$ and $\eta \in \mathfrak{A}$. By using the Δ_b-condition and the above inequality, for each $\eta \in \mathfrak{A}$, we get $\omega_{ms_\eta}(\mu, x_n, x_{n+1}) \leq r_\eta^n \delta_{\mathfrak{F}}(M)$ for each $\mu > 0$ and $n \in \mathbb{N}$. For each $m, p \in \mathbb{N}$, we get

$$\begin{aligned}\omega_{ms_\eta}(p, x_m, x_{m+p}) &\leq \sum_{i=m}^{m+p-1} \omega_{ms_\eta}(\frac{1}{s_\eta^i}, x_i, x_{i+1}) \leq \sum_{i=m}^{m+p-1} r_\eta^i \delta_{\mathfrak{F}}(M) \\ &\leq \sum_{i=m}^{\infty} r_\eta^i \delta_{\mathfrak{F}}(M) \to 0 \text{ as } m \to \infty \forall \eta \in \mathfrak{A}.\end{aligned}$$

Hence $\{x_n\}$ is ω_{ms_η}-Cauchy in M. As M is ω_{ms_η}-complete then there is $x^* \in M$ such that for each $\eta \in \mathfrak{A}$ we have $\lim_{n\to\infty} \omega_{ms_\eta}(\beta, x_n, x^*) = 0$ for some $\beta > 0$. Since the Δ_b-condition holds for the collection $\mathfrak{F}$ then, for each $\eta \in \mathfrak{A}$, we get $\lim_{n\to\infty} \omega_{ms_\eta}(\gamma, x_n, x^*) = 0 \ \forall \gamma > 0$. Hypothesis (iii) yields $\lambda(x_n, x^*) \geq 1 \ \forall n \in \mathbb{N}$. From (1), for $\lambda(x_n, x^*) \geq 1$ and $x_{n+1} \in Tx_n$

there is $v^* \in Tx^*$ such that

$$\begin{aligned}\omega_{ms_\eta}(1, x_{n+1}, v^*) &\leq r_\eta \max\Big\{\omega_{ms_\eta}(1, x_n, x^*), \frac{\omega_{ms_\eta}(1, x_n, x_{n+1}) + \omega_{ms_\eta}(1, x^*, v^*)}{2}, \\ &\quad \frac{\omega_{ms_\eta}(2s_\eta, x_n, v^*) + \omega_{ms_\eta}(1, x^*, x_{n+1})}{2}\Big\} + L_\eta \omega_{ms_\eta}(1, x^*, x_{n+1}) \\ &\leq r_\eta \max\Big\{\omega_{ms_\eta}(1, x_n, x^*), \frac{\omega_{ms_\eta}(1, x_n, x_{n+1}) + \omega_{ms_\eta}(1, x^*, v^*)}{2}, \\ &\quad \frac{\omega_{ms_\eta}(1, x_n, x^*) + \omega_{ms_\eta}(1, x^*, v^*) + \omega_{ms_\eta}(1, x^*, x_{n+1})}{2}\Big\} \\ &\quad + L_\eta \omega_{ms_\eta}(1, x^*, x_{n+1}) \ \forall \eta \in \mathfrak{A}. \end{aligned} \tag{5}$$

In the above inequality by keeping the Fatou property and the case as $n \to \infty$, we get

$$\omega_{ms_\eta}(1, x^*, v^*) \leq r_\eta \frac{\omega_{ms_\eta}(1, x^*, v^*)}{2} \ \forall \eta \in \mathfrak{A},$$

which is only possible if $\omega_{ms_\eta}(1, x^*, v^*) = 0 \ \forall \eta \in \mathfrak{A}$. As we know that the collection $\{\omega_{ms_\eta}$ with $s_\eta \geq 1 : \eta \in \mathfrak{A}\}$ is separating, thus we get $x^* = v^*$. Hence, $x^* \in Tx^*$. □

Theorem 2.2. *Consider a map $T \colon M \to CL(M)$ such that for each $x, y \in M$ with $\lambda(x, y) \geq 1$ and $u \in Tx$, there exists $v \in Ty$ satisfying the following inequality:*

$$\begin{aligned}\omega_{ms_\eta}(1, u, v) &\leq a_\eta \omega_{ms_\eta}(1, x, y) + b_\eta \omega_{ms_\eta}(1, x, u) + c_\eta \omega_{ms_\eta}(1, y, v) \\ &\quad + e_\eta \omega_{ms_\eta}(2s_\eta, x, v) + L_\eta \omega_{ms_\eta}(1, y, u) \ \forall \eta \in \mathfrak{A}\end{aligned} \tag{6}$$

where $a_\eta, b_\eta, c_\eta, e_\eta, L_\eta \geq 0$, and $s_\eta a_\eta + s_\eta b_\eta + s_\eta c_\eta + 2s_\eta^2 e_\eta < 1 \ \forall \eta \in \mathfrak{A}$. Further, assume the given below conditions hold:

- : *(i) there are two elements $x_0 \in M$ and $x_1 \in Tx_0$ with $\lambda(x_0, x_1) \geq 1$;*
- : *(ii) for $x \in M$ and $y \in Tx$ with $\lambda(x, y) \geq 1$, we have $\lambda(y, z) \geq 1$ for each $z \in Ty$;*
- : *(iii) if $\{x_n\}$ is a sequence in M with $x_n \to^{\mathfrak{F}} x \in M$ and $\lambda(x_n, x_{n+1}) \geq 1 \ \forall n \in \mathbb{N}$, then $\lambda(x_n, x) \geq 1 \ \forall n \in \mathbb{N}$.*

Then atleast one fixed point of T must exists.

Proof. Hypothesis (i) gives the existence of two elements $x_0 \in M$ and $x_1 \in Tx_0$ with $\lambda(x_0, x_1) \geq 1$. From (6), for $\lambda(x_0, x_1) \geq 1$ and $x_1 \in Tx_0$, we get $x_2 \in Tx_1$ such that

$$\begin{aligned}\omega_{ms_\eta}(1, x_1, x_2) &\leq a_\eta \omega_{ms_\eta}(1, x_0, x_1) + b_\eta \omega_{ms_\eta}(1, x_0, x_1) + c_\eta \omega_{ms_\eta}(1, x_1, x_2) \\ &\quad + e_\eta \omega_{ms_\eta}(2s_\eta, x_0, x_2) + L_\eta \omega_{ms_\eta}(1, x_1, x_1) \\ &\leq (a_\eta + b_\eta + e_\eta)\omega_{ms_\eta}(1, x_0, x_1) + (c_\eta + e_\eta)\omega_{ms_\eta}(1, x_1, x_2) \\ &\quad + L_\eta 0 \ \forall \ \eta \in \mathfrak{A}.\end{aligned}$$

By performing some necessary simplification we obtain $\omega_{ms_\eta}(1, x_1, x_2) \leq \xi_\eta \omega_{ms_\eta}(1, x_0, x_1)$ for all $\eta \in \mathfrak{A}$, here $\xi_\eta = \frac{a_\eta + b_\eta + e_\eta}{1 - c_\eta - e_\eta} < 1$. Since $x_0 \in M$ and $x_1 \in Tx_0$ with $\lambda(x_0, x_1) \geq 1$, then by hypothesis (ii), for $x_2 \in Tx_1$, we have $\lambda(x_1, x_2) \geq 1$. Thus from (6), for $\lambda(x_1, x_2) \geq 1$ and $x_2 \in Tx_1$, we have $x_3 \in Tx_2$ such that

$$\begin{aligned}\omega_{ms_\eta}(1, x_2, x_3) &\leq a_\eta \omega_{ms_\eta}(1, x_1, x_2) + b_\eta \omega_{ms_\eta}(1, x_1, x_2) + c_\eta \omega_{ms_\eta}(1, x_2, x_3) \\ &\quad + e_\eta \omega_{ms_\eta}(2s_\eta, x_1, x_3) + L_\eta \omega_{ms_\eta}(1, x_2, x_2) \\ &\leq (a_\eta + b_\eta + e_\eta)\omega_{ms_\eta}(1, x_1, x_2) + (c_\eta + e_\eta)\omega_{ms_\eta}(1, x_2, x_3) \\ &\quad + L_\eta 0 \ \forall \ \eta \in \mathfrak{A}.\end{aligned}$$

Again by performing necessary simplification, we obtain $\omega_{ms_\eta}(1, x_2, x_3) \leq (\xi_\eta)^2 \omega_{ms_\eta}(1, x_0, x_1)$ for all $\eta \in \mathfrak{A}$, here $\xi_\eta = \frac{a_\eta + b_\eta + e_\eta}{1 - c_\eta - e_\eta} < 1$. Proceeding with this pattern we obtain $\{x_n\}$ in M with $x_n \in Tx_{n-1}$, $\lambda(x_{n-1}, x_n) \geq 1$ and $\omega_{ms_\eta}(1, x_n, x_{n+1}) \leq (\xi_\eta)^n \omega_{ms_\eta}(1, x_0, x_1) \leq$

$(\xi_\eta)^n\delta_{\mathfrak{F}}(M)$ for all $\eta \in \mathfrak{A}$ and $n \in \mathbb{N}$. By considering the Δ_b-condition and the last inequality, for each $\eta \in \mathfrak{A}$, we get $\omega_{ms_\eta}(\gamma, x_n, x_{n+1}) \leq (\xi_\eta)^n\delta_{\mathfrak{F}}(M)$ for each $\gamma > 0$ and $n \in \mathbb{N}$. Following we will show $\{x_n\}$ is ω_{ms_η}-Cauchy. For each $p, m \in \mathbb{N}$ and $\eta \in \mathfrak{A}$, we get

$$\begin{aligned}\omega_{ms_\eta}(p, x_p, x_{p+m}) &\leq \sum_{i=p}^{p+m-1} \omega_{ms_\eta}(\frac{1}{s_\eta^i}, x_i, x_{i+1}) \leq \sum_{i=p}^{p+m-1} (\xi_\eta)^i\delta_{\mathfrak{F}}(M) \\ &\leq \sum_{i=p}^{\infty}(\xi_\eta)^i\delta_{\mathfrak{F}}(M) \to 0 \text{ as } p \to \infty.\end{aligned}$$

Hence $\{x_n\}$ is ω_{ms_η}-Cauchy in M. As M is ω_{ms_η}-complete then there is $x^* \in M$ such that for each $\eta \in \mathfrak{A}$ we have $\lim_{n\to\infty} \omega_{ms_\eta}(\beta, x_n, x^*) = 0$ for some $\beta > 0$. Since the Δ_b-condition holds for the collection $\mathfrak{F}$ then, for each $\eta \in \mathfrak{A}$, we get $\lim_{n\to\infty} \omega_{ms_\eta}(\gamma, x_n, x^*) = 0\ \forall\gamma > 0$. By considering hypothesis (iii) and the facts about $\{x_n\}$, we get $\lambda(x_n, x^*) \geq 1$ for each $n \in \mathbb{N}$. From (6), for $\lambda(x_n, x^*) \geq 1$ and $x_{n+1} \in Tx_n$, there is $v^* \in Tx^*$ with

$$\begin{aligned}\omega_{ms_\eta}(1, x_{n+1}, v^*) &\leq a_\eta\omega_{ms_\eta}(1, x_n, x^*) + b_\eta\omega_{ms_\eta}(1, x_n, x_{n+1}) + c_\eta\omega_{ms_\eta}(1, x^*, v^*) \\ &\quad + e_\eta\omega_{ms_\eta}(2s_\eta, x_n, v^*) + L_\eta\omega_{ms_\eta}(1, x^*, x_{n+1}) \\ &\leq a_\eta\omega_{ms_\eta}(1, x_n, x^*) + b_\eta\omega_{ms_\eta}(1, x_n, x_{n+1}) + c_\eta\omega_{ms_\eta}(1, x^*, v^*) \\ &\quad + e_\eta[\omega_{ms_\eta}(1, x_n, x^*) + \omega_{ms_\eta}(1, x^*, v^*)] \\ &\quad + L_\eta\omega_{ms_\eta}(1, x^*, x_{n+1})\ \forall\ \eta \in \mathfrak{A}.\end{aligned}$$

In the above inequality by applying the Fatou property and the limit as $n \to \infty$, we get

$$\omega_{ms_\eta}(1, x^*, v^*) \leq (c_\eta + e_\eta)\omega_{ms_\eta}(1, x^*, v^*) < \omega_{ms_\eta}(1, x^*, v^*)\ \forall\ \eta \in \mathfrak{A},$$

this is impossible if $\omega_{ms_\eta}(1, x^*, v^*) \neq 0$. Thus, $\omega_{ms_\eta}(1, x^*, v^*) = 0\ \forall\ \eta \in \mathfrak{A}$. Since the collection $\{\omega_{ms_\eta}$ with $s_\eta \geq 1 : \eta \in \mathfrak{A}\}$ is separating, thus we get $x^* = v^*$. Hence, $x^* \in Tx^*$. □

The above theorem implies to the following result, when we assume that $T\colon M \to M$ and $\lambda(x, y) = 1$ for each $x, y \in M$.

Corollary 2.1. *Consider a map $T\colon M \to M$ such that for each $x, y \in M$ we get*

$$\begin{aligned}\omega_{ms_\eta}(1, Tx, Ty) &\leq a_\eta\omega_{ms_\eta}(1, x, y) + b_\eta\omega_{ms_\eta}(1, x, Tx) + c_\eta\omega_{ms_\eta}(1, y, Ty) \\ &\quad + e_\eta\omega_{ms_\eta}(2s_\eta, x, Ty) + L_\eta\omega_{ms_\eta}(1, y, Tx)\ \forall\eta \in \mathfrak{A} \qquad (7)\end{aligned}$$

where, $s_\eta a_\eta + s_\eta b_\eta + s_\eta c_\eta + 2s_\eta^2 e_\eta < 1$ and $a_\eta, b_\eta, c_\eta, e_\eta, L_\eta \geq 0,\ \forall\eta \in \mathfrak{A}$. Then atleast one fixed point of T must exists.

Remark 2.1. *Note that*

- : *(i) if the collection of pseudomodular b-metrics $\mathfrak{F} = \{\omega_{ms_\eta}$ with $s_\eta \geq 1 : \eta \in \mathfrak{A}\}$ on X is such that $\omega_\eta(\gamma, x, y) < \infty$ (that is, finite number) $\forall\eta \in \mathfrak{A}$ and $\gamma > 0$, and every $x, y \in X$, then ω_{ms_η}-boundedness of M may be ignored from the results.*
- : *(ii) one may use continuous operator in the results instead of Hypothesis (iii).*

3. **Consequences**

In this section, we consider $G = (V, E)$ as directed graph with vertex set V equal to M and edge set E contains $\{(x, x) : x \in V\}$. Moreover, no parallel edges contained in G. Define a map $\lambda : M \times M \to [0, \infty)$ by

$$\lambda(x, y) = \begin{cases} 1, & \text{if } (x, y) \in E \\ 0, & \text{otherwise.} \end{cases}$$

Then Theorem 2.1 and 2.2 yields the following results, respectively.

Corollary 3.1. *Consider a map $T\colon M \to CL(M)$ such that for each $x, y \in M$ with $(x, y) \in E$ and $u \in Tx$, there is $v \in Ty$ satisfying the following inequality:*

$$\begin{aligned}\omega_\eta(1,u,v) \quad \leq \quad & r_\eta \max\Big\{\omega_{ms_\eta}(1,x,y), \frac{\omega_{ms_\eta}(1,x,u)+\omega_{ms_\eta}(1,y,v)}{2}, \\ & \frac{\omega_{ms_\eta}(2s_\eta,x,v)+\omega_{ms_\eta}(1,y,u)}{2}\Big\} + L_\eta\omega_{ms_\eta}(1,y,u) \ \forall \eta \in \mathfrak{A},\end{aligned}$$

where $r_\eta \in [0, 1/s_\eta)$ and $L_\eta \geq 0$ $\forall \eta \in \mathfrak{A}$. Further, assume the given below conditions hold:

- : *(i) there are two elements $x_0 \in M$ and $x_1 \in Tx_0$ with $(x_0, x_1) \in E$;*
- : *(ii) for $x \in M$ and $y \in Tx$ with $(x, y) \in E$, we have $(y, z) \in E$ for each $z \in Ty$;*
- : *(iii) if $\{x_n\}$ is a sequence in M with $x_n \to^{\mathfrak{F}} x \in M$ and $\lambda(x_n, x_{n+1}) \geq 1$ $\forall n \in \mathbb{N}$, then $\lambda(x_n, x) \geq 1$ $\forall n \in \mathbb{N}$.*

Then atleast one fixed point of T must exists.

Corollary 3.2. *Consider a map $T\colon M \to CL(M)$ such that for each $x, y \in M$ with $(x, y) \in E$ and $u \in Tx$, there is $v \in Ty$ satisfying the following inequality:*

$$\begin{aligned}\omega_{ms_\eta}(1,u,v) \quad \leq \quad & a_\eta\omega_{ms_\eta}(1,x,y) + b_\eta\omega_{ms_\eta}(1,x,u) + c_\eta\omega_{ms_\eta}(1,y,v) \\ & + e_\eta\omega_{ms_\eta}(2s_\eta,x,v) + L_\eta\omega_{ms_\eta}(1,y,u) \ \forall \eta \in \mathfrak{A}\end{aligned}$$

where $s_\eta a_\eta + s_\eta b_\eta + s_\eta c_\eta + 2s_\eta^2 e_\eta < 1$ and $a_\eta, b_\eta, c_\eta, e_\eta, L_\eta \geq 0$ $\forall \eta \in \mathfrak{A}$. Further, assume the given below conditions hold:

- : *(i) there are two elements $x_0 \in M$ and $x_1 \in Tx_0$ with $(x_0, x_1) \in E$;*
- : *(ii) for $x \in M$ and $y \in Tx$ with $(x, y) \in E$, we have $(y, z) \in E$ for each $z \in Ty$;*
- : *(iii) if $\{x_n\}$ is a sequence in M with $x_n \to^{\mathfrak{F}} x \in M$ and $\lambda(x_n, x_{n+1}) \geq 1$ $\forall n \in \mathbb{N}$, then $\lambda(x_n, x) \geq 1$ $\forall n \in \mathbb{N}$.*

Then atleast one fixed point of T must exists.

4. **Application and Example**

As an application, we prove the existence theorem for nonlinear integral equation of the below mentioned form:

$$x(t) = p(t) + \int_0^t S(t,u)g(u,x(u))du, \quad t \in Y \tag{8}$$

where $p : Y \to \mathbb{R}$, $g : Y \times \mathbb{R} \to \mathbb{R}$ are continuous functions and $S : Y \times Y \to [0, \infty)$ is such that $S(t, \cdot) \in L^1(Y)$ for all $t \in Y$.

Denote $X = (C[0, b], \mathbb{R})$ with the collection of all bounded and continuous realvalued functions on $Y = [0, b]$, where b is any fixed natural number. Consider a collection of pseudomodular b-metrics as

$$\omega_n(\gamma, x, y) = \frac{1}{\lceil\gamma\rceil} \max_{t\in[0,n]} (x(t) - y(t))^2.$$

Clearly, modular b-gauge space obtained by the collection $\mathfrak{F} = \{\omega_n$ with $s_n = 2 : n \in J = \{1, 2, \cdots, b\}\}$ on X is separating and ω_{ms_η}-complete. Also it satisfies the Δ_b-condition and Fatou property.

Theorem 4.1. *Let $X = (C[0, b], \mathbb{R})$ and let the operator*

$$T\colon X \to X, \quad Tx(t) = p(t) + \int_0^t S(t,u)g(u,x(u))du, \quad t \in Y = [0,b]$$

where $p : Y \to \mathbb{R}$, $g : Y \times \mathbb{R} \to \mathbb{R}$ are continuous functions and $S : Y \times Y \to [0, \infty)$ is such that $S(t, \cdot) \in L^1(Y)$ for all $t \in Y$, and $b > 1$. Further, assume the given below conditions hold, for each natural number $n \leq b$:

: *(i) for each $s \in [0, n]$ and $x, y \in X$, we obtain*

$$|g(s, x(s)) - g(s, y(s))|^2 \leq \frac{1}{2b} \max_{t \in [0,n]} |x(t) - y(t)|^2 \text{ for each } n \in J;$$

: *(ii) there is $\delta \in (0, 1)$ with*

$$\max_{t \in [0,n]} \left(\int_0^t S(t, u) du \right)^2 \leq n^{\delta}.$$

Then the integral equation (8) *has at least one solution.*

Proof. For any $x, y \in X$ and $t \in [0, n]$ for $n \in J$, we have

$$\begin{aligned} |Tx(t) - Ty(t)|^2 &\leq \left(\int_0^t S(t, u)[|g(u, x(u)) - g(u, y(u))|]du \right)^2 \\ &\leq \left(\sqrt{\frac{1}{2b} \max_{t \in [0,n]} |x(t) - y(t)|^2} \int_0^t S(t, u)du \right)^2 \\ &= \frac{1}{2b} \left(\int_0^t S(t, u)du \right)^2 \max_{t \in [0,n]} |x(t) - y(t)|^2. \end{aligned}$$

This yields the inequality $\omega_n(\gamma, Tx, Ty) \leq a_n \omega_n(\gamma, x, y)$ $\forall$ $x, y \in X$, $\gamma > 0$ and $n \in J$ with $a_n = \frac{n^\delta}{2b} < \frac{1}{2}$. Hence, we say (7) holds with $a_n = \frac{n^\delta}{2b}$, and $b_n = c_n = e_n = L_n = 0$ for each $n \in J$. Therefore, by Corollary 2.1, there exists a fixed point of T, that is, the integral equation (8) has at least one solution. □

Remark 4.1. *Consider the integral equation of the form:*

$$x(t) = \int_0^t (t - u)^{\gamma - 1} \frac{x(u)}{b + x(u)} du, \quad t \in Y = [0, b] \tag{9}$$

where $b > 1$ is a fixed natural number and $\gamma \in (0, 1)$. Note that Theorem 4.1 validate the existence of solution of the integral equation (9)*.*

Example 4.1. *Take $\mathcal{S}$ as the collection of all real sequences with $\omega_n(\gamma, x, y) = \frac{1}{\lceil \gamma \rceil}|x_n - y_n|^2$ $\forall$ $n \in N$ and $\gamma > 0$, where $x = \{x_n\}$, $y = \{y_n\} \in \mathcal{S}$. Define the mapping $T : \mathcal{S} \to CL(\mathcal{S})$ by*

$$T(\{x_n\}_{n \in \mathbb{N}}) = \begin{cases} \{\{\frac{x_n + 2}{3}\}_{n \in \mathbb{N}}, \{\frac{x_n + 3}{3}\}_{n \in \mathbb{N}}\}, & \text{if } \{x_n\}_{n \in \mathbb{N}} \subseteq [0, \infty) \\ \{0, \{{x_n}^2\}_{n \in \mathbb{N}}\}, & \text{otherwise} \end{cases}$$

and $\lambda : \mathcal{S} \times \mathcal{S} \to [0, \infty)$ by

$$\lambda(\{x_n\}_{n \in \mathbb{N}}, \{y_n\}_{n \in \mathbb{N}}) = \begin{cases} 1, & \text{if } \{x_n\}_{n \in \mathbb{N}}, \{y_n\}_{n \in \mathbb{N}} \subseteq [0, \infty) \\ 0, & \text{otherwise.} \end{cases}$$

Here, it can be seen that (1) *holds for each $x, y \in \mathcal{S}$ with $\lambda(x, y) = 1$, where $r_n = \frac{1}{9}$, $s_n = 2$ and $L_n = 0$ for each $n \in \mathbb{N}$. Also for $x_0 = \{\frac{1}{n}\}_{n \in \mathbb{N}} \in \mathcal{S}$ we obtain $x_1 = \{\frac{1+3n}{3n}\}_{n \in \mathbb{N}} \in Tx_0$ with $\lambda(\{\frac{1}{n}\}_{n \in \mathbb{N}}, \{\frac{1+3n}{3n}\}_{n \in \mathbb{N}}) = 1$. Further for each $x \in \mathcal{S}$ and $y \in Tx$ with $\lambda(x, y) = 1$ we get $\lambda(y, z) = 1$ for each $z \in Ty$. Moreover, every $\{x_n\}$ in $\mathcal{S}$ with $\lambda(x_n, x_{n+1}) = 1$ $\forall$ $n \in \mathbb{N}$ and ω_{ms_η}-converges to $x \in \mathcal{S}$, we have $\lambda(x_n, x) = 1$ $\forall$ $n \in \mathbb{N}$. Hence, Theorem 2.1 conclude that T has a fixed point.*

5. **Conclusion**

We conclude this article with these sentences: First, we defined the notion of modular b-gauge spaces and proved few fixed point results on this structure. Secondly, we applied our result to study about the existence of the solution of nonlinear integral equations. In last, we gave the examples of our results.

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