U.P.B. Sci. Bull., Series A, Vol. 74, Iss. 3, 2012 ISSN 1223-7027

VARIATIONAL PRINCIPLES AND A GENERALIZED DISTANCE

Georgiana GOGA[1]

Scopul acestui articol este de a prezenta un principiu variațional general de tip Ekeland, bazat pe un nou concept de distanță generalizată numită u-distanță, care a fost introdusă recent de către Ume. Ca aplicații, extindem principiul variațional al lui Zhong şi o teoremă de minimizare.

The aim of this paper is to present a general variational principle on Ekeland-type relied on a new concept of a generalized distance called u-distance, which was introduced recently by Ume. As applications, we extend Zhong's variational principle and a minimization theorem.

Key words: variational principles, *u*-distance, minimization theorem
2000 MSC: 54E45, 54E50, 58E30

1. Introduction

One of the most powerful tool in nonlinear analysis during the last four decades is surely given by Ekeland's variational principle for lower semi-continuous functionals on complete metric spaces, which has various applications. Let us begin with the original principle:

Theorem 1.1 ([1],[2]). *Let* (X,d) *be a complete metric space. Let* $f : X \to (-\infty,\infty]$ *be a proper lower semicontinuous function which is bounded from below. Then for every* $\varepsilon > 0, \lambda > 0$ *and* $u \in X$ *such that*

$$f(u) < \inf_{x\in X} f(x) + \varepsilon$$

there exists $v \in X$, *satisfying the following inequalities:*

(E_1) $f(v) \le f(u)$;

(E_2) $d(u,v) \le \lambda$;

(E_3) $f(w) > f(v) - \frac{\varepsilon}{\lambda} d(v,w)$ *for every* $w \in X \setminus \{v\}$.

Soon after its formulation, many extensions of Theorem 1.1. were proposed. One of these was obtained in 1997 by Zhong [3], [4]:

[1] Prof., Department. of Mathematics, "Nicolae Rotaru" College, Constanta , Romania; e-mail: georgia_goga@yahoo.com

Theorem 1.2 ([3],[4]). *Let* (X,d) *be a complete metric space and* $x_0 \in X$ *be fixed. Let* $f : X \to (-\infty,\infty]$ *be a proper lower semicontinuous function which is bounded from below and let* $h:[0,\infty)\to[0,\infty)$ *be a nondecreasing continuous function such that*

$$\int_0^\infty \frac{1}{1+h(r)}dr = +\infty.$$

Then, for every $\varepsilon > 0$, *every* $y \in X$ *so that*

$$f(y) < \inf_{x\in X} f(x) + \varepsilon,$$

and $\lambda > 0$, *there exists some point* $z \in X$ *such that*

(i) $f(z) \le f(y)$;

(ii) $d(x_0, z) \le r_0 + r^*$;

(iii) $f(x) \ge f(z) - \dfrac{\varepsilon}{\lambda[1+h(d(x_0,z))]} d(z,x)$ *for all* $x \in X$,

where $r_0 = d(x_0, y)$, *and* r^* *is a number so that*

$$\int_{r_0}^{r_0+r^*} \frac{1}{1+h(t)}dt \ge \lambda.$$

Recently, in 2010, Ume [5] presented a new concept of generalized distance called $u-$distance, which generalizes some distances anterior introduced (see, e.g., $\omega-$distance [6] , Tataru's distance [7] , $\tau-$distance [8]).

In this paper, starting by this concept of distance, we expand the Ekeland variational principle ([1],[2]) and prove a new minimization theorem by using this generalized variational principle. Our results extend and improve other known results due to Zhong ([3], [4]), Ekeland ([1], [2]), and Takahashi ([9]).

2. Preliminaries

The purpose of this section is to present former results necessarily in our approach. First, we recall the above-mentioned concept of Ume's [5] generalized distance in a metric space.

Definition 2.1. *Let* (X,d) *be a metric space. A function* $p : X \times X \to R_+$ *is called* $u-$*distance on* X *if there exists a map* $\Theta : X \times X \times R_+ \times R_+ \to R_+$ *such that the following condition hold:*

(u_1) $p(x,z) \le p(x,y) + p(y,z), \ \forall x,y,z \in X;$

(u_2) $\Theta(x,y,0,0) = 0$ *and* $\Theta(x,y,s,t) \ge \min\{s,t\}$ *for all* $x,y \in X$, $s,t \in R_+$, *and, for every* $x \in X$ *and* $\varepsilon > 0$, *there is* $\delta > 0$ *such that*

$$|\Theta(x,y,s,t) - \Theta(x,y,s_0,t_0)| < \varepsilon$$

if $|s - s_0| < \delta, |t - t_0| < \delta,\ s,t,s_0,t_0 \in R_+$ *whatever* $y \in X$;

(u_3) $\lim_n x_n = x$ *and* $\lim_n \sup\{\Theta(w_n, z_n, p(w_n, x_m), p(z_n, x_m)) : m \ge n\} = 0$ *imply*

$$p(y,x) \le \liminf_{n\to\infty} p(y,x_n) \ \textit{for } y \in X;$$

(u_4)

$$\left.\begin{array}{l}\lim_n \sup\{p(x_n, w_m) : m \ge n\} = 0,\\ \lim_n \sup\{p(y_n, z_m) : m \ge n\} = 0,\\ \lim_n \Theta(x_n, w_n, s_n, t_n) = 0,\\ \lim_n \Theta(y_n, z_n, s_n, t_n) = 0\end{array}\right\} \Rightarrow \lim_n \Theta(w_n, z_n, s_n, t_n) = 0,$$

or

$$\left.\begin{array}{l}\lim_n \sup\{p(w_m, x_n) : m \ge n\} = 0,\\ \lim_n \sup\{p(z_m, y_n) : m \ge n\} = 0,\\ \lim_n \Theta(x_n, w_n, s_n, t_n) = 0,\\ \lim_n \Theta(y_n, z_n, s_n, t_n) = 0\end{array}\right\} \Rightarrow \lim_n \Theta(w_n, z_n, s_n, t_n) = 0,$$

(u_5)

$$\left.\begin{array}{l}\lim_n \Theta(w_n, z_n, p(w_n, x_n), p(z_n, x_n)) = 0,\\ \lim_n \Theta(w_n, z_n, p(w_n, y_n), p(z_n, y_n)) = 0\end{array}\right\} \Rightarrow \lim_n d(x_n, y_n) = 0,$$

or

$$\left.\begin{array}{l}\lim_n \Theta(a_n, b_n, p(x_n, a_n), p(x_n, b_n)) = 0,\\ \lim_n \Theta(a_n, b_n, p(y_n, a_n), p(y_n, b_n)) = 0\end{array}\right\} \Rightarrow \lim_n d(x_n, y_n) = 0.$$

Example 2.2 ([5]). *Let* X *be a normed space with norm* $\|.\|$. *Then the function* $p : X \times X \to R_+$ *defined by* $p(x,y) = \|x\|$ *is a* $u-$*distance on* X *but it is not a* $\tau-$ *distance on* X.

Example 2.3 ([5]). *Let* p *be a* $u-$*distance on a metric space* (X,d) *and let* c *be a positive real number. Then the function* $q: X\times X\to R_+$ *defined by* $q(x,y)=c\cdot p(x,y)$ *for every* $x,y\in X$ *is also a* $u-$*distance on* X*.*

By means of the generalized $u-$distance, Ume obtained in [5] the following version of Ekeland's variational principle. This result will play an important role in the proof of our main theorem.

Theorem 2.4 ([5]). *Let* (X,d) *be a complete metric space, let* $f: X\to(-\infty,\infty]$ *be a proper lower semicontinuous function, which is bounded from below, and let* $p: X\times X\to R_+$ *be a* $u-$*distance on X. Then the following assertions hold:*

(A) *For each* $x\in X$ *with* $f(x)<\infty$, *there exists* $v\in X$ *such that*

$$f(v)\le f(x) \text{ and } f(w)>f(v)-p(v,w) \quad \text{for all } w\in X\setminus\{v\}.$$

(B) *For each* $\varepsilon>0, \lambda>0$, *and* $x\in X$ *with* $p(x,x)=0$ *and* $f(x)<\inf_{a\in X} f(a)+\varepsilon$, *there exists* $v\in X$ *such that* $f(v)\le f(x)$,

$$p(x,v)\le\lambda \text{ and}$$

$$f(w)>f(v)-\frac{\varepsilon}{\lambda}p(v,w) \text{ for all } w\in X\setminus\{v\}.$$

3. The main statements

We begin this section by extending a result of Suzuki [10] using the $u-$distance.

Proposition 3.1. *Let* (X,d) *be a complete metric space, and let* $p: X\times X\to R_+$ *be a* $u-$*distance on* X*. Let* $q: X\times X\to R_+$ *be a function with the properties:*

(a) $q(x,z)\le q(x,y)+q(y,z)$, $\forall x,y,z\in X$;

(b) q *is lower semicontinuous in the second variable;*

(c) $q(x,y)\ge p(x,y)$ *for all* $x,y\in X$.

Then q *is also* $u-$ *distance on* X*.*

Proof. The assumption (a) is equivalent with $(u_1)_q$.

Let $\Theta : X \times X \times R_+ \times R_+ \to R_+$ be a function satisfying $(u_2) \sim (u_5)$. Clearly, $(u_3)_q$ follows from (b). Now, we assume that

$$\begin{aligned} &\lim_n \sup\{q(x_n, w_m) : m \geq n\} = 0, \\ &\lim_n \sup\{q(y_n, z_m) : m \geq n\} = 0, \\ &\lim_n \Theta(x_n, w_n, s_n, t_n) = 0, \\ &\lim_n \Theta(y_n, z_n, s_n, t_n) = 0. \end{aligned} \tag{1}$$

By (1) and (c), we have

$$\lim_n \sup\{p(x_n, w_m) : m \geq n\} = 0$$

and

$$\lim_n \sup\{p(y_n, z_m) : m \geq n\} = 0.$$

Therefore, by (u_4), we find $\lim_n \Theta(w_n, z_n, s_n, t_n) = 0$, and derive $(u_4)_q$.

Next, we assume that

$$\lim_n \Theta(w_n, z_n, q(w_n, x_n), q(z_n, x_n)) = 0, \tag{2}$$

and

$$\lim_n \Theta(w_n, z_n, q(w_n, y_n), q(z_n, y_n)) = 0. \tag{3}$$

Apply again (c) in (2) and (3) to obtain

$$\lim_n \Theta(w_n, z_n, p(w_n, x_n), p(z_n, x_n)) = 0,$$

And

$$\lim_n \Theta(w_n, z_n, q(w_n, y_n), q(z_n, y_n)) = 0.$$

By virtue of (u_5), we have $\lim_n d(x_n, y_n) = 0$ and $(u_5)_q$ is also verified. ■

Next, we establish a general variational principle ([11], [12]), which is an extension of both Ekeland's and Zhong's variational principles.

Theorem 3.2. *Let (X, d) be a complete metric space, $a \in X$ a fixed element and let $p : X \times X \to R_+$ be a $u-$distance on X, lower semicontinuous in its second variable. Let $f : X \to (-\infty, \infty]$ be a proper lower semicontinuous function which is bounded from below and let $b : [0, \infty) \to (0, \infty)$ be a nonincreasing continuous function such that*

$$B(t) = \int_0^t b(r)dr,$$

is a C^1- *function from* R_+ *into itself such that* $B(\infty)=\infty$. *Let* $y\in X$ *be such that* $p(y,y)=0$ *and*

$$f(y)>\inf_{x\in X} f(x). \tag{4}$$

Then, for $\varepsilon_0>0$, *there exists* $z\in X$ *such that*

(i) $f(z)\le f(y)$,

(ii) $p(a,z)\le \beta(y)+\beta^*$,

(iii) $f(x)>f(z)-\frac{\varepsilon_0}{\lambda}b(\beta(z))p(z,x)$, *for all* $x\in X$,

where $\beta(.)=p(a,.)$, *and the number* β^* *is such that*

$$\int_{\beta(y)}^{\beta(y)+\beta^*} b(t)dt\ge\alpha(y), \tag{5}$$

with $\alpha(y)=f(y)-\inf_{x\in X} f(x)\ge\lambda>0.$

Proof. First, we define a function $q:X\times X\to R_+$ by

$$q(x,y):=\int_{p(a,x)}^{p(a,x)+p(x,y)} b(t)dt.$$

Since b is nonincreasing, for $(x,z)\in X\times X$, we deduce

$$\begin{aligned}q(x,z)&=\int_{p(a,x)}^{p(a,x)+p(x,z)} b(t)dt\le\int_{p(a,x)}^{p(a,x)+p(x,y)+p(y,z)} b(t)dt=\\&=\int_{p(a,x)}^{p(a,x)+p(x,y)} b(t)dt+\int_{p(a,x)+p(x,y)}^{p(a,x)+p(x,y)+p(y,z)} b(t)dt\le\\&\le\int_{p(a,x)}^{p(a,x)+p(x,y)} b(t)dt+\int_{p(a,y)}^{p(a,y)+p(y,z)} b(t)dt=q(x,y)+q(y,z).\end{aligned}$$

In addition, q is obviously lower semicontinuous in its second variable.

On the other hand, we have

$$\begin{aligned}q(x,y)&=B(p(a,x)+p(x,y))-B(p(a,x))\ge\\&\ge b(p(a,x)+p(x,y))p(x,y).\end{aligned} \tag{6}$$

Taking into account of the definition of function b,

$$b(p(a,x)+p(x,y))>b(\infty)=M\ge 0. \tag{7}$$

Combining (6) and (7), we deduce

$$q(x,y) \geq M \cdot p(x,y),$$

with $M \geq 0$. The assumptions of *Proposition 3.1* are verified, since $M \cdot p(x,y)$ is a $u-$distance. Hence, $q(x,y)$ is also $u-$distance.

Now, from (4) and (5), we obtain

$$\begin{aligned} 0 < \lambda \leq f(y) - \inf_{x \in X} f(x) = \alpha(y) \leq \int_{\beta(y)}^{\beta(y)+\beta^*} b(t)dt = \\ = \int_0^{\beta^*} b(u+\beta(y))du \leq \int_0^{\beta^*} b(u)du = B(\beta^*). \end{aligned} \tag{8}$$

So, by (8), we have

$$f(y) \leq \inf_{x \in X} f(x) + B(\beta^*),$$

and the *Theorem 2.4* is applicable to $q(x,y)$ for $\varepsilon = B(\beta^*) > 0$ and $\lambda = \alpha(y) > 0$. Therefore, there exists $z \in X$ such that

$$f(z) \leq f(y), \tag{9}$$

$$q(y,z) \leq \alpha(y), \tag{10}$$

and

$$f(x) > f(z) - \frac{B(\beta^*)}{\alpha(y)} q(z,x), \forall x \in X \setminus \{z\}. \tag{11}$$

By (u_1), we know that

$$p(a,z) \leq p(a,y) + p(y,z) = \beta(y) + p(y,z). \tag{12}$$

On the other hand, from (5) and (10), it follows that

$$B(\beta(y) + p(y,z)) - B(\beta(y)) \leq \alpha(y) \leq B(\beta(y) + \beta^*) - B(\beta(y)).$$

Thereby, we find that

$$p(y,z) \leq \beta^*, \tag{13}$$

since B is a nondecreasing function. Thus, (ii) follows from (12) and (13). Moreover, since

$$q(z,x) = \int_{p(a,z)}^{p(a,z)+p(z,x)} b(t)dt \leq b(p(a,z))p(z,x) = b(\beta(z))p(z,x), \tag{14}$$

by multiplying (14) with (-1) and, using (8) and (11), for $0 < B(\beta^*) \le \varepsilon_0$, we get

$$f(x) > f(z) - \frac{B(\beta^*)}{\alpha(y)} q(z,x) \ge f(z) - \frac{\varepsilon_0}{\lambda} b(\beta(z)) p(z,x),$$

for all $x \in X$ and (iii) is verified. This completes the proof. ■

Remark 3.3. *Let a, f, b, p,* $\alpha(y)$, $\beta(y)$, β^* *and X be as in Theorem 3.2.*

(i) *When* $a = y$, $b(t) = 1$, $\beta^* = \lambda$, $\varepsilon_0 > \alpha(y) \ge \lambda > 0$, *and* $p(x,y) = d(x,y)$, *Theorem 3.2 reduces to Theorem 1.1.*

(ii) *Take* $a = x_0$, $b(t) = \dfrac{1}{1+h(t)}$, *where* $h : [0,\infty) \to [0,\infty)$ *is a continuous nondecreasing function such that*

$$\int_0^\infty \frac{1}{1+h(r)} dr = +\infty,$$

where $\varepsilon_0 > \alpha(y) \ge \lambda > 0$, $\beta(y) = d(x_0, y) = r_0$, $\beta^* = r^*$ *and* $p(x,y) = d(x,y)$. *Then, Theorem 3.2 implies Theorem 1.2.*

We give now our minimization theorem.

Theorem 3.4. *Let* X, a, f, p, b *and* B *be as in Theorem 3.2. We assume that for any* $\varepsilon > 0$ *and for every* $y \in X$ *with*

$$f(y) > \inf_{x \in X} f(x),$$

there exists $w \in X \setminus \{y\}$ *such that*

$$f(w) \le f(y) - \varepsilon b(\beta(y)) p(y,w). \tag{15}$$

Then, there exists $x_0 \in X$ *such that* $\inf_{x \in X} f(x) = f(x_0)$.

Proof. Suppose that

$$f(y) > \inf_{x \in X} f(x),$$

for every $y \in X$. By *Theorem* 3.2, for any $\varepsilon > 0$, there exists $z \in X$ such that

$$f(x) > f(z) - \varepsilon b(\beta(z)) p(z,x), \text{ for all } x \in X. \tag{16}$$

Besides, the point z will satisfy

$$\inf_{x\in X} f(x) = f(z).$$

Otherwise, assuming that

$$f(y) \geq f(z) > \inf_{x\in X} f(x),$$

by (15) and (16), we get to

$$f(z) \geq f(w) + \varepsilon b(\beta(z)) p(z,w) > f(z),$$

which is a contradiction. This completes the proof. ■

In the following, we illustrate how to satisfying the condition (15) of Theorem 3.4 by an elementary example.

Example 3.5. *Consider the function* $f: [0,\infty) \to [0,\infty)$ *defined by* $f(x) = \sqrt[3]{x}$ *and* $a = 0$. *Clearly,* $f(0) = \inf_{x\in[0,\infty)} f(x)$. *For any* $\varepsilon > 0$ *and each* $y \in (0,\varepsilon]$, *we have* $w \in (0,y)$ *such that*

$$p(y,w) = |y - w| = y - w = \left(\sqrt[3]{y} - \sqrt[3]{w}\right)\left(\sqrt[3]{y^2} + \sqrt[3]{yw} + \sqrt[3]{w^2}\right) <$$
$$< 3\sqrt[3]{y^2+1}\left(\sqrt[3]{y} - \sqrt[3]{w}\right) = \frac{f(y) - f(w)}{\varepsilon b(\beta(y))},$$

where $b(x) = \dfrac{1}{3\varepsilon\sqrt[3]{x^2+1}}$, *and* $\beta(y) = p(0,y) = |y| = y$. *Then, we can apply Theorem 3.4 to conclude that there exists* $x_0 \in [0,\infty)$ *such that* $f(x_0) = \inf_{x\in[0,\infty)} f(x)$. ■

REFERENCES

[1] *I. Ekeland*, "On the variational principle", J. Math. Anal. Appl. **47** (1974), pp. 324 – 353.

[2] *I. Ekeland*, "Nonconvex minimization problems", Bull. Amer. Math. Soc. **1** (1979), pp.443 – 474.

[3] *C.-K. Zhong*, "A generalization of Ekeland's variational principle and applications to the study of the relation between the weak P.S. condition and coercivity", Nonlinear Anal. **29** (1997), pp.1421-1431.

[4] *C.-K. Zhong*, "On Ekeland's variational principle and a minimax theorem", J. Math. Anal. Appl. **205** (1997), pp. 239-250.

[5] *J.S. Ume*, "Existence theorems for generalized distance on complete metric spaces", Hindawi Publ. Corp., Fixed Point Theory Appl., 2010, Art. ID 397150, 21 pp.

[6] *O. Kada, T. Suzuki, W. Takahashi*, "Nonconvex minimization theorems and fixed point theorems in complete metric spaces", Math. Japan. **44** (1996), pp. 381 – 391.

[7] *D. Tataru*, "Viscosity solutions of Hamilton- Jacobi equations with unbounded nonlinear terms", J. Math. Anal. Appl. **163** (1992), pp. 354 – 392.
[8] *T. Suzuki*, "Generalized distances and existence theorems in complete metric spaces", J. Math. Anal. Appl. **253** (2001), pp. 440 – 458.
[9] *W. Takahashi*, Minimization theorems and fixed points theorems, Nonlinear Analysis and Mathematical Economics, Edited by T. Maruyama, RIMS Kokyuroku **829** (1991), pp.175-191.
[10] *T. Suzuki*, "On the relation between the weak Palais-Smale condition and coercivity given by Zhong", Nonlinear Anal. **68** (2008), pp. 2471 – 2478.
[11] *T.Q. Bao, P.Q. Khanh*, "Are several recent generalizations of Ekeland's variational principle more general than the original principle ? ", Acta Math. Vietnam **28** (2003), pp. 345-350.
[12] *M. Turinici*, "Function variational principles and coercivity", J. Math. Anal. Appl. **304** (2005), pp. 236-248.
[13] *I. Meghea*, "Ekeland variational principle with generalization and variants", Old City Publ., Philadelphia, PA, & Ed. Archives Contemp., Paris, 2009.
[14] *Z. Wu*, "Equivalent extensions to Caristi-Kirk's fixed point theorem, Ekeland's variational principle and Takahashi's minimization theorem", Hindawi Publ.Corp., Fixed Point Theory and Appl., vol. 2010, ID 970579.