SUFFICIENT OPTIMALITY CONDITIONS AND DUALITY FOR NONSMOOTH INTERVAL-VALUED OPTIMIZATION PROBLEMS VIA L-INVEX-INFINE FUNCTIONS

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This work focuses on an interval-valued optimization problem with both inequality and equality constraints. Utilizing the concept of LU optimal solution, sufficient optimality conditions are established involving L-invex-infine functions, defined with reference to the limiting/Mordukhovich subdifferential. Furthermore, appropriate duality results are presented for a Wolfe type dual problem.

Keywords: Limiting subdifferential, L-invex-infine function, interval-valued programming, LU-optimal, sufficiency, duality.

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1. Introduction

An interval-valued optimization problem is introduced to tackle uncertainty in the optimization problems. In recent years, interval-valued optimization has become an active field of research in applied mathematics. This is due to the fact that the theory about the parameters of a physical world system is uncertain in many cases and these parameters cannot evaluate with accuracy. The applications of interval based models are in financial, corporate planning healthcare, production planning and hospital planning and other diverse fields. For basic aspects that relate to theory and a large area of applications of interval-valued optimization programming, we refer the reader to the books [5, 6, 11, 12, 17] and to some recent papers [1, 2, 7, 8, 9, 13, 15, 18, 19].

Nonsmooth calculus refers to differential calculus in the nonappearance of differentiability and can be considered as a sub field of the nonlinear analysis has developed rapidly over the past decades. It is to be observed that, not all experimental problems, formulated as an interval-valued optimization problems, meet the requirements of differentiability. Since many experimental problems come across in management science, economics and engineering can be formulated only by nonsmooth functions and modelled as an interval-valued optimization problem. Therefore, the field of nonsmooth interval-valued optimization problems, in which every involved function is locally Lipchitz, has attracted the researchers to discuss the optimality results, see, for example [7, 9, 18, 19] and references therein.

Li et al. [9] studied the relation between interval-valued invex and interval-valued weakly invex functions. Further, they derived sufficient optimality conditions for an interval-valued optimization problem under proposed invexity assumptions. Chuong [3] proposed the notion of L-invex-infine functions on the lines of Sach et al. [16] and also discussed that

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the class of L-invex-infine functions is significantly higher than the one of invex-infine functions. Moreover, Choung [3] applied these functions to examine the Gordan type alternative theorem, and Geoffrion's properly efficient solutions of a multiple objective programming problem with reference to the Lagrange multiplier rules and limiting/Mordukhovich subdifferential of real-valued functions.

In this study, our attention is focused to provide the sufficient optimality conditions for LU optimal solution of an interval-valued optimization problem by taking L-invex-infine functions defined with reference to the limiting subdifferential. Moreover, we propose the Wolfe type dual problem, and examine duality relations under L-invexity-infineness.

We now moving forward to discuss the contents of this paper. Section 2 consists of some basic definitions and background material. Section 3 represents the necessary and sufficient optimality conditions for an interval-valued optimization problem under L-invex-infine functions defined with reference to the limiting subdifferential. In Section 4, Wolfe type dual is presented and appropriate duality results are also discussed. Finally, the paper is concluded in Section 5.

2. **Preliminaries**

The aim of this section is to provide some basic concepts and auxiliary results that will be used often throughout the paper. We denote by I the class of all bounded closed intervals in R. Suppose $I_1 = [\alpha^L, \alpha^U], I_2 = [\beta^L, \beta^U] \in I$, then we write

(i) $I_1 + I_2 = \{\alpha + \beta \mid \alpha \in I_1 \text{ and } \beta \in I_2\} = [\alpha^L + \beta^L, \alpha^U + \beta^U]$,

(ii) $-I_1 = \{-\alpha \mid \alpha \in I_1\} = [-\alpha^U, -\alpha^L]$,

(iii) $I_1 - I_2 = I_1 + (-I_2) = [\alpha^L - \beta^U, \alpha^U - \beta^L]$,

(iv) $k + I_1 = \{k + \alpha \mid \alpha \in I_1\} = [k + \alpha^L, k + \alpha^U]$,

(v) $kI_1 = \{k\alpha \mid \alpha \in I_1\} = \begin{cases} [k\alpha^L, k\alpha^U], & \text{if } k \geq 0, \\ [k\alpha^U, k\alpha^L], & \text{if } k < 0, \end{cases}$

where k is a real number.

For $I_1 = [\alpha^L, \alpha^U]$ and $I_2 = [\beta^L, \beta^U]$, the partial ordering $\leq_{LU}$ on I is defined as $I_1 \leq_{LU} I_2$ if and only if $\alpha^L \leq \beta^L$ and $\alpha^U \leq \beta^U$. Moreover, we write $I_1 <_{LU} I_2$ if and only if $I_1 \leq_{LU} I_2$ and $I_1 \neq I_2$. In the other words, $I_1 <_{LU} I_2$ if and only if

$$\begin{array}{lll} & \alpha^L < \beta^L, & \alpha^U < \beta^U, \\ \text{or,} & \alpha^L \leq \beta^L, & \alpha^U < \beta^U, \\ \text{or,} & \alpha^L < \beta^L, & \alpha^U \leq \beta^U. \end{array}$$

Let R^n be the n-dimensional Euclidean space and R^n_+ be its non-negative orthant. Unless otherwise specified, all the spaces considered in the paper are Banach whose norms are always denoted by $\|.\|$ and T^* is dual of a given space T. The canonical pairing between T and its dual T^* is denoted by $\langle ., . \rangle$, and $\mathbb{S}^\circ = \{a^* \in T^* \mid \langle a^*, a \rangle \leq 0, \ \forall a \in \mathbb{S}\}$ is the polar cone of a set $\mathbb{S} \subset T$. As usual, the symbol cl$\mathbb{S}$ stand for the closure of $\mathbb{S}$ and $\mathbb{N}$ denotes the set of all natural numbers. Now, we recall the following Definitions 2.1-2.5 from Mordukhovich [10].

Definition 2.1. Let $\ell : T \rightrightarrows T^*$ be a multifunction. Then the sequential Painlevé-Kuratowski upper/outer limit with respect to the norm topology of T and the weak* topology of T^* is given by

$$\operatorname*{Lim\,sup}_{a \to \bar{a}} \ell(a) = \{a^* \in T^* \mid \exists \text{ sequences } a_n \to \bar{a} \text{ and } a_n^* \xrightarrow{w^*} a^* \\ \text{with } a_n^* \in \ell(a_n) \text{ for all } n \in \mathbb{N}\},$$

where the symbol $\overset{w^*}{\to}$ represents the convergence in the weak* topology of T^*.

Definition 2.2. For a given $\epsilon \geq 0$ and $\mathbb{S}$, the collection of ϵ-normals to $\mathbb{S}$ at $\bar{a} \in \mathbb{S}$ defined by

$$\widehat{N}_\epsilon(\bar{a}, \mathbb{S}) = \{a^* \in T^* \mid \operatorname*{Lim\ sup}_{a \overset{\mathbb{S}}{\to} \bar{a}} \frac{\langle a^*, a - \bar{a}\rangle}{\|a - \bar{a}\|} \leq \epsilon\}, \tag{1}$$

where $a \overset{\mathbb{S}}{\to} \bar{a}$ means that $a \to \bar{a}$ with $a \in \mathbb{S}$.

In the above definition, for all $\epsilon \geq 0$, if $\bar{a} \notin \mathbb{S}$, we write $\widehat{N}_\epsilon(\bar{a}, \mathbb{S}) = \emptyset$. For suppose $\epsilon = 0$ in (1), then the set $\widehat{N}_0(\bar{a}, \mathbb{S})$ is called the Fréchet normal cone to $\mathbb{S}$ at $\bar{a}$.

Definition 2.3. The limiting/Mordukhovich normal cone $N(\bar{a}, \mathbb{S})$ to $\mathbb{S}$ at $\bar{a} \in \mathbb{S}$ is obtained from $\widehat{N}_\epsilon(a, \mathbb{S})$ by taking the sequential Painlevé-Kuratowski upper limits as

$$N(\bar{a}, \mathbb{S}) = \operatorname*{Lim\ sup}_{\substack{a \overset{\mathbb{S}}{\to} \bar{a} \\ \epsilon \downarrow 0}} \widehat{N}_\epsilon(a, \mathbb{S}) \tag{2}$$

If $\bar{a} \notin \mathbb{S}$, we put $N(\bar{a}, \mathbb{S}) = \emptyset$. Note that, if $\mathbb{S}$ is (locally) closed around $\bar{a}$, i.e., there is a neighborhood U of $\bar{a}$ such that $\mathbb{S} \cap \mathrm{cl}U$ is closed then one can put $\epsilon = 0$ in (2) (see Mordukhovich [10], Theorem 1.6).

Definition 2.4. Let $f : T \to \bar{R} = [-\infty, \infty]$ be an extended real-valued function. Then the limiting/Mordukhovich subdifferential of f at $\bar{a} \in T$ with $|f(\bar{a})| < \infty$ is defined by

$$\partial f(\bar{a}) = \{a^* \in T^* : (a^*, -1) \in N((\bar{a}, f(\bar{a})), \mathrm{epi} f)\},$$

where $\mathrm{epi} f = \{(a, \rho) \in T \times R : \rho \geq f(a)\}$.

If $|f(\bar{a})| = \infty$, then one puts $\partial f(\bar{a}) = \emptyset$. It is clear from Mordukhovich [10] that, if f is a convex function, then above-defined limiting/Mordukhovich subdifferential coincides with the subdifferential in the sense of convex analysis (cf. Rockafellar [14]).

Definition 2.5. A set $\mathbb{S} \subset T$ is sequentially normally compact at $\bar{a} \in \mathbb{S}$ if for any sequence $(\epsilon_n, a_n, a_n^*) \in [0, \infty) \times \mathbb{S} \times T^*$ satisfying $\epsilon_n \downarrow 0,\ a_n \overset{\mathbb{S}}{\to} \bar{a}$ and $a_n^* \overset{w^*}{\to} 0$ with $a_n^* \in \hat{N}_{\epsilon_n}(a_n, \mathbb{S})$, one has $\|a_n^*\| \to 0$ as $n \to \infty$.

When $\mathbb{S}$ is closed around $\bar{a}$ in the above definition then ϵ_n can be neglected. We refer [10] for more results, discussions and various sufficient conditions ensuring the fulfillment of the sequentially normally compact property.

Let $\mathbb{S}$ be a nonempty locally closed subset of T, and let $\mathbb{J} = \{1, 2, ..., q\}$ and $\mathbb{K} = \{1, 2, ..., r\}$ be index sets. In what follows, $\mathbb{S}$ is always assumed to be sequentially normally compact at the point under consideration.

Let us consider an *Optimization Problem* with interval-valued objective function of the form:

$$\text{(IVP)} \qquad \min_{x \in \mathbb{F}} \quad \Psi(x) = [\Psi^L(x), \Psi^U(x)]$$

$$\text{subject to}$$

$$G_u(x) \leq 0,\ u \in \mathbb{J}, \quad H_v(x) = 0, \quad v \in \mathbb{K},$$

where $\Psi : \mathbb{S} \to I$ is an interval-valued function satisfying the condition $\Psi^L(x) \leq \Psi^U(x)$ and $\Psi^L,\ \Psi^U,\ G_u,\ u \in \mathbb{J}$ and $H_v,\ v \in \mathbb{K}$ are locally Lipschitz on $\mathbb{S}$. The feasible set of (IVP) is given by $\mathbb{F} = \{x \in \mathbb{S} | G_u(x) \leq 0,\ u \in \mathbb{J},\ H_v(x) = 0,\ v \in \mathbb{K}\}$.

Definition 2.6 (Sun and Wang[18])**.** We say that $x^* \in \mathbb{F}$ is a LU optimal solution of (IVP), if there exists no $x_0 \in \mathbb{F}$ such that $\Psi(x_0) <_{LU} \Psi(x^*)$.

For $\bar{a} \in \mathbb{S}$, we define

$$\mathbb{J}(\bar{a}) = \{u \in \mathbb{J} : G_u(\bar{a}) = 0\}, \quad \mathbb{K}(\bar{a}) = \{v \in \mathbb{K} : H_v(\bar{a}) = 0\}.$$

Definition 2.7 (Mordukhovich [10])**.** The Constraint Qualification (CQ) is said to hold at $\bar{a} \in \mathbb{S}$ if there do not exist $\mu_u \geq 0$, $u \in \mathbb{J}(\bar{a})$ and $\gamma_v \geq 0, v \in \mathbb{K}(\bar{a})$, such that $\sum\limits_{u\in\mathbb{J}(\bar{a})} \mu_u + \sum\limits_{v\in\mathbb{K}(\bar{a})} \gamma_v \neq 0$ and

$$0 \in \sum_{u\in\mathbb{J}(\bar{a})} \mu_u \partial G_u(\bar{a}) + \sum_{v\in\mathbb{K}(\bar{a})} \gamma_v(\partial H_v(\bar{a}) \cup \partial(H_v)(\bar{a})) + N(\bar{a}, \mathbb{S}).$$

We now turn our attention to define the concept of L-invexity-infineness for locally Lipschitz functions on the lines of Chuong [3]. Suppose that $G = (G_1, G_2, ..., G_q)$ and $H = (H_1, H_2, ..., H_r)$.

Definition 2.8. For any $x \in \mathbb{S}$, $(\Psi, G; H)$ is said to be L-(strictly) invex-infine at $\bar{a} \in \mathbb{S}$ on $\mathbb{S}$, if for all $x^{*L} \in \partial\Psi^L(\bar{a})$, $x^{*U} \in \partial\Psi^U(\bar{a})$, $y_u^* \in \partial(G_u)(\bar{a})$, $u \in \mathbb{J}$, and $z_v^* \in \partial H_v(\bar{a}) \cup \partial(-H_v)(\bar{a})$, $v \in \mathbb{K}$, there exists $\eta \in N(\bar{a}, \mathbb{S})^\circ$ such that

$$\Psi^L(x) - \Psi^L(\bar{a})(>) \geq \left\langle x^{*L}, \eta \right\rangle, \ (x \neq \bar{a}),$$

$$\Psi^U(x) - \Psi^U(\bar{a})(>) \geq \left\langle x^{*U}, \eta \right\rangle, \ (x \neq \bar{a}),$$

$$G_u(x) - G_u(\bar{a}) \geq \langle y_u^*, \eta \rangle, \ u \in \mathbb{J},$$

$$H_v(x) - H_v(\bar{a}) = \zeta_v \langle z_v^*, \eta \rangle, \ v \in \mathbb{K},$$

where $\zeta_v = 1$ (*respectively*, $\zeta_v = -1$) whenever $z_v^* \in \partial H_v(\bar{a})$ (*respectively*, $z_v^* \in \partial(-H_v)(\bar{a})$).

Hereafter, we assume that $\zeta_v = 1$ (respectively, $\zeta_v = -1$) whenever $z_v^* \in \partial H_v(\bar{a})$ (respectively, $z_v^* \in \partial(-H_v)(\bar{a})$) and $\bar{a} \in \mathbb{S}$.

3. **Optimality conditions**

Chuong and Kim ([4] Theorem 3.3) established the necessary optimality conditions for (weakly) efficient solutions of a multiobjective optimization problem with equality and inequality constraints. In the view point of Chuong and Kim ([4] Theorem 3.3), if we consider $m = 2$, then we obtain the following Karush-Kuhn-Tucker type necessary optimality conditions for an interval-valued problem (IVP) as follow:

Theorem 3.1 (Karush-Kuhn-Tucker Type Necessary Conditions)**.** *Let the constraints qualification* (CQ) *be satisfied at* $\tilde{x}$*. If* $\tilde{x}$ *is a LU optimal solution of problem* (IVP)*, then there exist* $0 < \lambda^L, \lambda^U \in R$*,* $\bar{\mu} \in R_+^q$*, and* $\bar{\gamma} \in R_+^r$ *such that*

$$0 \in \lambda^L \partial\Psi^L(\tilde{x}) + \lambda^U \partial\Psi^U(\tilde{x}) + \sum_{u\in\mathbb{J}} \bar{\mu}_u \partial G_u(\tilde{x})$$

$$+ \sum_{v\in\mathbb{K}} \bar{\gamma}_v \left(\partial H_v(\tilde{x}) \cup \partial(-H_v)(\tilde{x})\right) + N(\tilde{x}, \mathbb{S}), \quad (3)$$

$$\bar{\mu}_u G_u(\tilde{x}) = 0, \ u \in \mathbb{J}. \quad (4)$$

We now prove that above necessary conditions are sufficient under L-invex-infine functions.

Theorem 3.2 (Karush-Kuhn-Tucker Type Sufficient Optimality Conditions)**.** *Let* $\tilde{x} \in \mathbb{F}$ *and assume that there exist* $0 < \lambda^L, \lambda^U \in R$*,* $\bar{\mu} \in R_+^q$*,* $\bar{\gamma} \in R_+^r$ *such that*

(i) $0 \in \lambda^L \partial \Psi^L(\tilde{x}) + \lambda^U \partial \Psi^U(\tilde{x}) + \sum_{u \in \mathbb{J}} \bar{\mu}_u \partial G_u(\tilde{x})$
$+ \sum_{v \in \mathbb{K}} \bar{\gamma}_v \left(\partial H_v(\tilde{x}) \cup \partial(-H_v)(\tilde{x})\right) + N(\tilde{x}, \mathbb{S})$,
(ii) $\bar{\mu}_u G_u(\tilde{x}) = 0,\ u \in \mathbb{J}$,
(iii) $(\Psi, G; H)$ *is L-invex-infine on* $\mathbb{S}$ *at* $\tilde{x}$.

Then $\tilde{x}$ *is a LU optimal solution of problem* (IVP).

Proof. By hypothesis (i), it is obvious that there exist $x^{*L} \in \partial \Psi^L(\tilde{x})$, $x^{*U} \in \partial \Psi^U(\tilde{x})$, $y_u^* \in \partial G_u(\tilde{x})$, $u \in \mathbb{J}$ and $z_v^* \in \partial H_v(\tilde{x}) \cup \partial(-H_v)(\tilde{x})$, $v \in \mathbb{K}$ such that

$$-\left(\lambda^L x^{*L} + \lambda^U x^{*U} + \sum_{u \in \mathbb{J}} \bar{\mu}_u y_u^* + \sum_{v \in \mathbb{K}} \bar{\gamma}_v z_v^*\right) \in N(\tilde{x}, \mathbb{S}). \tag{5}$$

Suppose on the contrary, $\tilde{x}$ is not a LU optimal solution for the considered optimization problem with interval valued objective function (IVP). Then, by Definition (2.6) there exists a feasible solution $x_0 \in \mathbb{F}$ such that

$$\Psi(x_0) <_{LU} \Psi(\tilde{x}).$$

That is,

$$\begin{cases} \Psi^L(x_0) < \Psi^L(\tilde{x}) \\ \Psi^U(x_0) < \Psi^U(\tilde{x}) \end{cases}, \text{ or } \begin{cases} \Psi^L(x_0) \le \Psi^L(\tilde{x}) \\ \Psi^U(x_0) < \Psi^U(\tilde{x}) \end{cases}, \text{ or } \begin{cases} \Psi^L(x_0) < \Psi^L(\tilde{x}) \\ \Psi^U(x_0) \le \Psi^U(\tilde{x}) \end{cases}.$$

Since $\lambda^L > 0,\ \lambda^U > 0$, then above inequalities together yield

$$\lambda^L \Psi^L(x_0) + \lambda^U \Psi^U(x_0) < \lambda^L \Psi^L(\tilde{x}) + \lambda^U \Psi^U(\tilde{x}). \tag{6}$$

On the other hand, $(\Psi, G; H)$ is a L-invex-infine on $\mathbb{S}$ at $\tilde{x}$, then by Definition (2.8), there exists $\eta \in N(\tilde{x}, \mathbb{S})^\circ$ such that the following inequalities

$$\Psi^L(x_0) - \Psi^L(\tilde{x}) \ge \left\langle x^{*L}, \eta \right\rangle, \tag{7}$$

$$\Psi^U(x_0) - \Psi^U(\tilde{x}) \ge \left\langle x^{*U}, \eta \right\rangle, \tag{8}$$

$$G_u(x_0) - G_u(\tilde{x}) \ge \langle y_u^*, \eta \rangle,\ u \in \mathbb{J}, \tag{9}$$

$$H_v(x_0) - H_v(\tilde{x}) = \zeta_v \langle z_v^*, \eta \rangle,\ v \in \mathbb{K}, \tag{10}$$

hold for any $x_0 \in \mathbb{F}$, $x^{*L} \in \partial \Psi^L(\tilde{x})$, $x^{*U} \in \partial \Psi^U(\tilde{x})$, $y_u^* \in \partial(G_u)(\tilde{x})$, $u \in \mathbb{J}$ and $z_v^* \in \partial H_v(\tilde{x}) \cup \partial(-H_v)(\tilde{x})$, $v \in \mathbb{K}$.

The inequalities (7) and (8) together with the positivity of λ^L and λ^U, gives

$$[\lambda^L \Psi^L(x_0) + \lambda^U \Psi^U(x_0)] - [\lambda^L \Psi^L(\tilde{x}) + \lambda^U \Psi^U(\tilde{x})] \ge \left\langle \lambda^L x^{*L} + \lambda^U x^{*U}, \eta \right\rangle. \tag{11}$$

Multiplying (9) by $\bar{\mu}_u,\ u \in \mathbb{J}$ and (10) by $\bar{\gamma}_v,\ v \in \mathbb{K}$, then adding the resultant inequalities, we get

$$\begin{aligned} &\sum_{u \in \mathbb{J}} \bar{\mu}_u \left[G_u(x_0) - G_u(\tilde{x})\right] + \sum_{v \in \mathbb{K}} \frac{\bar{\gamma}_v}{\zeta_v} \left[H_v(x_0) - H_v(\tilde{x})\right] \\ &\qquad \ge \sum_{u \in \mathbb{J}} \bar{\mu}_u \langle y_u^*, \eta \rangle + \sum_{v \in \mathbb{K}} \bar{\gamma}_v \langle z_v^*, \eta \rangle. \end{aligned} \tag{12}$$

On adding (11) and (12), we get

$$\begin{aligned} &[\lambda^L \Psi^L(x_0) + \lambda^U \Psi^U(x_0)] - [\lambda^L \Psi^L(\tilde{x}) + \lambda^U \Psi^U(\tilde{x})] \\ &\quad + \sum_{u \in \mathbb{J}} \bar{\mu}_u \left[G_u(x_0) - G_u(\tilde{x})\right] + \sum_{v \in \mathbb{K}} \frac{\bar{\gamma}_v}{\zeta_v} \left[H_v(x_0) - H_v(\tilde{x})\right] \end{aligned}$$

$$\geq \left\langle \lambda^L x^{*L} + \lambda^U x^{*U}, \eta \right\rangle + \sum_{u\in\mathbb{J}} \bar{\mu}_u \left\langle y_u^*, \eta \right\rangle + \sum_{v\in\mathbb{K}} \bar{\gamma}_v \left\langle z_v^*, \eta \right\rangle. \tag{13}$$

Now by the meaning of polar cone, we have from (5) and $\eta \in N(\tilde{x}, \mathbb{S})^\circ$ that

$$\left\langle \lambda^L x^{*L} + \lambda^U x^{*U}, \eta \right\rangle + \sum_{u\in\mathbb{J}} \bar{\mu}_u \left\langle y_u^*, \eta \right\rangle + \sum_{v\in\mathbb{K}} \bar{\gamma}_v \left\langle z_v^*, \eta \right\rangle \geq 0. \tag{14}$$

By the hypothesis (ii), (13), (14) and the fact $x_0 \in \mathbb{F}, \tilde{x} \in \mathbb{F}$, we see that

$$[\lambda^L \Psi^L(x_0) + \lambda^U \Psi^U(x_0)] - [\lambda^L \Psi^L(\tilde{x}) + \lambda^U \Psi^U(\tilde{x})] \geq 0,$$

which contradicts (6). Thus, the theorem is proved. □

The following simple example expose that the L-invex-infine property of $(\Psi, G; H)$ on $\mathbb{S}$ invoked in the above *Sufficient Optimality Conditions* is essential.

Example 3.1. *Consider the optimization problem of the form*

$$\text{(IVP-1)} \qquad \min_{x\in\mathbb{F}} \quad \Psi(x) = [\Psi^L(x), \Psi^U(x)]$$

$$\text{subject to} \quad G_1(x) \leq 0, \quad H_1(x) = 0,$$

where

$$\Psi^L(x) = \begin{cases} x^3, & x \geq 0 \\ x^5, & \text{otherwise} \end{cases}, \quad \Psi^U(x) = \begin{cases} \frac{x}{2}, & x \geq 0 \\ x, & \text{otherwise} \end{cases},$$

$$G_1(x) = -|x|, \ x \in R \quad \text{and } H_1(x) = x^2 - x, \ x \in R.$$

Let us consider $\mathbb{S} = R$. Then the feasible region to (IVP-1) is $\mathbb{F} = \{1, 0\}$. Taking $\tilde{x} = 0 \in \mathbb{F}$, we posses $N(\tilde{x}, \mathbb{S}) = \{0\}$ and $N(\tilde{x}, \mathbb{S})^\circ = R$. It is easy to observe that the hypotheses (i) *and* (ii) *of Theorem 3.2 hold at $\tilde{x} = 0$. Even though, $\tilde{x} = 0$ is not a LU optimal solution of problem* (IVP-1)*. This means that the conclusion of Theorem 3.2 fails to holds. The reason is that $(\Psi, G_1; H_1)$ is not a L-invex-infine at $\tilde{x} = 0$ on $\mathbb{S}$.*

4. **Wolfe type duality**

We consider the following Wolfe type dual for (IVP):

$$\text{(IWD)} \qquad \max \left[\Psi(y) + \sum_{u\in\mathbb{J}} \mu_u G_u(y) + \sum_{v\in\mathbb{K}} \gamma_v H_v(y) \right]$$

$$\text{subject to}$$

$$0 \in \lambda^L \partial\Psi^L(y) + \lambda^U \partial\Psi^U(y) + \sum_{u\in\mathbb{J}} \mu_u \partial G_u(y) + \sum_{v\in\mathbb{K}} \gamma_v \left(\partial H_v(y) \cup \partial(-H_v)(y) \right) + N(y, \mathbb{S}), \tag{15}$$

$$y \in \mathbb{S}, 0 < \lambda^L, \lambda^U \in R, \ \lambda^L + \lambda^U = 1, \mu \in R_+^q, \gamma \in R_+^r, H(y) \in (\gamma - \Omega(0, \|\gamma\|))^\circ, \tag{16}$$

where $\Omega(0, \|\gamma\|) = \{\tau \in R^r : \|\tau\| = \|\gamma\|\}$ and $\Psi(y) + \sum_{u\in\mathbb{J}} \mu_u G_u(y) + \sum_{v\in\mathbb{K}} \gamma_v H_v(y) = \left[\Psi^L(y) + \sum_{u\in\mathbb{J}} \mu_u G_u(y) + \sum_{v\in\mathbb{K}} \gamma_v H_v(y), \quad \Psi^U(y) + \sum_{u\in\mathbb{J}} \mu_u G_u(y) + \sum_{v\in\mathbb{K}} \gamma_v H_v(y) \right]$ is an interval-valued function. We denote by $\mathbb{W}_1$ the set of all feasible solutions $(y, \lambda^L, \lambda^U, \mu, \gamma) \in \mathbb{S} \times R_+ \setminus \{0\} \times R_+ \setminus \{0\} \times R_+^q \times R_+^r$ of problem (IWD).

Definition 4.1. *We say that $(\tilde{y}, \tilde{\lambda}^L, \tilde{\lambda}^U, \tilde{\mu}, \tilde{\gamma}) \in \mathbb{W}_1$ is a LU optimal solution of (IWD), if there exists no feasible solution $(y, \tilde{\lambda}^L, \tilde{\lambda}^U, \tilde{\mu}, \tilde{\gamma})$ to (IWD), such that $\Psi(\tilde{y}) + \sum_{u\in\mathbb{J}} \tilde{\mu}_u G_u(\tilde{y}) + \sum_{v\in\mathbb{K}} \tilde{\gamma}_v H_v(\tilde{y}) <_{LU} \Psi(y) + \sum_{u\in\mathbb{J}} \tilde{\mu}_u G_u(y) + \sum_{v\in\mathbb{K}} \tilde{\gamma}_v H_v(y)$.*

Now, we discuss the relationships between (IWD) and (IVP).

Theorem 4.1 (Weak Duality)**.** *Let* $x \in \mathbb{F}$ *and* $(y, \lambda^L, \lambda^U, \mu, \gamma) \in \mathbb{W}_1$*. Assume that* $(\Psi, G; H)$ *is L-invex-infine on* $\mathbb{S}$ *at* y*, then* $\Psi(x) \geq_{LU} \Psi(y) + \sum\limits_{u\in\mathbb{J}} \mu_u G_u(y) + \sum\limits_{v\in\mathbb{K}} \gamma_v H_v(y)$*.*

Proof. Since $(y, \lambda^L, \lambda^U, \mu, \gamma) \in \mathbb{W}_1$ satisfy the relations (15)-(16), there exist $x^{*L} \in \partial\Psi^L(y)$, $x^{*U} \in \partial\Psi^U(y)$, $y_u^* \in \partial G_u(y)$, $u \in \mathbb{J}$ and $z_v^* \in \partial H_v(y) \cup \partial(-H_v)(y)$, $v \in \mathbb{K}$ such that

$$-\left(\lambda^L x^{*L} + \lambda^U x^{*U} + \sum_{u\in\mathbb{J}} \mu_u y_u^* + \sum_{v\in\mathbb{K}} \gamma_v z_v^*\right) \in N(y, \mathbb{S}), \tag{17}$$

$$\langle \gamma - \tau, H(y)\rangle \leq 0, \quad \forall \tau \in R^r \text{ with } \|\tau\| = \|\gamma\| . \tag{18}$$

Suppose on the contrary to the result that

$$\Psi(x) <_{LU} \Psi(y) + \sum_{u\in\mathbb{J}} \mu_u G_u(y) + \sum_{v\in\mathbb{K}} \gamma_v H_v(y).$$

That is,

$$\begin{cases} \Psi^L(x) < \Psi^L(y) + \sum\limits_{u\in\mathbb{J}} \mu_u G_u(y) + \sum\limits_{v\in\mathbb{K}} \gamma_v H_v(y) \\ \Psi^U(x) < \Psi^U(y) + \sum\limits_{u\in\mathbb{J}} \mu_u G_u(y) + \sum\limits_{v\in\mathbb{K}} \gamma_v H_v(y), \end{cases}$$

$$\text{or} \begin{cases} \Psi^L(x) \leq \Psi^L(y) + \sum\limits_{u\in\mathbb{J}} \mu_u G_u(y) + \sum\limits_{v\in\mathbb{K}} \gamma_v H_v(y) \\ \Psi^U(x) < \Psi^U(y) + \sum\limits_{u\in\mathbb{J}} \mu_u G_u(y) + \sum\limits_{v\in\mathbb{K}} \gamma_v H_v(y), \end{cases}$$

$$\text{or} \begin{cases} \Psi^L(x) < \Psi^L(y) + \sum\limits_{u\in\mathbb{J}} \mu_u G_u(y) + \sum\limits_{v\in\mathbb{K}} \gamma_v H_v(y) \\ \Psi^U(x) \leq \Psi^U(y) + \sum\limits_{u\in\mathbb{J}} \mu_u G_u(y) + \sum\limits_{v\in\mathbb{K}} \gamma_v H_v(y) \end{cases}.$$

Since $\lambda^L > 0,\ \lambda^U > 0$ and $\lambda^L + \lambda^U = 1$, then above inequalities together yield

$$\begin{aligned} &\lambda^L\Psi^L(x) + \lambda^U\Psi^U(x) \\ &< \lambda^L\Psi^L(y) + \lambda^U\Psi^U(y) + \sum_{u\in\mathbb{J}} \mu_u G_u(y) + \sum_{v\in\mathbb{K}} \gamma_v H_v(y). \end{aligned} \tag{19}$$

On the other hand, $(\Psi, G; H)$ is a L-invex-infine on $\mathbb{S}$ at y, then by Definition (2.8), there exists $\eta \in N(y, \mathbb{S})^\circ$ such that the following inequalities

$$\Psi^L(x) - \Psi^L(y) \geq \left\langle x^{*L}, \eta \right\rangle, \tag{20}$$

$$\Psi^U(x) - \Psi^U(y) \geq \left\langle x^{*U}, \eta \right\rangle, \tag{21}$$

$$G_u(x) - G_u(y) \geq \langle y_u^*, \eta\rangle,\ u \in \mathbb{J}, \tag{22}$$

$$H_v(x) - H_v(y) = \zeta_v \langle z_v^*, \eta\rangle,\ v \in \mathbb{K}, \tag{23}$$

hold for any $x \in \mathbb{F}$, $x^{*L} \in \partial\Psi^L(y)$, $x^{*U} \in \partial\Psi^U(y)$, $y_u^* \in \partial(G_u)(y)$, $u \in \mathbb{J}$ and $z_v^* \in \partial H_v(y) \cup \partial(-H_v)(y)$, $v \in \mathbb{K}$.

Now, proceeding in exactly the same manner as in the proof of Theorem 3.2, it follows from the relations (20)-(23) that

$$[\lambda^L\Psi^L(x) + \lambda^U\Psi^U(x)] - [\lambda^L\Psi^L(y) + \lambda^U\Psi^U(y)] + \sum_{u\in\mathbb{J}} \mu_u [G_u(x) - G_u(y)]$$

$$+ \sum_{v\in\mathbb{K}} \frac{\gamma_v}{\zeta_v} [H_v(x) - H_v(y)] \geq 0.$$

Thus, by taking $\tau_v = \dfrac{\gamma_v}{\zeta_v}, v \in \mathbb{K}$, we have

$$[\lambda^L \Psi^L(x) + \lambda^U \Psi^U(x)] - [\lambda^L \Psi^L(y) + \lambda^U \Psi^U(y)] + \sum_{u\in\mathbb{J}} \mu_u \left[G_u(x) - G_u(y)\right]$$

$$+ \sum_{v\in\mathbb{K}} \tau_v \left[H_v(x) - H_v(y)\right] \geq 0.$$

On the basis of $x \in \mathbb{F}$, above inequality yields

$$[\lambda^L \Psi^L(x) + \lambda^U \Psi^U(x)] - [\lambda^L \Psi^L(y) + \lambda^U \Psi^U(y)] - \sum_{u\in\mathbb{J}} \mu_u \left[G_u(y)\right]$$

$$- \sum_{v\in\mathbb{K}} \tau_v \left[H_v(y)\right] \geq 0,$$

equivalently,

$$[\lambda^L \Psi^L(x) + \lambda^U \Psi^U(x)] - [\lambda^L \Psi^L(y) + \lambda^U \Psi^U(y)] - \sum_{u\in\mathbb{J}} \mu_u \left[G_u(y)\right]$$

$$- \sum_{v\in\mathbb{K}} \gamma_v H_v(y) + \langle \gamma - \tau, H(y)\rangle \geq 0, \tag{24}$$

where $\tau = (\tau_1, \tau_2, ..., \tau_r) \in R^r$. It is to be observed that $\|\tau\| = \|\gamma\|$ and thus, by (18) and (24), we get

$$[\lambda^L \Psi^L(x) + \lambda^U \Psi^U(x)] - [\lambda^L \Psi^L(y) + \lambda^U \Psi^U(y)] - \sum_{u\in\mathbb{J}} \mu_u G_u(y) - \sum_{v\in\mathbb{K}} \gamma_v H_v(y) \geq 0,$$

which contradicts (19). Thus, the theorem is proved. □

Theorem 4.2 (Strong Duality). *Let $\tilde{x}$ be a LU optimal solution of* (IVP) *and the constraint qualification* (CQ) *be satisfied at $\tilde{x}$. Then there exist $\tilde{\lambda}^L > 0, \tilde{\lambda}^U > 0$, $\tilde{\mu} \geq 0$, and $\tilde{\gamma} \geq 0$ such that $(\tilde{x}, \tilde{\lambda}^L, \tilde{\lambda}^U, \tilde{\mu}, \tilde{\gamma})$ is a feasible solution to the problem* (IWD) *and the objective values of (IVP) and (IWD) are same. Furthermore, if all the conditions of weak duality Theorem 4.1 are satisfied, then $(\tilde{x}, \tilde{\lambda}^L, \tilde{\lambda}^U, \tilde{\mu}, \tilde{\gamma})$ is a LU optimal solution of* (IWD).

Proof. By assumption $\tilde{x}$ is a LU optimal solution of (IVP), and the constraint qualification (CQ) is satisfied at $\tilde{x}$. Then by Theorem 3.1 there exist $\tilde{\lambda}^L > 0, \tilde{\lambda}^U > 0$, $\tilde{\mu} \in R^q_+$, and $\tilde{\gamma} \in R^r_+$ such that

$$0 \in \tilde{\lambda}^L \partial \Psi^L(\tilde{x}) + \tilde{\lambda}^U \partial \Psi^U(\tilde{x}) + \sum_{u\in\mathbb{J}} \tilde{\mu}_u \partial G_u(\tilde{x}) + \sum_{v\in\mathbb{K}} \tilde{\gamma}_v \left(\partial H_v(\tilde{x}) \cup \partial(-H_v)(\tilde{x})\right) + N(\tilde{x}, \mathbb{S}),$$

$$\tilde{\mu}_u G_u(\tilde{x}) = 0, \ u \in \mathbb{J}.$$

In addition, since $H_v(\tilde{x}) = 0, v \in \mathbb{K}$ for $\tilde{x} \in \mathbb{F}$. This implies that $\langle \tilde{\gamma} - \tau, H(\tilde{x})\rangle = 0$, for all $\tau \in R^r$ with $\|\tau\| = \|\tilde{\gamma}\|$. That is $H(\tilde{x}) \in (\tilde{\gamma} - \Omega(0, \|\tilde{\gamma}\|))^\circ$ and so $(\tilde{x}, \tilde{\lambda}^L, \tilde{\lambda}^U, \tilde{\mu}, \tilde{\gamma})$ is a feasible solution of (IWD), moreover, the two objective values of (IVP) and (IWD) are equal. Further, if $(\tilde{x}, \tilde{\lambda}^L, \tilde{\lambda}^U, \tilde{\mu}, \tilde{\gamma})$ is not a LU optimal solution of (IWD), then there exists a feasible solution $(\tilde{y}, \tilde{\lambda}^L, \tilde{\lambda}^U, \tilde{\mu}, \tilde{\gamma})$ of (IWD) such that

$$\Psi(\tilde{x}) <_{LU} \Psi(\tilde{y}) + \sum_{u\in\mathbb{J}} \tilde{\mu}_u G_u(\tilde{y}) + \sum_{v\in\mathbb{K}} \tilde{\gamma}_v H_v(\tilde{y}),$$

which contradicts Theorem 4.1. Hence $(\tilde{x}, \tilde{\lambda}^L, \tilde{\lambda}^U, \tilde{\mu}, \tilde{\gamma})$ is a LU optimal solution of (IWD). □

Theorem 4.3 (Strict Converse Duality)**.** *Let* $\tilde{x}$ *and* $(\tilde{y}, \tilde{\lambda}^L, \tilde{\lambda}^U, \tilde{\mu}, \tilde{\gamma})$ *be the feasible solutions of* (IVP) *and* (IWD)*, respectively. Assume that* $(\Psi, G; H)$ *is L-strictly invex-infine on* $\mathbb{S}$ *at* $\tilde{y}$ *and*

$$\tilde{\lambda}^L \Psi^L(\tilde{x}) + \tilde{\lambda}^U \Psi^U(\tilde{x}) \leq \tilde{\lambda}^L \Psi^L(\tilde{y}) + \tilde{\lambda}^U \Psi^U(\tilde{y}) + \sum_{u \in \mathbb{J}} \tilde{\mu}_u G_u(\tilde{y}) + \sum_{v \in \mathbb{K}} \tilde{\gamma}_v H_v(\tilde{y}), \tag{25}$$

then $\tilde{x} = \tilde{y}$.

Proof. Suppose on the contrary, $\tilde{x} \neq \tilde{y}$. Since $(\tilde{y}, \tilde{\lambda}^L, \tilde{\lambda}^U, \tilde{\mu}, \tilde{\gamma}) \in \mathbb{W}_1$ satisfy the relations (15)-(16), there exist $x^{*L} \in \partial \Psi^L(\tilde{y})$, $x^{*U} \in \partial \Psi^U(\tilde{y})$, $y_u^* \in \partial G_u(\tilde{y})$, $u \in \mathbb{J}$ and $z_v^* \in \partial H_v(\tilde{y}) \cup \partial(-H_v)(\tilde{y})$, $v \in \mathbb{K}$ such that

$$-\left(\tilde{\lambda}^L x^{*L} + \tilde{\lambda}^U x^{*U} + \sum_{u \in \mathbb{J}} \tilde{\mu}_u y_u^* + \sum_{v \in \mathbb{K}} \tilde{\gamma}_v z_v^*\right) \in N(\tilde{y}, \mathbb{S}). \tag{26}$$

$$\langle \tilde{\gamma} - \tau, H(\tilde{y}) \rangle \leq 0, \quad \forall \tau \in R^r \text{ with } \|\tau\| = \|\tilde{\gamma}\|. \tag{27}$$

Now, proceeding in exactly the same manner as shown in the proof of Theorem 4.1, it follows immediately from L-strictly invex-infine of $(\Psi, G; H)$ at $\tilde{y}$ on $\mathbb{S}$, that

$$[\tilde{\lambda}^L \Psi^L(\tilde{x}) + \tilde{\lambda}^U \Psi^U(\tilde{x})] - [\tilde{\lambda}^L \Psi^L(\tilde{y}) + \tilde{\lambda}^U \Psi^U(\tilde{y})] - \sum_{u \in \mathbb{J}} \tilde{\mu}_u G_u(\tilde{y}) - \sum_{v \in \mathbb{K}} \tilde{\gamma}_v H_v(\tilde{y}) > 0,$$

which contradicts (25). Thus, the theorem is proved. □

5. Conclusions

In this study, we have obtained Karush-Kuhn-Tucker type sufficient optimality conditions for an interval-valued optimization problem under L-invex-infine functions defined with reference to the limiting subdifferential of locally Lipschitz functions. Usual duality theorems are discussed for a Wolfe type dual model. It may be enjoyable to investigate that the second and higher order duality results for an interval-valued optimization problem hold or not in terms of the limiting/ Mordukhovich subdifferential. This will reflect the forthcoming research of the authors.

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