U.P.B. Sci. Bull., Series A, Vol. 77, Iss. 2, 2015 ISSN 1223-7027

SUB-RIEMANNIAN GEOMETRY AND OPTIMAL CONTROL ON LORENZ-INDUCED DISTRIBUTIONS

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Within the framework of optimal control on non-holonomic distributions of sub-Riemannian manifolds, several problems are studied, which are further applied to Lorenz-induced distributions. The developed approach is original, and considers specific geometric control objects, like: distributions and the related Riemann-Vrănceanu metrics, moving frames, auto-parallelism, infinitesimal deformations and adjointness. The results concern single-time optimal control problems, and the optimal control problem of nonholonomic geodesics. As illustrative example, the article develops a non-holonomic analysis of the Lorenz dynamical system, which includes the study of the Lorenz - Pfaff form, of the Lorenz moving frame and co-frame, and of the related Riemann-Vrănceanu metric.

Keywords: non-holonomic distributions, sub-Riemannian geometry, horizontal geodesics, Riemann-Vrănceanu metrics, optimal control, Lorenz distribution.

MSC2010: 53C17, 49K15.

1. Statement of problems

In this paper we formulate and solve problems involving four issues:

(i) the non-holonomic distributions optimal control theory; (ii) non-holonomic analysis of dynamical systems; (iii) the geometrization of dynamical systems in the sense of bringing them to non-holonomic distributions and including sub-Riemannian theory; (iv) adapting optimal control problems on non-holonomic geodesics.

Most interesting original results focuses on the following topics: non-holonomic distributions and parallelism, optimal control on distributions, optimal control via non-holonomic geodesics, non-holonomic analysis of the Lorenz dynamical system, geodesics on Lorenz distributions.

We are convinced that the intersection of differential geometry, dynamical systems theory and optimal control theory produce unexpected representations for the solutions of concrete problems. Besides, all the references [1]-[31] at the end of the work are based on ideas such as control theory from geometric viewpoint, non-holonomic kinematics, sub-Riemannian geometry, stochastic optimal control, multitime optimal control etc.

2. Non-holonomic distributions and parallelism

Consider a pair of linearly independent vector fields X_1, X_2 defined on an open set $\mathbb{D} \subset \mathbb{R}^3$, which generate a *nonholonomic (non-integrable) distribution*

$$\mathcal{D} = \mathrm{span}\{X_\alpha(x) |\, \alpha = 1, 2\}.$$

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To find a Pfaff equation that describes this distribution, we introduce the 1-form ω associated to the vector field $X_* = X_1 \times X_2$,

$$\omega = \omega_i \, dx^i, \;\; \omega_i = \delta_{ij} X_*^j. \tag{2.1}$$

Due to the non-holonomy, the 1-form ω satisfies $\omega \wedge d\omega \neq 0$. Consequently, the distribution $\mathcal{D}$ is defined alternatively by a Pfaff equation, i.e.,

$$\mathcal{D} = \text{ Ker } \omega = \{X \in \mathcal{X}(\mathbb{D}) \mid \omega(X) = 0\}.$$

A vector field X which belongs to the distribution $\mathcal{D}$ is uniquely decomposed as $X(x) = u^\alpha(x) X_\alpha(x)$. In this case, the trajectories (the field lines) of X are *horizontal curves* (integral curves of the distribution $\mathcal{D}$). We shall further use both the fact that a non-vanishing Pfaffian form ω defines an integrable distribution, if and only if $\omega \wedge d\omega = 0$,
and the expression of the exterior differential

$$d\omega(Y, Z) = Y\omega(Z) - Z\omega(Y) - \omega([Y, Z]),$$

where ω is a differential 1-form and $Y, Z \in \mathcal{X}(\mathbb{D})$. We shall further find a geometrical structure compatible with the distribution $\mathcal{D} = \text{ Span}\{X_1, X_2\}$. To this aim, we use the following result

Lemma 2.1. *There exists a vector field $X_3 \in \mathcal{X}(\mathbb{D})$, such that the vector fields X_1, X_2, X_3 are linearly independent and*

$$[X_1, X_2, X_3]^{-1} = \begin{bmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{bmatrix}, \tag{2.2}$$

where $\omega^3 = \omega$ given in (2.1).

Proof. The last row of the inverse matrix is proportional to the cross product $X_* = X_1 \times X_2$. We select $X_3 = ||X_*||^{-2} X_*$, such that the proportionality factor be 1. □

The vector fields $X_a^i(x), x = (x^i),\ i,\ a = \overline{1,3}$ determine a moving frame on $(\mathbb{D}, \delta_{ab})$. Using Vrănceanu's idea ([30]), we associate to this frame the Riemannian-type structure $(\mathbb{D}, g_{ij}, \{X_a^i\})$, determined by a positive definite metric and a moving frame on $\mathbb{D}$, the latter being assumed to be orthonormal relative to the metric. The contravariant *Riemann-Vrănceanu metric* g^{ij} is then the tensor field defined by

$$\delta^{ab} X_a \otimes X_b \equiv \delta^{ab} X_a^i \frac{\partial}{\partial x^i} \otimes X_b^j \frac{\partial}{\partial x^j} = g^{ij}(x) \, \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j},$$

which has the explicit form

$$g^{ij} = \delta^{ab} X_a^i X_b^j. \tag{2.3}$$

Using (2.2) and (2.3), it follows that the moving co-frame $\{\omega^1, \omega^2, \omega^3\}$ is orthonormal with respect to the (contravariant) metric g^{ij}, and the moving frame $\{X_1, X_2, X_3\}$ is orthonormal with respect to the (covariant) metric $g_{ij} = \delta_{ab} \omega_i^a \omega_j^b$. Let

$$\Gamma^i_{jk} = \frac{1}{2} g^{ih} \left(\frac{\partial g_{kh}}{\partial x^j} + \frac{\partial g_{jh}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^h} \right)$$

be the components of the *Levi-Civita connection* determined by the metric g_{ij} on $\mathbb{D}$, let $\omega_{j,k}$ be the covariant derivative of ω_i[1], and let

$$\Gamma^{*i}_{jk} = \Gamma^i_{jk} + \Omega^i_{jk}, \;\; \Omega^i_{jk} = \frac{\omega^i}{2} (\omega_{j,k} + \omega_{k,j}),$$

be the components of the *induced (horizontal) linear connection* on the distribution $\mathcal{D}$. The auto-parallelism of the two connections is subject to the following

[1]We shall further assume that ω is normalized, i.e., $g^{ij} \omega_i \omega_j = 1$

Theorem 2.2. *The auto-parallelism of a vector field in the distribution, with respect to the connection* Γ^i_{jk}*, implies the auto-parallelism with respect to the connection* ${\Gamma^*}^i_{jk}$.

Proof. We have to prove that the auto-parallelism of a vector field in the distribution, regarded as a vector field on the Riemannian manifold $(\mathbb{D}, g_{ij})$, implies its horizontal auto-parallelism. A vector field $X = (X^i)$ on the Riemannian manifold $(\mathbb{D}, g_{ij})$ is *autoparallel with respect to the Levi-Civita connection* (or *geodesic*), if $\nabla_X X = 0$, or

$$X^i_{,j} X^j \equiv \frac{\partial X^i}{\partial x^j} X^j + \Gamma^i_{jk} X^j X^k = 0. \tag{2.4}$$

Assume that the vector field $X = (X^i)$ is in the distribution $\mathcal{D}$, i.e, $\omega_j X^j = 0$. It is *horizontally autoparallel* if $\nabla^*_X X = 0$ or, taking into account (2.4),

$$\frac{\partial X^i}{\partial x^j} X^j + {\Gamma^*}^i_{jk} X^j X^k = 0 \quad \Leftrightarrow \quad X^i_{,j} X^j + \omega^i \omega_{j,k} X^j X^k = 0.$$

On the other hand, the condition $\omega_j X^j = 0$ implies

$$\omega_{j,k} X^j + \omega_j X^j_{,k} = 0 \quad \Rightarrow \quad \omega_{j,k} X^j X^k + \omega_j X^j_{,k} X^k = 0.$$

The horizontal auto-parallelism condition can alternatively be written as

$$(\delta^i_j - \omega^i \omega_j) X^j_{,k} X^k = 0 \quad \Leftrightarrow \quad (g_{ij} - \omega_i \omega_j) X^j_{,k} X^k = 0$$

and hence the statement holds true. □

Corollary 2.3. *Suppose the vector field* $X_* = X_1 \times X_2$ *is a Killing vector field on* $(\mathbb{D}, g_{ij})$*. Then the horizontal auto-parallelism on the distribution defined by the associated* 1*-from* ω *is the auto-parallelism on the Riemannian manifold* $(\mathbb{D}, g_{ij})$.

Proof. The relations $\omega_{j,k} + \omega_{k,j} = 0$, straightforward infer ${\Gamma^*}^i_{jk} = \Gamma^i_{jk}$. □

Open problems. 1. Being considered a vector field X, there exist an infinity of connections Γ^i_{jk} such that X be autoparallel. Which of them are produced by a metric g_{ij}?

2. Being given a vector field X, do there exist two linearly independent vector fields X_1, X_2, with $X \in \text{Span}\{X_1, X_2\}$, such that X be a geodesic vector field[2] with respect to the horizontal connection ${\Gamma^*}^i_{jk}$?

3. Optimal control on distributions

In time, a battery of geometrical methods has been developed to address from a new perspective some old problems in control theory [1]-[16], [29], [31]. We conduct our research work in this direction being convinced that a lot of important open problems are on the crossroads of Differential Geometry, Dynamical Systems, and Optimal Control Theory (see also, [17]-[24]).

3.1. Infinitesimal deformations and adjointness

The distribution $\mathcal{D}$ can be described in terms of smooth vector fields (or generators),

$$\mathcal{D} = \text{span}\{X_\alpha(x) \,|\, \omega_i(x) X^i_\alpha(x) = 0,\, \alpha = 1, 2,\, X_1 \times X_2 \neq 0\}, \tag{3.1}$$

where $\omega_i = \delta_{ij} X^j_*$, $X_* = X_1 \times X_2$. Any vector field $X \in \mathcal{D}$ can be written in the form $X(x) = u^\alpha(x) X_\alpha(x)$.

Let $x(t)$ be a curve solution of the differential system $\dot{x}(t) = X(x(t))$ or

$$\dot{x}(t) = u^\alpha(x(t)) X_\alpha(x(t)).$$

[2]Here X_α are subject to the three parallelism conditions, ω satisfies the algebraic condition $g^{ij}\omega_i\omega_j = 1$, while the number of unknowns (the components of X_α) is six.

Let $x(t;\epsilon)$ be a differentiable variation of $x(t)$, i.e.,

$$\dot{x}(t;\epsilon) = u^\alpha(x(t;\epsilon))X_\alpha(x(t;\epsilon)),\ x(t;0) = x(t).$$

Denoting $y^i(t) = \dfrac{\partial x^i}{\partial \epsilon}(t;0)$, we find both the *single-time infinitesimal deformation system*

$$\dot{y}^j(t) = A^j_k(x(t))y^k(t) \tag{3.2}$$

and the *single-time adjoint (dual) system*

$$\dot{p}_k(t) = -A^j_k(x(t))p_j(t), \tag{3.3}$$

where $A^j_k = \frac{\partial u^\alpha}{\partial x^k}X^j_\alpha + u^\alpha \frac{\partial X^j_\alpha}{\partial x^k}$, with all the factors depending on $x(t)$. The solution $p = (p_k)$ of (3.3) is called the *costate vector*. The foregoing PDE systems (3.2) and (3.3) are *adjoint* (*dual*), in the sense of *constant inner product of solutions*, i.e., the scalar product $p_k\, y^k$ is a first integral.

3.2. The single-time maximum principle

We further consider the 2-dimensional distribution $\mathcal{D}$ on $\mathbb{D}$, determined by $\{X_1, X_2\}$ as in (3.1). Let $x(t)$, $t \in I = [t_0, t_1]$ be an integral curve of the driftless control system

$$dx(t) = u^\alpha(x(t))X_\alpha(x(t))dt,\ \alpha = 1, 2.$$

An *optimal control problem* consists of maximizing the functional

$$I(u(\cdot)) = \int_{t_0}^{t_1} L(t, x(t), u(t))dt + h(x(t_1)) \tag{3.4}$$

subject to the constraints

$$dx(t) = u^\alpha(x(t))X_\alpha(x(t))dt \ \text{ a.e.}, \quad t \in I = [t_0, t_1],\ x(t_0) = x_0. \tag{3.5}$$

We hereby assume that $L : I \times A \times U \to \mathbb{R}$ is a C^2 function, $h : \mathbb{D} \to \mathbb{R}$ is a C^1 function, $X_{1,2} \in \mathcal{X}(\mathbb{D})$ are C^2 vector fields, where $A \subset \mathbb{D} \subset \mathbb{R}^3$ is a bounded and closed subset which contains the trajectory of the controlled system for $t \in I$, and x_0 and x_1 are the initial and final states of the trajectory $x(t)$ in the controlled system. Also, U is a bounded and closed subset of $\mathbb{R}^2$, in which the control functions u^α take values, and the mapping $u = (u^1, u^2)$ (called *admissible*) is piecewise smooth or piecewise analytic; the space $\mathcal{U}$ of all such maps forms *the set of admissible controls.*

We further determine first order necessary conditions for an optimal pair (x, u). The *infinitesimal (Pfaff) deformation equation* of the constraint $dx(t) = u^\alpha(x(t))X_\alpha(x(t))dt$ is the system (3.2), and the *adjoint Pfaff equation* is the system (3.3).

The control variables may be *open-loop* - i.e., of the form $u^\alpha(t)$, directly depending on the time variable t, or *closed-loop, feedback* - i.e., of the form $u^\alpha(x(t))$, depending on t by means of the state $x(t)$. In both cases, the following result holds true:

Theorem 3.1 (Single-time maximum principle)**.** *Consider the Lagrangian* 1*-form*

$$\begin{aligned}\mathcal{L}(t, x(t), u(x(t)), p(t)) = \ & L(t, x(t), u(x(t)))dt \\ & + p_i(t)\,[u^\alpha(x(t))X^i_\alpha(x(t))dt - dx^i(t)],\end{aligned}$$

and the associated Hamiltonian 1*-form*

$$\mathcal{H} = [L(t, x(t), u(x(t))) + p_i(t)u^\alpha(x(t))X^i_\alpha(x(t))]dt.$$

Assume that the problem of maximizing the functional (3.4) *constrained by* (3.5) *has an interior optimal solution* $\hat{u}(t)$*, which determines the optimal evolution* $x(t)$*. Then along* $x(t)$ *there exists a costate vector* $p(t) = (p_i(t))$*, such that*

$$dx^i = \frac{\partial \mathcal{H}}{\partial p_i},\ x(t_0) = x_0. \tag{3.6}$$

Moreover, the 1-form $p(t)$ *is the unique solution of the following Pfaff adjoint system*

$$dp_i = -\frac{\partial \mathcal{H}}{\partial x^i},\ p(t_0) = p(t_1) = 0 \tag{3.7}$$

and the following critical point conditions are identically satisfied

$$\mathcal{H}_{u^\alpha}\left(t, x(t), u(t), p(t)\right) = 0,\ \alpha = 1, 2. \tag{3.8}$$

4. Optimal control via nonholonomic geodesics

Let $X_\alpha,\ \alpha = 1, 2$ be two orthonormal vector fields which describe a distribution on the Riemannian manifold $(\mathbb{R}^3, g_{ij})$. Informally, sub-Riemannian geometry is a type of geometry in which the trajectories evolve tangent to a horizontal plane inside the tangent plane only. The main theme of this subject is the study of geodesics which arise in such a geometry.

4.1. Open-loop control variables

A solution of the system $\dot{x}(t) = u^\alpha(t) X_\alpha(x(t))$ is a horizontal curve. Since the kinetic energy has the form

$$\frac{1}{2} g_{ij}(x(t)) \dot{x}^i(t) \dot{x}^j(t) = \frac{1}{2} \delta_{\alpha\beta} u^\alpha(t) u^\beta(t),$$

the optimal control problem of nonholonomic geodesics can be written as a **non-linear-quadratic regulator problem**,

$$\min_{u(\cdot)} J(u(\cdot)) = \frac{1}{2} \int_{t_0}^{t_1} \delta_{\alpha\beta} u^\alpha(t) u^\beta(t)\, dt$$

subject to the subject to the *polydynamical driftless control system*:

$$\dot{x}(t) = u^\alpha(t) X_\alpha(x(t)),\ u(t) = (u^\alpha(t)),\ t \in [t_0, t_1];\ x(t_0) = x_0.$$

We introduce the nonholonomic dual vector $P = (P_a)$, $a = 1, 2, 3$ of the dual vector $p = (p_i)$, $i = 1, 2, 3$ as $P_a(t, x) = p_i(t) X^i_a(x)$. To solve this problem we apply the maximum principle for the Hamiltonian

$$H(x, p, u) := -\frac{1}{2}\, \delta_{\alpha\beta} u^\alpha u^\beta + P_\alpha u^\alpha,\ \alpha, \beta = 1, 2.$$

Theorem 4.1. *A horizontal vector field* X*, i.e., a vector field in a distribution* $\{X_1, X_2\}$*, is a geodesic vector field if and only if there exists a costate vector function* $p(t) = (p_i(t))$ *such that the adjoint system*

$$\frac{dp_j}{dt}(t) = \delta^{\alpha\beta}\, P_\alpha(t, x(t))\, \frac{\partial P_\beta}{\partial x^j}(t, x(t)),\ p(t_0) = p(t_1) = 0 \tag{4.1}$$

the critical point condition

$$u^\alpha = P_\alpha, \tag{4.2}$$

and the following initial dynamics are satisfied:

$$\dot{x}^i(t) = \delta^{\alpha\beta} P_\alpha(t, x(t)) X^i_\beta(x(t)),\ x(t_0) = x_0. \tag{4.3}$$

Proof. The adjoint dynamics

$$\frac{dp_j}{dt} = -\frac{\partial H}{\partial x^j} = -u^\alpha \frac{\partial P_\alpha}{\partial x^j}$$

is changed into nonholonomic adjoint dynamics

$$\frac{dP_\alpha}{dt} = \frac{dp_i}{dt} X^i_\alpha + p_i \frac{\partial X^i_\alpha}{\partial x^j} \dot{x}^j = -u^\beta \frac{\partial P_\beta}{\partial x^j} X^j_\alpha + p_i \frac{\partial X^i_\alpha}{\partial x^j} \dot{x}^j$$

$$= -u^\beta \frac{\partial P_\beta}{\partial x^j} X^j_\alpha + \frac{\partial P_\alpha}{\partial x^j} u^\beta X^j_\beta = u^\beta \left(\frac{\partial P_\alpha}{\partial x^j} X^j_\beta - \frac{\partial P_\beta}{\partial x^j} X^j_\alpha \right) = u^\beta p([X_\beta, X_\alpha]).$$

We have to maximize the Hamiltonian $H(x, p, u)$ with respect to the control variable u. Since the critical points u of H are solutions of the algebraic system

$$\frac{\partial H}{\partial u^\alpha} = -u^\alpha + P_\alpha = 0,$$

the adjoint ODE system rewrites

$$\frac{dp_j}{dt}(t) = -\delta^{\alpha\beta} P_\alpha(t, x(t)) \frac{\partial P_\beta}{\partial x^j}(t, x(t)),$$

and the initial dynamics becomes $\dot{x}^i(t) = \delta^{\alpha\beta} P_\alpha(t, x(t)) X^i_\beta(x(t))$. Looking for a solution of the form $p(t) = K(t)x(t)$ or $p_i(t) = K_{ij}(t)x^j(t)$, we find the feed-back control law (the Kalman vector) $u^\alpha = K_{ij} X^i_\alpha x^j$. Then, denoting $K_{ij} X^i_\alpha = K_{\alpha j}$ and using $P_\alpha(t, x(t)) = p_i(t) X^i_\alpha(x(t))$, we find the feed-back control law $u^\alpha(t) = K_{\alpha j}(t) x^j(t)$ and the initial ODE system can be written

$$\dot{x}^i(t) = \delta^{\alpha\beta} X^i_\alpha(x(t)) K_{\beta j}(t) x^j(t),$$

whence the claim follows. □

Theorem 4.2 (optimal control is unitary, see also [16])**.** *The optimal controls* $u^\alpha = p_i X^i_\alpha$ *satisfy the ODE system* $\dot{u}^\alpha = p_i [X_\beta, X_\alpha]^i u^\beta$. *Consequently, the control* $u = (u^\alpha)$ *is unitary, i.e.,* $\frac{1}{2}\delta_{\alpha\beta} u^\alpha u^\beta$ *is a first integral.*

Proof. Direct differentiation and use the adjoint dynamics and the initial dynamics. The unitarity of the controls follows from the fact that the evolution $\dot{u}^\alpha$ of u^α is described by a linear ODE system with skew-symmetric right-hand side. □

Theorem 4.3. *The distribution $\mathcal{D}$ is controllable (accessible) by geodesics (shortest paths).*

Proof. Since the vector fields X_α, $\alpha = 1, 2$ are linearly independent, the distribution $\mathcal{D}$ is bracket generating. The Chow-Sussmann Theorem [16] shows that any two sufficiently close points can be joined by a minimizing geodesic, i.e., a solution of the foregoing optimal problem. □

4.2. Closed-loop control variables

A solution of the system $\dot{x}(t) = u^\alpha(x(t)) X_\alpha(x(t))$ is a horizontal curve. Since the kinetic energy can be written

$$\frac{1}{2} g_{ij}(x(t)) \dot{x}^i(t) \dot{x}^j(t) = \frac{1}{2} \delta_{\alpha\beta} u^\alpha(x(t)) u^\beta(x(t)),$$

the optimal control problem of nonholonomic geodesics can be written as a **non-linear-quadratic regulator problem**,

$$\min_{u(\cdot)} J(u(\cdot)) = \frac{1}{2} \int_{t_0}^{t_1} \delta_{\alpha\beta} u^\alpha(x(t)) u^\beta(x(t))\, dt$$

subject to the polydynamical driftless control system:

$$\dot{x}(t) = u^\alpha(x(t)) X_\alpha(x(t)),\ u(x(t)) = (u^\alpha(x(t))), \quad t \in [t_0, t_1];\ x(t_0) = x_0.$$

Theorem 4.4. *A trajectory $x(t)$ of the vector field $X = (X^j)$ is a geodesic in the distribution X_1, X_2 if and only if there exists a costate vector function $p(t) = (p_i(t))$ such that the adjoint system*

$$\frac{dp_k}{dt}(t) = -p_j(t)\frac{\partial X^j}{\partial x^k}(x(t)),\ p(t_0) = p(t_1) = 0$$

and the following decomposition holds:

$$X^j(x(t)) = \left(p_i(t)X_1^i(x(t))\right) X_1^j(x(t)) + \left(p_i(t)X_2^i(x(t))\right) X_2^j(x(t)).$$

Proof. We introduce the nonholonomic dual vector $P = (P_a)$, $a = 1, 2, 3$ of the dual vector $p = (p_i)$, $i = 1, 2, 3$ as $P_a(t, x) = p_i(t)X_a^i(x)$. To apply the maximum principle, we use the Hamiltonian

$$H(x, p, u) := -\frac{1}{2}\,\delta_{\alpha\beta}u^\alpha u^\beta + P_\alpha u^\alpha,\ \ \alpha, \beta = 1, 2.$$

We have to maximize the Hamiltonian $H(x, p, u)$ with respect to the control variable u. The critical points u of H are solutions of the algebraic system

$$\frac{\partial H}{\partial u^\alpha} = -u^\alpha + P_\alpha = 0.$$

It follows the adjoint dynamics

$$\frac{dp_k}{dt} = -\frac{\partial H}{\partial x^k} = -\frac{1}{2}\,p_i(t)p_j(t)\frac{\partial}{\partial x^k}(\delta^{\alpha\beta}X_\alpha^i(x(t))X_\beta^j(x(t)))$$

which can be written as

$$\frac{dp_k}{dt}(t) = -p_j(t)\frac{\partial X^j}{\partial x^k}(x(t)),\ p(t_0) = p(t_1) = 0.$$

Then the initial dynamics becomes

$$\dot{x}^j(t) = p_i(t)\delta^{\alpha\beta}X_\alpha^i(x(t))X_\beta^j(x(t))$$

or, $\dot{x}^j(t) = \left(p_i(t)X_1^i(x(t))\right) X_1^j(x(t)) + \left(p_i(t)X_2^i(x(t))\right) X_2^j(x(t)), x(t_0) = x_0$. The right hand member of this ODE system is just the orthogonal projection of the adjoint vector $p(t)$ on the distribution.

Conversely, let X be a vector field on $\mathbb{R}^3$ in the distribution generated by two linearly independent vector fields X_1 and X_2. The decomposition $X(x) = u^1(x)X_1(x) + u^2X_2(x)$ is unique, since $u^1(x) = \langle X_1(x), X(x)\rangle$ and $u^2(x) = \langle X_2(x), X(x)\rangle$, where the scalar product is given by the metric g_{ij}. □

The existence and uniqueness Theorem shows that the previous initial and terminal values problems have solutions. Therefore, our initial problem is solved via maximum principle theory.

Corollary 4.5. *The vector field $X = (X^j)$ is a horizontally autoparallel in the distribution X_1, X_2 if and only if*

$$X^j(x) - \left(p_iX_1^i(x))\right) X_1^j(x) - \left(p_iX_2^i(x)\right) X_2^j(x) = 0$$

constrained by the system made from the original dynamical system and the adjoint system.

Corollary 4.6. *(i) If the costate vector $p = (p_i)$ is of the form*

$$p_i(t) = g_{ij}(x(t))X^j(x(t))$$

and if the vector field X is unitary, i.e., $g_{ij}X^iX^j = 1$, then X is autoparallel (geodesic) with respect to the connection Γ^i_{jk}.

(ii) If the vector field X is autoparallel with respect to the connection Γ^i_{jk} and unitary with respect to g_{ij}, then $p_i(t) = g_{ij}(x(t))X^j(x(t))$ satisfies the adjoint system.

Proof. (i) Assume that the costate vector has the form

$$p_i(t) = g_{ij}(x(t))X^j(x(t)).$$

Replacing it in the dual system, we obtain

$$g_{ki}\frac{\partial X^i}{\partial x^j}X^j + g_{ij}X^i\frac{\partial X^j}{\partial x^k} + \frac{\partial g_{kj}}{\partial x^i}X^iX^j = 0.$$

We add the hypotheses $g_{ij}X^iX^j = 1$. Taking the partial derivative with respect to x^k, we find $g_{ij}X^i\frac{\partial X^j}{\partial x^k} = -\frac{1}{2}\frac{\partial g_{ij}}{\partial x^k}X^iX^j$. Consequently,

$$g_{ki}\frac{\partial X^i}{\partial x^j}X^j + \frac{\partial g_{kj}}{\partial x^i}X^iX^j - \frac{1}{2}\frac{\partial g_{ij}}{\partial x^k}X^iX^j = 0,$$

i.e., $\frac{\partial X^i}{\partial x^j}X^j + \Gamma^i_{jk}X^jX^k = 0$, which infers that the vector field X is autoparallel with respect to the connection Γ^i_{jk}.

(ii) The second statement is straightforward. □

Remark 4.7. The vector field $X(x) = u^\alpha(x)X_\alpha(x)$ is unitary with respect to the metric g_{ij} if and only if $\delta_{\alpha\beta}u^\alpha(x)u^\beta(x) = 1$. Consequently, for a unitary vector field X, the auto-parallelism of X with respect to Γ^i_{jk} is equivalent to the adjointness of p.

Corollary 4.8. *A given vector field X, in a distribution $\mathcal{D} = span\{X_1, X_2\}$, is a geodesic vector field (i.e., auto-parallel with respect to the horizontal connection ${\Gamma^*}^i_{jk}$), if and only if*

$$(g_{ij}X^iX^j)_{,k} = \omega_k(\omega_{i,j} + \omega_{j,i})X^iX^j.$$

Proof. Replacing $p_i(t) = g_{ij}(x(t))X^j(x(t))$ in the dual system, we obtain

$$g_{ki}\frac{\partial X^i}{\partial x^j}X^j + g_{ij}X^i\frac{\partial X^j}{\partial x^k} + \frac{\partial g_{kj}}{\partial x^i}X^iX^j = 0.$$

On the other hand, the auto-parallelism with respect to the horizontal connection ${\Gamma^*}^i_{jk}$ means $\frac{\partial X^i}{\partial x^j}X^j + {\Gamma^*}^i_{jk}X^jX^k = 0$, or,

$$g_{ki}\frac{\partial X^i}{\partial x^j}X^j + \frac{\partial g_{kj}}{\partial x^i}X^iX^j - \frac{1}{2}\frac{\partial g_{ij}}{\partial x^k}X^iX^j + \frac{1}{2}\omega_k(\omega_{i,j} + \omega_{j,i})X^iX^j = 0,$$

Eliminating the common terms, we find

$$-g_{ij}X^i\frac{\partial X^j}{\partial x^k} - \frac{1}{2}\frac{\partial g_{ij}}{\partial x^k}X^iX^j + \frac{1}{2}\omega_k(\omega_{i,j} + \omega_{j,i})X^iX^j = 0$$

or $(g_{ij}X^iX^j)_{,k} = 2\omega_k\omega_{i,j}X^iX^j$, which completes the proof. □

Corollary 4.9. *A unit vector field X, in a distribution $\mathcal{D} = span\{X_1, X_2\}$, is a geodesic vector field (i.e., auto-parallel with respect to the horizontal connection ${\Gamma^*}^i_{jk}$), if and only if $\omega_{i,j}X^iX^j = 0$.*

Corollary 4.10. *If a vector field X belongs to the distribution $\mathcal{D}$ and it is autoparallel with respect to the connection Γ^i_{jk} , then X is autoparallel with respect to the horizontal connection ${\Gamma^*}^i_{jk}$.*

Proof. Suppose $\nabla_X X = 0$ or $X^i_{,j}X^j = 0$. Since $\omega_iX^i = 0$, we obtain $\omega_{i,j}X^i + \omega_iX^i_{,j} = 0$. Contracting by X^j, we find $\omega_{i,j}X^iX^j = 0$. □

5. Applications: the nonholonomic analysis of the Lorenz dynamical system

5.1. Special Lorentz control distributions

The autonomous differential Lorenz system

$$\dot{x} = \sigma(y - x),\ \dot{y} = -xz + rx - y,\ \dot{z} = xy - bz,$$

where σ, r, b are real parameters, is the mathematical model for the *dynamic turbulence of a fluid*. More precisely, the fluid evolves between two parallel plates which have distinct temperature levels. The evolution of the fluid is described by the Navier-Stokes equation. Developing in Fourier series in terms of the spatial coordinates and making the coefficients equal to zero, we get the previous differential system, in which x is proportional to the convective motion, y is proportional to the temperature, z is proportional to the drift from linearity of the vertical temperature profile. Generally, we impose the conditions $\sigma > 0, r > 0, b > 0$, but usually $\sigma = 10$, $b = \frac{8}{3}$, and r varies. Let

$$L = \sigma(y - x)\frac{\partial}{\partial x} + (-xz + rx - y)\frac{\partial}{\partial y} + (xy - bz)\frac{\partial}{\partial z} \tag{5.1}$$

be the Lorenz vector field. Since $\operatorname{div} L = -(1 + \sigma + b) < 0$, the Lorenz flow is contractive.

We remark first that the Lorenz system is invariant relative to the symmetry $(x, y, z) \to (-x, -y, z)$. The $z-$axis is invariant, since $x(t) = 0$, $y(t) = 0$, $z(t) = z_0\, e^{-bt}$, $t \in \mathbb{R}$ is a solution of the system.

The Lorenz system exhibits chaotic behavior for $r = 28$ but displays knotted periodic orbits for other values of r. For example, with $r = 99.96$, this becomes a $T(3, 2)$ torus knot. A saddle-node bifurcation occurs in the case when $b(r - 1) = 0$.

When $\sigma \neq 0$ and $b(r - 1) \geq 0$, the Lorenz flow has three equilibrium points. The equilibrium point $(0, 0, 0)$ corresponds to no convection, and the equilibrium points $(\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r - 1)$ correspond to steady convections. This pair is stable only if

$$r < \sigma\frac{\sigma + b + 3}{\sigma - b - 1}, \tag{5.2}$$

which can hold only for positive r if $\sigma > b + 1$.

When $r = 28$, $\sigma = 10$, and $b = \frac{8}{3}$, the Lorenz system has chaotic solutions (but not all solutions are chaotic). In this case $\sigma > b + 1$ and the right hand side in (5.2) is 470/19, less than r. The Lorentz vector field (5.1) can be decomposed as $L = u^1 X_1 + U_2 X_2$, $u^1 = x, u^2 = 1$, with

$$X_1 = (-\sigma, -z, y),\quad X_2 = (\sigma y, rx - y, -bz).$$

The domain $\mathbb{D}$ on which the 2-dimensional distribution $\mathcal{D} = Span\{X_1, X_2\}$ generated by these vector fields is of maximal rank, is

$$\mathbb{D} = \mathbb{R}^3 \setminus \alpha(\mathbb{R}),\quad \alpha(t) = \left(\frac{8t + 3t^3}{224}, t, \frac{3t^2}{8}\right). \tag{5.3}$$

We note that on $\alpha(\mathbb{R})$ the Riemannian structure degenerates, and the dimension of $\mathcal{D}$ collapses to one. The set of chaotic solutions make up the Lorenz attractor, a strange attractor and a fractal of Hausdorff dimension between 2 and 3. Grassberger (1983) has estimated the Hausdorff dimension to be 2.06 ± 0.01 and the correlation dimension to be 2.05 ± 0.01.

5.2. Lorenz distributions - 1

5.2.1. *Lorenz* 1*-forms.* The Lorenz vector field

$$L = \sigma(y - x)\frac{\partial}{\partial x} + (-xz + rx - y)\frac{\partial}{\partial y} - (xy + bz)\frac{\partial}{\partial z}$$

is a linear combination of the form

$$L(x,y,z) = xX_1(x,y,z) + X_2(x,y,z),$$

where

$$X_1 = -\sigma\tfrac{\partial}{\partial x} - z\tfrac{\partial}{\partial y} + y\tfrac{\partial}{\partial z}, \quad X_2 = \sigma y\tfrac{\partial}{\partial x} + (rx-y)\tfrac{\partial}{\partial y} - bz\tfrac{\partial}{\partial z}.$$

Then

$$X_1 \times X_2 = (bz^2 - rxy + y^2)\frac{\partial}{\partial x} + \sigma(y^2 - bz)\frac{\partial}{\partial y} + \sigma(-rx + y + yz)\frac{\partial}{\partial z}.$$

The cross product $X_1 \times X_2$ generates the *Lorenz* 1-*form*

$$\begin{aligned}\omega &= \omega_1(x,y,z)dx + \omega_2(x,y,z)dy + \omega_3(x,y,z)dz\\ &= (bz^2 - rxy + y^2)dx + \sigma(y^2 - bz)dy + \sigma(-rx + y + yz)dz.\end{aligned}$$

This 1-form satisfies $\omega \wedge d\omega \neq 0$. Consequently, the Lorenz distribution

$$\mathcal{D}_1 = \text{Ker}\,(\omega) = \text{span}\{X_\alpha(x,y,z)|\,\omega_i(x,y,z)X^i_\alpha(x,y,z) = 0,\ \alpha = 1,2\}$$

is nonholonomic.

5.**2**.2. *Lorenz moving frames.* To the vector fields X_1, X_2 we add the vector field

$$X_3 = \frac{1}{\sigma(bz-y^2)}\frac{\partial}{\partial y}.$$

We identify these vector fields X^i_a with the columns of the matrix

$$\left(X^i_a\right) = \begin{pmatrix} -\sigma & \sigma y & 0\\ -z & rx-y & \frac{1}{\sigma(bz-y^2)}\\ y & -bz & 0\end{pmatrix}.$$

These are linearly independent vector fields on the subset $D \subset \mathbb{R}^3$, defined by the condition $\det\left(X^i_a\right) = bz - y^2 \neq 0$, determining a *Lorenz moving frame.* These vector fields are not orthogonal in the Euclidean space $\left(D \subset \mathbb{R}^3, \delta_{ab}\right)$. On the region $D \subset \mathbb{R}^3$, we introduce the inverse matrix

$$(\omega^a_i) = \begin{pmatrix} -\frac{bz}{\sigma(bz-y^2)} & 0 & \frac{-y}{bz-y^2}\\ \frac{-y}{\sigma(bz-y^2)} & 0 & \frac{-1}{bz-y^2}\\ -(bz^2 - rxy + y^2) & \sigma(bz-y^2) & \sigma(rx - y - yz)\end{pmatrix}.$$

The rows of the matrix (ω^a_i) determine the Lorenz 1-forms (*Lorenz moving co-frame*):

$$\begin{aligned}\omega^1 &= -\tfrac{bz}{\sigma(bz-y^2)}\,dx - \tfrac{y}{bz-y^2}\,dz,\ \omega^2 = -\tfrac{y}{\sigma(bz-y^2)}\,dx - \tfrac{1}{bz-y^2}\,dz\\ \omega^3 &= \omega = -(bz^2 - rxy + y^2)dx + \sigma(bz - y^2)dy + \sigma(rx - y - yz)dz.\end{aligned}$$

5.**2**.3. *Riemann-Vrănceanu metrics.* Let $X^i_a(x), x = (x^i),\ i,\ a = \overline{1,3}$ be the Lorenz moving frame on the Euclidean manifold $\left(D \subset \mathbb{R}^3, \delta_{ab}\right)$. Vrănceanu's idea [30], associates to this frame only one Riemannian manifold with a positive definite metric and a moving orthonormal frame $\left(D \subset \mathbb{R}^3, g_{ij}, X^i_a\right)$. The contravariant *Riemann-Vrănceanu metric* g^{ij} is the tensor field given by

$$\delta^{ab}X_a \otimes X_b = \delta^{ab}X^i_a\frac{\partial}{\partial x^i} \otimes X^j_b\frac{\partial}{\partial x^j} = g^{ij}(x)\,\frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}.$$

Its symmetric associated matrix has the explicit form

$$(g^{ij}) = (\delta^{ab}X^i_aX^j_b) = \begin{pmatrix} \sigma^2 + y^2 + z^2 & -\sigma^2 y - z(rx-y) - byz & -\frac{z}{\sigma(bz-y^2)}\\ \cdots & \sigma^2y^2 + (rx-y)^2 + b^2z^2 & \frac{rx-y}{\sigma(bz-y^2)}\\ \cdots & \cdots & \frac{1}{\sigma^2(bz-y^2)^2}\end{pmatrix}.$$

The Lorenz moving co-frame is orthonormal with respect to the contravariant metric g^{ij}, while the Lorenz moving frame is orthonormal with respect to the covariant metric[3] g_{ij}.

5.3. Lorenz distributions - 2

5.3.1. *Lorenz 1-forms.* The Lorenz vector field

$$L = \sigma(y-x)\frac{\partial}{\partial x} + (-xz + rx - y)\frac{\partial}{\partial y} + (xy - bz)\frac{\partial}{\partial z}$$

is a linear combination of the form

$$L(x,y,z) = Y_1(x,y,z) + xY_2(x,y,z),$$

where

$$Y_1 = \sigma(y-x)\tfrac{\partial}{\partial x} + (rx-y)\tfrac{\partial}{\partial y} - bz\tfrac{\partial}{\partial z}, \quad Y_2 = -z\tfrac{\partial}{\partial y} + y\tfrac{\partial}{\partial z}.$$

Then

$$Y_1 \times Y_2 = (-bz^2 + rxy - y^2)\frac{\partial}{\partial x} - \sigma y(y-x)\frac{\partial}{\partial y} + \sigma z(x-y)\frac{\partial}{\partial z}.$$

The cross product $Y_1 \times Y_2$ generates the *Lorenz 1-form*

$$\begin{aligned}\eta &= \eta_1 dx + \eta_2 dy + \eta_3 dz\\ &= (-bz^2 + rxy - y^2)dx - \sigma y(y-x)dy + \sigma z(x-y)dz.\end{aligned}$$

This 1-form satisfies $\eta \wedge d\eta \neq 0$. Consequently, the Lorenz distribution

$$\mathcal{D}_2 = \operatorname{Ker}\eta = \operatorname{span}\{Y_\alpha(x,y,z) \,|\, \eta_i(x,y,z)Y^i_\alpha(x,y,z) = 0,\ \alpha = 1,2\}$$

is nonholonomic.

5.3.2. *Lorenz moving frames.* To the vector fields Y_1, Y_2 we supplement the vector field

$$Y_3 = \frac{1}{\sigma y(x-y)}\frac{\partial}{\partial y}.$$

We identify these vector fields Y^i_a with the columns of the matrix

$$\left(Y^i_a\right) = \begin{pmatrix} -\sigma(x-y) & 0 & 0\\ rx-y & -z & \frac{1}{\sigma y(x-y)}\\ -bz & y & 0\end{pmatrix}.$$

These are linearly independent vector fields on the subset $D \subset \mathbb{R}^3$, defined by the condition $\det\left(Y^i_a\right) = y(x-y) \neq 0$, determining a *Lorenz moving frame.* These vector fields are not orthogonal in the Euclidean space $\left(D \subset \mathbb{R}^3, \delta_{ab}\right)$.

On the region $D \subset \mathbb{R}^3$, we introduce the inverse matrix

$$(\eta^a_i) = \begin{pmatrix} -\frac{1}{\sigma(x-y)} & 0 & 0\\ -\frac{bz}{\sigma y(x-y)} & 0 & \frac{1}{y}\\ -bz^2 + rxy - y^2 & \sigma y(x-y) & \sigma z(x-y)\end{pmatrix}.$$

The lines of the matrix (η^a_i) determine the Lorenz 1-forms (*Lorenz moving co-frame*):

$$\eta^1 = -\frac{1}{\sigma(x-y)}\,dx,\ \eta^2 = -\frac{bz}{\sigma y(x-y)}\,dx + \frac{1}{y}\,dz$$

$$\eta^3 = \eta = (-bz^2 + rxy - y^2)dx + \sigma y(x-y)dy + \sigma z(x-y)dz.$$

[3] In the present case, the components of the covariant metric have a much more complicated form than the contravariant one.

5.3.3. *Riemann-Vrănceanu metrics.* Let Y_a^i, $i, a = \overline{1,3}$ be the Lorenz moving frame on the Euclidean manifold $\left(D \subset \mathbb{R}^3, \delta_{ab}\right)$. Using Vrănceanu's idea [30], we associate to this frame only one Riemannian manifold $\left(D \subset \mathbb{R}^3, g^{ij}, Y_a^i\right)$ (with a positive definite metric and a moving frame), in which the frame is orthonormal. The contravariant *Riemann-Vrănceanu metric* g^{ij} is the tensor field defined by

$$\delta^{ab} Y_a \otimes Y_b = \delta^{ab} Y_a^i \frac{\partial}{\partial x^i} \otimes Y_b^j \frac{\partial}{\partial x^j} = g^{ij}(x) \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j},$$

where

$$(g^{ij}) = (\delta^{ab} Y_a^i Y_b^j)$$

$$= \begin{pmatrix} \sigma^2(x-y)^2 + (rx-y)^2 + b^2 z^2 & -z(rx-y) - byz & \frac{rx-y}{\sigma y(x-y)} \\ -z(rx-y) - byz & z^2 + y^2 & -\frac{z}{\sigma y(x-y)} \\ \frac{rx-y}{\sigma y(x-y)} & \frac{1}{\sigma(x-y)} & \frac{1}{\sigma^2 y^2 (x-y)^2} \end{pmatrix}.$$

The Lorenz moving co-frame is orthonormal with respect to the (contravariant) metric g^{ij}. The Lorenz moving frame is orthonormal with respect to the covariant metric g_{ij}.

5.4. Geodesics on Lorenz distributions

Let $\mathcal{D}_1$ be the Lorenz distribution with the related geometry as in Section 4.2. Let L be the Lorenz vector field (in the Lorenz distribution) and its length $\|L\|_g^2 = 1 + x^2$ with respect to the Riemannian metric g in subsection 4.2.3. The associated unit vector field is $\xi = \frac{1}{\|L\|_g} L$. As any unit vector field, ξ is a complete vector field, i.e., its field lines are defined on $\mathbb{R}$. It is also a horizontal vector field and hence the solutions of the dynamical system $\dot{x}(t) = \xi(x(t))$, including the chaotic orbit, are horizontal curves. We shall further address the question whether these curves are geodesics of a certain Riemannian-Vrănceanu metric structure.

To this aim, we first consider the decomposition

$$\xi = v^1 X_1 + v^2 X_2,\; v^1 = \frac{u^1}{\sqrt{{u^1}^2 + {u^2}^2}},\; v^2 = \frac{u^2}{\sqrt{{u^1}^2 + {u^2}^2}}$$

of the Lorenz unit vector field, where X_1 and X_2 are linearly independent excepting a set of measure zero. According the Theorem 3.4, a normalized Lorenz field line is a geodesic in the distribution $\{X_1, X_2\}$ if and only if

$$p_i(t) X_1^i(x(t)) = v^1(x(t)),\; p_i(t) X_2^i(x(t)) = v^2(x(t)), \tag{4.4}$$

where $x(t), p(t)$ is a solution of the ODE system made by the Lorenz dynamical system and the adjoint system.

According the Corollary 3.9, the field lines of the Lorenz unit vector field ξ are not horizontal geodesics since $\omega_{i,j}\, \xi^i \xi^j \neq 0$. The computations in coordinates being "blocked" by the difficulty of concretely writing the connection components Γ^i_{jk}, we perform them by using the components with respect to the orthonormal frame $\{X_a \mid a = 1, 2, 3\}$. Particularly, the components of the 1-form ω are $\omega_1 = 0, \omega_2 = 0, \omega_3 = 1$ and the Ricci coefficients are $\gamma^a_{bc} = \omega^a(\nabla_{X_b} X_c)$. In this way, we find

$$\omega_{i,j}\, \xi^i \xi^j = \omega_{\alpha,\beta}\, v^\alpha v^\beta = -\gamma^3_{\alpha\beta}\, v^\alpha v^\beta \neq 0,\; \alpha, \beta = 1, 2.$$

Similar statements are true for the Lorenz distribution $\mathcal{D}_2$.

Let us change our point of view. We further consider the decomposition $L = u^1 X_1 + u^2 X_2$ of the Lorenz vector field L, where X_1 and X_2 are linearly independent excepting a set of measure zero. According the Theorem 3.4, a Lorenz field line is a geodesic in the distribution $\{X_1, X_2\}$ if and only if

$$p_i(t) X_1^i(x(t)) = u^1(x(t)),\; p_i(t) X_2^i(x(t)) = u^2(x(t)), \tag{4.4}$$

where $x(t), p(t)$ is a solution of the ODE system made by the Lorenz dynamical system and the adjoint system.

All Lorenz field lines are horizontal geodesics if and only if we can find a decomposition such that

$$L^j(x) - \left(p_i X_1^i(x)\right) X_1^j(x) - \left(p_i X_2^i(x)\right) X_2^j(x) = 0,$$

if $x(t), p(t)$ is a solution of the ODE system made by the Lorenz dynamical system and the adjoint system.

There exists an infinity of decompositions of the Lorenz vector field. Theorem 3.4 suggests a possible procedure to find such a decomposition and its associated geometry, in which the Lorenz trajectories are horizontal geodesics. From (4.4) we find, for instance, $p_1 = p_1(x, u^1, u^2, p_3)$ and $p_2 = p_2(x, u^1, u^2, p_3)$. We introduce them in the adjoint ODEs and we obtain equations determining $u^1(x(t)), u^2(x(t)), p_3(t)$.

6. **Conclusions and further research**

In the framework of sub-Riemannian geometry, we presented several associated optimal control problems and developed an original general nonholonomic setting for the Lorenz system. In this framework, we use the Pontryaguin Maximum Principle, which both states necessary conditions for optimal trajectories, and yields explicit expressions for the Hamiltonian system in terms of the optimal controls.

Section 1 and Subsection 3.2 describe a possible procedure to find a decomposition and its associated geometry, in which all Lorenz trajectories are horizontal geodesics.

In this respect, physics, mechanics, robotics, automation, economics, chemistry and biology provide a remarkable source of problems, which can be reformulated within the foregoing formalism. The applications of the described geometric techniques can eventually unveil properties for the solutions of such problems, especially regarding their control design and stabilization.

Acknowledgements. The work was partially supported by University Politehnica of Bucharest, and by the Academy of Romanian Scientists.

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