BOUNDED AND CONTINUOUS FUNCTIONS ON THE CLOSED UNIT BALL OF A NORMED VECTOR SPACE EQUIPPED WITH A NEW PRODUCT

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Assume that A is a non-zero normed vector space and φ is a non-zero element of A^* with $\|\varphi\| \leq 1$. We consider the Banach algebra $C^{b\varphi}(K)$ with a new product, where $K = \overline{B_1^{(0)}}$ is the closed unit ball of A. We wish to investigate and characterize miscellaneous algebraic properties of the Banach algebra $C^{b\varphi}(K)$ such as, idempotent and nilpotent elements, zero divisor elements, bounded approximate identities. Also we characterize some relations between the character spaces $\triangle(C^{b\varphi}(K))$ and $\triangle(C^b(K))$. Finally the derivations on $C^{b\varphi}(K)$ and $C^b(K)$ are investigated.

Keywords: Normed vector space, idempotent element, nilpotent element, character space, derivation.

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1. INTRODUCTION AND PRELIMINARIES

Suppose A is a non-zero normed vector space and φ is a non-zero element of A^* such that $\|\varphi\| \leq 1$. Also let $K = \overline{B_1^{(0)}}$ be the closed unit ball of A. It is well known that

$$C^b(K) = \left\{ f : K \longrightarrow \mathbb{C} \quad | \quad f \quad \text{is continuous and bounded} \right\}$$

is a Banach algebra with respect to the pointwise operations and the norm

$$\|f\|_\infty = \sup \left\{ |f(k)| \quad | \quad k \in K \right\}$$

for all $f \in C^b(K)$.

On the Banach space $C^b(K)$ we define the product

$$(f \cdot g)(k) = f(k)\varphi(k)g(k), \quad k \in K$$

for all $f, g \in C^b(K)$. We shall show that $(C^b(K), \cdot)$ is a non-unital commutative Banach algebra that we denote it by $C^{b\varphi}(K)$. Clearly

$$\|\varphi\|_\infty = \sup \left\{ |\varphi(k)| \quad k \in K \right\} = \|\varphi\| \leq 1$$

and φ is an element of the closed unit ball of the Banach algebra $C^{b\varphi}(K)$.

Let A be a Banach algebra and let $\psi : A \longrightarrow \mathbb{C}$ be an algebraic homomorphism. Then ψ is called a character on A. The set of all non-zero characters on A will be denoted by $\triangle(A)$. Also the character space of A is $\triangle(A) \cup \{0\}$.

Let A be a Banach algebra and let X be a Banach $A-$bimodule. A bounded linear map $D : A \longrightarrow X$ is a derivation if, $D(ac) = a \cdot D(c) + D(a) \cdot c$ for all $a, c \in A$. Clearly each

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Banach algebra A is a Banach $A-$bimodule by the product on A. So a bounded linear map $D : A \longrightarrow A$ is a derivation if, $D(ac) = aD(c) + D(a)c$ for all $a, c \in A$.

A bounded net $(a_\alpha)_\alpha$ in a Banach algebra A is said to be a bounded approximate identity if, $\|aa_\alpha - a\| \longrightarrow 0$ and $\|a_\alpha a - a\| \longrightarrow 0$ for all $a \in A$.

A non-zero element a of a commutative Banach algebra A is said to be a zero divisor if, there exists a non-zero $b \in A$ such that $ba = 0$. $a \in A$ is said to be a nilpotent element if $a^n = 0$ for some $n > 0$. Also $a \in A$ is said to be an idempotent element if $a^2 = a$.

Let A be a Banach algebra. In [1] R. A. Kamyabi-Gol and M. Janfada defined a new product " $\cdot$ " on A by $a \cdot c = a\varepsilon c$ for all $a, c \in A$, where ε is a fixed element of the closed unit ball $\overline{B_1^{(0)}}$ of A. $(A, \cdot)$ is an associative Banach algebra which is denoted by A_ε. Recall that the Banach algebra $C^{b\varphi}(K)$ is a special case of the algebra A_ϵ. Some properties such as, Arens regularity, amenability and derivations on A_ϵ are investigated in [1]. Also biflatness, biprojectivity, $\varphi-$amenability and $\varphi-$contractibility of A_ϵ are investigated in [6].

The notion and many basic properties of strongly zero-product preserving maps on normed algebras are investigated in [2, 3, 4, 5]. A class of non-equivalent norms and also strongly zero-product preserving maps on $C^{b\varphi}(K)$ are investigated in [7].

In this paper we characterize the idempotent, nilpotent, zero divisor elements and the bounded approximate identities of $C^{b\varphi}(K)$. Also we characterize some relations between the character spaces $\triangle(C^{b\varphi}(K))$ and $\triangle(C^b(K))$. Finally the derivations on $C^{b\varphi}(K)$ and $C^b(K)$ are investigated.

2. Main results and miscellaneous algebraic properties of $C^{b\varphi}(K)$

In this section let A be a non-zero normed vector space and let φ be a non-zero linear functional on A with $\|\varphi\| \leq 1$. Also let $K = \overline{B_1^{(0)}}$ be the closed unit ball of A. 1_K is the constant function on K such that $1_K(k) = 1$ for all $k \in K$.

Proposition 2.1. *$C^{b\varphi}(K)$ is a non-unital commutative Banach algebra.*

Proof. Clearly $C^{b\varphi}(K)$ is a commutative Banach algebra. For each $f \in C^{b\varphi}(K)$, $f \cdot 1_K \neq 1_K$. Indeed,

$$\begin{aligned} f \cdot 1_K(0) &= f(0)\varphi(0)1_K(0) = 0 \\ &\neq 1_K(0) \\ &= 1. \end{aligned}$$

This shows that $C^{b\varphi}(K)$ is not unital. □

Set $A^*\Big|_K = \left\{ f\Big|_K : K \longrightarrow \mathbb{C} \quad \Big| \quad f \in A^* \right\}$. So we can present the following result.

Proposition 2.2. $A^*\Big|_K \subseteq C^{b\varphi}(K)$ *and for each* $f \in A^*$, $\|f\| = \|f\|_\infty$.

Proof. As each element of A^* is continuous so, $f\Big|_K$ is continuous on K for all $f \in A^*$. Also for each $f \in A^*$,

$$\begin{aligned} \|f\| &= \sup\left\{ |f(k)| \quad \Big| \quad \|k\| \leq 1 \right\} \\ &= \|f\|_\infty. \end{aligned}$$

□

In the following Proposition we characterize the idempotent elements of $C^{b\varphi}(K)$.

Proposition 2.3. *The only idempotent element of $C^{b\varphi}(K)$ is $f = 0$.*

Proof. Let $f \in C^{b\varphi}(K)$ be an idempotent element. So $f \cdot f = f$. It follows that $f(k)\varphi(k)f(k) = f(k)$ for all $k \in K$. Hence $f(k)(f(k)\varphi(k) - 1) = 0$. This shows that $f(k) = 0$ or $f(k)\varphi(k) = 1$ for all $k \in K$. So,

$$f(k) = \begin{cases} 0 & k \in f^{-1}(\{0\}) \\ \frac{1}{\varphi(k)} & k \notin f^{-1}(\{0\}). \end{cases}$$

Clearly $K = f^{-1}(\{0\}) \cup (K - f^{-1}(\{0\}))$ and $f^{-1}(\{0\})$ is closed in K. We shall show that $f^{-1}(\{0\})$ is also open in K. For this end, we shall show that $K - f^{-1}(\{0\})$ is closed in K. Let $x \in \overline{K - f^{-1}(\{0\})}$. So there exists a sequence $\{k_n\}_n \in K - f^{-1}(\{0\})$ such that $k_n \longrightarrow x$. As f is continuous so, $\frac{1}{\varphi(k_n)} = f(k_n) \longrightarrow f(x)$. But φ is continuous. So $\frac{1}{\varphi(x)} = f(x)$. This shows that $x \in K - f^{-1}(\{0\})$. As K is connected and $f^{-1}(\{0\})$ is both open and closed in K so we can conclude that $f^{-1}(\{0\}) = \emptyset$ or $f^{-1}(\{0\}) = K$. But $f(0) = 0$. This shows that $f^{-1}(\{0\}) = K$ and consequently $f = 0$. □

Proposition 2.4. *The only nilpotent element of $C^{b\varphi}(K)$ is $f = 0$.*

Proof. Let $f \in C^{b\varphi}(K)$ be a nilpotent element. So there exists an $m \in \mathbb{N}$ such that $\overbrace{f \cdot f \cdots f}^{m \text{ times}} = 0$. One can easily show that $f^m(k)\varphi^{m-1}(k) = 0$ for all $k \in K$. It follows that $f\Big|_{K - \ker\varphi} = 0$. Let $x \in K \cap \ker\varphi$. Also let $e \in A$ be an element such that $\varphi(e) = 1$. So $\frac{e}{\|e\|} \in K$. As K is convex, we can conclude that $\frac{n}{n+1}x + \frac{1}{n+1}\frac{e}{\|e\|} \in K$ for all $n \in \mathbb{N}$. Clearly $\frac{n}{n+1}x + \frac{1}{n+1}\frac{e}{\|e\|} \in (K - \ker\varphi)$ and $\frac{n}{n+1}x + \frac{1}{n+1}\frac{e}{\|e\|} \longrightarrow x$. The continuity of f implies that $0 = f(\frac{n}{n+1}x + \frac{1}{n+1}\frac{e}{\|e\|}) \longrightarrow f(x)$. This shows that $f = 0$ on $K \cap \ker\varphi$. So $f = 0$ on K. □

We give a result concerning bounded approximate identity of $C^{b\varphi}(K)$.

Proposition 2.5. *There is no bounded approximate identity in $C^{b\varphi}(K)$.*

Proof. Assume by absurd that $(f_\alpha)_\alpha$ is a bounded approximate identity in $C^{b\varphi}(K)$. So $f_\alpha \cdot 1_K \longrightarrow 1_K$. It follows that

$$\begin{aligned} 0 &= f_\alpha(0)\varphi(0)1_K(0) \\ &= f_\alpha \cdot 1_K(0) \longrightarrow 1_K(0) \\ &= 1, \end{aligned}$$

that is a contradiction. So there is no bounded approximate identity in $C^{b\varphi}(K)$. □

In the following result we shall show that the two Banach algebras $C^b(K)$ and $C^{b\varphi}(K)$ are not isomorphic algebras. The first is unital whereas the second is not.

Proposition 2.6. *The map $\phi : C^{b\varphi}(K) \longrightarrow C^b(K)$ defined by, $\phi(f) = f\varphi$ is an injective homomorphism that is not surjective.*

Proof. The linearity of ϕ is obvious. For $f, g \in C^{b\varphi}(K)$,

$$\begin{aligned} \phi(f \cdot g) &= \phi(f\varphi g) \\ &= f\varphi g\varphi \\ &= \phi(f)\phi(g). \end{aligned}$$

Let $\phi(f) = 0$. Then $f\varphi = 0$. It follows that $f\Big|_{K - \ker\varphi} = 0$. Inspired by the proof of Proposition 2.4 we can conclude that $f = 0$. This shows that ϕ is injective. Clearly $1_K \notin \text{Rang}\phi$. So ϕ is not surjective. □

Clearly each ideal I of $C^b(K)$ is an ideal of $C^{b\varphi}(K)$. Indeed, for each $f \in I$ and $g \in C^{b\varphi}(K)$, $g \cdot f = (g\varphi)f \in I$. But the converse is not the case in general. The following example shows this fact.

Example 2.1.

$$I = \langle 1_K \rangle = \left\{ f \cdot 1_K + n1_K \;\middle|\; f \in C^{b\varphi}(K), n \in \mathbb{Z} \right\}$$

is a proper ideal of $C^{b\varphi}(K)$*. Indeed,* $i1_K \in C^{b\varphi}(K)$ *and there are no* $f \in C^{b\varphi}(K)$ *and* $n \in \mathbb{Z}$ *such that* $i1_K = f \cdot 1_K + n1_K$*. So* $i1_K \notin I$*. This shows that* I *is a proper ideal of* $C^{b\varphi}(K)$ *and so it is not an ideal of* $C^b(K)$*.*

Remark 2.1. *Let* $n \in \mathbb{N}$*. Then for each ideal* I *of* $C^{b\varphi}(K)$*,*

$$I \supseteq I\varphi \supseteq I\varphi^2 \supseteq \cdots \supseteq \cdots I\varphi^n \supseteq \cdots .$$

In the following result we give the relation between zero divisor elements of $C^{b\varphi}(K)$ and $C^b(K)$.

Proposition 2.7. *g is a zero divisor in $C^{b\varphi}(K)$ if and only if g is a zero divisor in $C^b(K)$.*

Proof. Let g be a zero divisor in $C^{b\varphi}(K)$. So there exists a non-zero element $f \in C^{b\varphi}(K)$ such that $f \cdot g = 0$. It follows that $f\varphi g = 0$. Hence $fg\Big|_{K-\ker\varphi} = 0$. An argument similar to the proof of Proposition 2.4 reveals that $fg = 0$. So g is a zero divisor in $C^b(K)$. Obviously each zero divisor element in $C^b(K)$ is a zero divisor element in $C^{b\varphi}(K)$. □

3. Some relations between $\triangle(C^{b\varphi}(K))$ and $\triangle(C^b(K))$

In this section we characterize some relations between character spaces of the two Banach algebras $C^{b\varphi}(K)$ and $C^b(K)$. For each $k \in K$, define $\hat{k} : C^{b\varphi}(K) \longrightarrow \mathbb{C}$ by $\hat{k}(f) = f(k), f \in C^{b\varphi}(K)$. Obviously $K \subseteq C^{b\varphi}(K)^*$.
It is well known that for each $k \in K, \hat{k}$ is a character on $C^b(K)$. But $\hat{k}$ is not a character on $C^{b\varphi}(K)$ in general. For example, $\hat{0} \notin \triangle(C^{b\varphi}(K)) \cup \{0\}$. Indeed, as $1_K \cdot 1_K = \varphi\Big|_K$ then $\hat{0}(1_K \cdot 1_K) = \hat{0}(\varphi\Big|_K) = \varphi(0) = 0 \neq \hat{0}(1_K)\hat{0}(1_K) = 1$.

Proposition 3.1. $K \cap \varphi^{-1}(\{1\}) \subseteq \triangle(C^{b\varphi}(K))$.

Proof. Let $k \in K \cap \varphi^{-1}(\{1\})$. Then $\varphi(k) = 1$. So,

$$\begin{aligned} \hat{k}(f \cdot g) &= (f \cdot g)(k) \\ &= f(k)\varphi(k)g(k) \\ &= f(k)g(k) \\ &= \hat{k}(f)\hat{k}(g), \end{aligned}$$

for all $f, g \in C^{b\varphi}(K)$. So $\hat{k} \in \triangle(C^{b\varphi}(K))$. □

Remark 3.1. *If* $k \in K \cap \ker\varphi$ *then* $0 = \hat{k}(1_K \cdot 1_K) \neq \hat{k}(1_K)\hat{k}(1_K) = 1$*. So* $\hat{k} \notin \triangle(C^{b\varphi}(K)) \cup \{0\}$*.*

The following Proposition is a result similar to [1, Proposition 2.4 (i)].

Proposition 3.2. *If* $\psi \in \triangle(C^b(K))$ *then* $\psi(\varphi)\psi \in \triangle(C^{b\varphi}(K))$*.*

Proof. Let $\psi \in \triangle(C^b(K))$. Then

$$\begin{aligned}(\psi(\varphi)\psi)(f \cdot g) &= (\psi(\varphi)\psi)(f\varphi g) \\ &= \psi(\varphi)\psi(f\varphi g) \\ &= \psi(\varphi)\psi(f)\psi(\varphi)\psi(g) \\ &= (\psi(\varphi)\psi)(f)(\psi(\varphi)\psi)(g).\end{aligned}$$

□

Proposition 3.3. *Let $\psi \in \triangle(C^{b\varphi}(K))$. Then $\psi = 0$ if and only if $\psi(1_K) = 0$.*

Proof. Let $\psi(1_K) = 0$. Then for each $f \in C^{b\varphi}(K)$, $f \cdot f = f^2 \cdot 1_K$. So,

$$\begin{aligned}\psi(f)\psi(f) &= \psi(f \cdot f) \\ &= \psi(f^2 \cdot 1_K) \\ &= \psi(f^2)\psi(1_K) \\ &= 0.\end{aligned}$$

Hence $\psi^2(f) = 0$ and so $\psi(f) = 0$. □

Proposition 3.4. *If $\psi \in \triangle(C^{b\varphi}(K))$ then $\frac{1}{\psi(1_K)}\psi \in \triangle(C^b(K))$.*

Proof. Let $\psi \in \triangle(C^{b\varphi}(K))$ and let $f, g \in C^b(K)$. As $(fg) \cdot 1_K = f \cdot g$, so

$$\begin{aligned}\psi(fg)\psi(1_K) &= \psi((fg) \cdot 1_K) \\ &= \psi(f \cdot g) \\ &= \psi(f)\psi(g).\end{aligned}$$

It follows that $\frac{1}{\psi(1_K)}\psi(fg) = \frac{1}{\psi(1_K)}\psi(f)\frac{1}{\psi(1_K)}\psi(g)$.
Hence,
$(\frac{1}{\psi(1_K)}\psi)(fg) = (\frac{1}{\psi(1_K)}\psi)(f)(\frac{1}{\psi(1_K)}\psi)(g)$. This shows that $\frac{1}{\psi(1_K)}\psi \in \triangle(C^b(K))$. □

4. The derivations on $C^{b\varphi}(K)$ and $C^b(K)$

In this section we present some relations between the derivations on $C^{b\varphi}(K)$ and $C^b(K)$. One side of the following Proposition is similar to
[1, Proposition 4.2 (i)].

Proposition 4.1. *Let $D : C^b(K) \longrightarrow C^b(K)$ be a bounded linear map such that $D(\varphi) = 0$. Then D is a derivation on $C^b(K)$ if and only if D is a derivation on $C^{b\varphi}(K)$.*

Proof. Let $D : C^b(K) \longrightarrow C^b(K)$ be a derivation and let $f, g \in C^{b\varphi}(K)$.

$$\begin{aligned}D(f \cdot g) &= D(f\varphi g) \\ &= fD(\varphi g) + D(f)\varphi g \\ &= f(\varphi D(g) + D(\varphi)g) + D(f)\varphi g \\ &= f\varphi D(g) + D(f)\varphi g \\ &= f \cdot D(g) + D(f) \cdot g.\end{aligned}$$

So $D : C^{b\varphi}(K) \longrightarrow C^{b\varphi}(K)$ is a derivation.
Conversely, let $D : C^{b\varphi}(K) \longrightarrow C^{b\varphi}(K)$ be a derivation and let $f, g \in C^b(K)$. Then

$$\begin{aligned} 0 &= D(\varphi) \\ &= D(1_K \cdot 1_K) \\ &= 1_K \cdot D(1_K) + D(1_K) \cdot 1_K \\ &= 2D(1_K)\varphi. \end{aligned}$$

So $D(1_K)\varphi = 0$. It follows that $D(1_K)\Big|_{K-\ker\varphi} = 0$. Hence $D(1_K) = 0$. As $f \cdot g = (fg) \cdot 1_K$ so,

$$\begin{aligned} D(f)\varphi g + f\varphi D(g) &= D(f) \cdot g + f \cdot D(g) \\ &= D(f \cdot g) \\ &= D((fg) \cdot 1_K) \\ &= D(fg) \cdot 1_K + (fg) \cdot D(1_K) \\ &= D(fg) \cdot 1_K \\ &= D(fg)\varphi. \end{aligned}$$

Therefore $(D(f)g + fD(g) - D(fg))\varphi = 0$. It follows that $D(f)g+fD(g)-D(fg)\Big|_{K-\ker\varphi} = 0$. Inspired by the proof of Proposition 2.4 we can conclude that $D(f)g + fD(g) - D(fg) = 0$. So D is a derivation on $C^b(K)$. □

Remark 4.1. *One can easily check that the Banach space $C^b(K)$ is a $C^{b\varphi}(K)-$bimodule with the actions defined by,*

$$\begin{aligned} f \bullet g &= g \bullet f \\ &= f\varphi g \end{aligned}$$

for all $f \in C^{b\varphi}(K)$ and $g \in C^bK$.

Proposition 4.2. *Let $D : C^{b\varphi}(K) \longrightarrow C^b(K)$ be a bounded linear map such that $D(\varphi) = 0$. If $D : C^{b\varphi}(K) \longrightarrow C^b(K)$ is a derivation then $D : C^b(K) \longrightarrow C^b(K)$ is a derivation.*

Proof. An argument similar to Proposition 4.1 can be applied to show that $D : C^b(K) \longrightarrow C^b(K)$ is a derivation. □

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