A CLASS OF DIFFERENTIAL SYSTEMS OF DEGREE $4k+1$ WITH ALGEBRAIC AND NON ALGEBRAIC LIMIT CYCLES

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For a given family of planar differential equations it is a very difficult problem to determine an upper bound for the number of its limit cycles. In this paper we give a family of planar polynomial differential systems of degree $4k+1$ whose limit cycles can be explicitly described using polar coordinates. The given family of planar polynomial differential systems can have at most two explicit limit cycles, one of them algebraic and the other one non–algebraic.

Keywords: Planar polynomial differential system, algebraic and non–algebraic limit cycle, hyperbolicity, Riccati differential equation.

2010 Mathematics Subject Clasification: 34A05, 34C07.

1. Introduction

We consider here two-dimensional polynomial differential systems of the form

$$\begin{cases} \dot{x} = \dfrac{dx}{dt} = P(x,y), \\ \dot{y} = \dfrac{dy}{dt} = Q\left(x,y\right), \end{cases} \tag{1}$$

where P and Q are two polynomials of $R[x,y]$ ($R\left[x,y\right]$ denotes the ring of polynomials in the variables x and y with real coefficients). By definition, the degree of the system (1) is $n = \max(\deg(P), \deg(Q))$. A limit cycle of system (1) is an isolated periodic orbit and it is said to be algebraic if it is contained in the zero set of an algebraic curve, otherwise it is called non-algebraic.

An important problem of the qualitative theory of differential equations is to determine the limit cycles of a system of the form (1). We usually only ask for the number of such limit cycles, but their location as orbits of the system is also an interesting problem, and an even more difficult problem is to give an explicit expression of them. We are able to solve this last problem for a given family of systems of the form (1).

Let us recall some useful notions (for more details see [8]). For $U \in \mathbb{R}\left[x,y\right]$, the algebraic curve $U = 0$ is called an invariant curve of (1) if for some polynomial $K \in \mathbb{R}\left[x,y\right]$ called the cofactor of the algebraic curve, we have

$$P(x,y)\frac{\partial U}{\partial x} + Q(x,y)\frac{\partial U}{\partial y} = K\left(x,y\right)U\left(x,y\right).$$

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The curve $\Gamma = \{(x,y) \in \mathbb{R}[x,y] : U(x,y) = 0\}$ is non-singular of system (1) if the equilibrium points of the system that satisfy

$$\begin{cases} P(x,y) = 0 \\ Q(x,y) = 0 \end{cases},$$

are not contained on the curve Γ.

Until recently, the only limit cycles known in an explicit way where algebraic (see for instance [1], [3], [4], [7], [12], [14] and references therein). On the other hand, it seems intuitively clear that "most" limit cycles of planar polynomial vector fields have to be non algebraic. Nevertheless, until 1995 it was not proved that the limit cycle of the van der Pol equation is not algebraic (see K. Odani [13]) . In the chronological order the first explicit non-algebraic limit cycle due to Gasull, Giacomini and Torregrosa [9], was for a polynomial differential system of degree 5, after this first paper appeared the paper of Al-Dosary [2] inspired by [9], providing a similar polynomial differential system of degree 5 exhibiting an explicit non-algebraic limit cycle. In [6], an example of an explicit limit cycle which is not algebraic is given for $n = 3$.

The first result for the coexistence of algebraic and non-algebraic limit cycles goes back to J. Giné and M. Grau [10] for $n = 9$. These last authors transform their system into a Ricatti equation which is itself transformed into a variable coefficients second order linear differential equation using the classic linearization method. From the principal result of an earlier work (see details from page 5 of their paper) they obtain a first integral and also the explicit equations of the possible limit cycles. Bendjeddou and al [5] provide a polynomial differential system of degree 5 exhibiting simultaneously two explicit limit cycles one algebraic and another non-algebraic.

The aim of this paper is to show that there exist a class polynomial differential systems of degree $5, 9, ..., 4k+1, k \in \mathbb{N}^*$ exhibiting two explicit limit cycles, one of them algebraic and the other one non–algebraic.

2. Main result

As a main result, we shall prove the following theorem.

Theorem 2.1. *The differential polynomial system of degree* $n = 4k+1, k \in \mathbb{N}^*$

$$\begin{aligned} \dot{x} &= \left(\gamma x - x\left(x^2+y^2\right)^k - 2k\gamma y\right)\left(a\left(x^2+y^2\right)^k + bP_{2k}(x,y)\right) - x\left(\left(x^2+y^2\right)^k - \gamma\right)^2 \\ \dot{y} &= \left(\gamma y - y\left(x^2+y^2\right)^k + 2k\gamma x\right)\left(a\left(x^2+y^2\right)^k + bP_{2k}(x,y)\right) - y\left(\left(x^2+y^2\right)^k - \gamma\right)^2 \end{aligned} \quad (2)$$

where $a, b, \gamma \in \mathbb{R}_+^*$ *and* P_{2k} *a polynomial of degree* $2k$ *such that:*

$$P_{2k}(x,y) = \sum_{s=0}^{k-1} (-1)^s \binom{2k}{2s+1} x^{2k-2s-1} y^{2s+1}, \text{ where } \binom{2k}{2s+1} = \frac{2k!}{(2s+1)!\,(2k-2s-1)!},$$

possesses exactly two limit cycles: the circle $(\Gamma_1) : \left(x^2+y^2\right)^k - \gamma = 0$ *surrounding a transcendental and unstable limit cycle* (Γ_2) *explicitly given in polar coordinates* (r,θ) *by the equation*

$$r(\theta, r_*) = \left(\gamma + \gamma \frac{e^{-\theta}}{\frac{r_*^{2k}}{r_*^{2k}-\gamma} - e^{-\theta} + f(\theta)}\right)^{\frac{1}{2k}},$$

with $f(\theta) = \int_0^\theta \dfrac{e^{-s}}{a + b\sin 2ks} ds$ *and* $r_* = \left(\gamma \frac{f(2\pi)}{1-e^{-2\pi}+f(2\pi)}\right)^{\frac{1}{2k}}$, *when the following condition is assumed :*

$$b^2 - a^2 < 0.$$

Proof. Firstly, we have

$$y\dot{x} - x\dot{y} = -2k\gamma\left(bP_{2k}\left(x,y\right) + a\left(x^2+y^2\right)^k\right)\left(x^2+y^2\right),$$

thus, the equilibrium points of system (2) are present in the curve

$$\left(bP_{2k}\left(x,y\right) + a\left(x^2+y^2\right)^k\right)\left(x^2+y^2\right) = 0. \tag{3}$$

In polar coordinates (r,θ), defined by $x = r\cos\theta$ and $y = r\sin\theta$, (2.1) reads as

$$\begin{aligned} P_{2k}(r\cos\theta, r\sin\theta) &= \sum_{s=0}^{k-1} C_{2k}^{2s+1}\left(-1\right)^s\left(r\cos\theta\right)^{2k-2s-1}\left(r\sin\theta\right)^{2s+1} \\ &= r^{2k}\sum_{s=0}^{k-1} C_{2k}^{2s+1}\left(-1\right)^s\left(\cos\theta\right)^{2k-2s-1}\left(\sin\theta\right)^{2s+1} \\ &= r^{2k}\sin\left(2k\theta\right), \end{aligned} \tag{4}$$

then, the curve's equation (3) can be written as

$$r^{2k+2}\left(b\sin\left(2k\theta\right) + a\right) = 0.$$

Since $b^2 - a^2 < 0$ and $a \in \mathbb{R}_+^*$, $b \in \mathbb{R}_+$, it follows that $0 < a - b$ and

$$a - b < b\sin\left(2k\theta\right) + a < a + b, for\ all\ \theta \in \mathbb{R}, \tag{5}$$

according to this equation, we deduce that $0 < b\sin\left(2k\theta\right) + a$, then $r = 0$, thus the origin is the unique critical point at finite distance.

We prove that $(\Gamma_1) : \left(x^2+y^2\right)^k - \gamma = 0$ is an invariant algebraic curve of the differential system (2). Indeed, if we put

$$P\left(x,y\right) = \left(\gamma x - x\left(x^2+y^2\right)^k - 2k\gamma y\right)\left(a\left(x^2+y^2\right)^k + bP_{2k}(x,y)\right) - x\left(\left(x^2+y^2\right)^k - \gamma\right)^2,$$

$$Q\left(x,y\right) = \left(\gamma y - y\left(x^2+y^2\right)^k + 2k\gamma x\right)\left(a\left(x^2+y^2\right)^k + bP_{2k}(x,y)\right) - y\left(\left(x^2+y^2\right)^k - \gamma\right)^2,$$

$$U\left(x,y\right) = \left(x^2+y^2\right)^k - \gamma.$$

Immediately we have

$$P(x,y)\frac{\partial U(x,y)}{\partial x} + Q(x,y)\frac{\partial U(x,y)}{\partial y} = K\left(x,y\right)U\left(x,y\right),$$

where

$$K\left(x,y\right) = -2k\left(x^2+y^2\right)^k Q_{2k}\left(x,y\right),$$

and

$$Q_{2k}\left(x,y\right) = bP_{2k}\left(x,y\right) + (a+1)\left(x^2+y^2\right)^k - \gamma,$$

therefore, the circle $(\Gamma_1) : \left(x^2+y^2\right)^k - \gamma = 0$ is an invariant curve of system (2).
The curve (Γ_1) is a periodic orbit of system (2) if and only if there does not exist any singular point on (Γ_1), since the origin is the unique equilibrium point of the system of system (2) and $(\Gamma_1) : \left(x^2+y^2\right)^k - \gamma = 0$ do not pass through the origin, then (Γ_1) is a periodic orbits of system (2).
To see that Γ_1 is in fact a limit cycle, we use a classic result characterizing limit cycles among other periodic orbits (see for instance [14] for more details), which means that $\Gamma_1 = \{(x(t), y(t)), t \in [0,T]\}$ is a limit cycle when $\int_0^T div(\Gamma_1)dt \neq 0$, stable if $\int_0^T div(\Gamma_1)dt < 0$, and instable if $\int_0^T div(\Gamma_1)dt > 0$, where T be the priod of Γ_1. We use also a practical result of J. Giné and al [11], which asserts that $\int_0^T div(\Gamma_1)dt = \int_0^T K\left(x,y\right)dt$.
Note that if a periodic curve (Γ_1) is invariant for a differential system with a cofactor

$K(x, y)$ of constant sign for $(x, y) \in Int(\Gamma_1)$ where $Int(\Gamma_1)$ denotes the interior of (Γ_1), then $\int_0^T K(x, y)dt$, is automatically different from zero.
Now we shall prove that $K(x, y)$ does not intersect the orbit Γ_1, to show this, we prove that the system

$$\begin{cases} -2k\left(x^2+y^2\right)^k \left(bP_{2k} + a\left(x^2+y^2\right)^k + \left(x^2+y^2\right)^k - \gamma\right) = 0, \\ \left(x^2+y^2\right)^k - \gamma = 0, \end{cases} \tag{6}$$

has no solutions.
In polar coordinates (r, θ), system (6) reads as

$$\begin{cases} -2kr^{2k}\left(br^{2k}\sin(2k\theta) + ar^{2k} + r^{2k} - \gamma\right) = 0, \\ r^{2k} - \gamma = 0, \end{cases}$$

this system can be written as

$$-2k\gamma^2\left(b\sin(2k\theta) + a\right) = 0.$$

Since $b^2 - a^2 < 0$, then $b\sin(2k\theta) + a > 0$ for all $\theta \in \mathbb{R}$, thus,

$$-2k\gamma^2\left(b\sin(2k\theta) + a\right) < 0, \forall\theta \in \mathbb{R}, \tag{7}$$

then, the curve $K(x, y) = 0$ do not cross (Γ_1).
But $Q_{2k}(0, 0) = -\gamma < 0$, hence $Q_{2k}(x, y) < 0$ inside (Γ_1) and $K(x, y) = -2k\left(x^2+y^2\right)^k Q_{2k}(x, y) > 0$ inside $(\Gamma_1)\setminus\{(0, 0)\}$, so $\int_0^T K(x, y)\,dt > 0$, where T be the period of the periodic solution (Γ_1). Consequently (Γ_1) defines a unstable algebraic limit cycle for system (2).

The search for the non-algebraic limit cycle, requires the integration of our system. Taking into account (3), then in polar coordinates (r, θ), defined by $x = r\cos\theta$ and $y = r\sin\theta$, the system (2) can be written as the system

$$\begin{aligned} \dot{r} &= r\left(r^{2k} - \gamma\right)\left(r^{2k} + ar^{2k} + br^{2k}\sin 2k\theta - \gamma\right), \\ \dot{\theta} &= -2k\gamma\left(a + b\sin 2k\theta\right)r^{2k}. \end{aligned} \tag{8}$$

Taking as θ independent variable, we obtain the equation

$$2kr^{2k-1}\frac{dr}{d\theta} = -\frac{\left(r^{2k} - \gamma\right)\left(r^{2k} + ar^{2k} + br^{2k}\sin 2k\theta - \gamma\right)}{\gamma\left(a + b\sin 2k\theta\right)}. \tag{9}$$

Note that since we have $\gamma > 0$ and $b\sin(2k\theta) + a > 0$ for all $\theta \in \mathbb{R}$, then

$$\dot{\theta} = -2k\gamma\left(a + b\sin 2k\theta\right)r^{2k} < 0, \tag{10}$$

so, the orbits $r(\theta)$ of the differential equation (9) has reversed their orientation with respect to the orbits $(r(t), \theta(t))$ or $(x(t), y(t))$ of the differential systems (8) and (2), respectively.
Via the change of variables $\rho = r^{2k}$, the equation (9) is transformed into the Riccati equation

$$\begin{aligned} \frac{d\rho}{d\theta} &= -\frac{(\rho - \gamma)\left((1 + a + b\sin 2k\theta)\rho - \gamma\right)}{\gamma\left(a + b\sin 2k\theta\right)} \qquad (11) \\ &= -(\rho - \gamma)\left(\frac{1 + a + b\sin 2k\theta}{\gamma\left(a + b\sin 2k\theta\right)}\rho - \frac{1}{(a + b\sin 2k\theta)}\right). \end{aligned}$$

Fortunately, this equation is integrable, since it possesses the particular solution $\rho = \gamma$ corresponding of course to the limit cycle (Γ_1), the general solution of equation (11) is given by

$$\rho = \left(\gamma + \frac{1}{R}\right), \tag{12}$$

where R is a function of the variable θ. Indeed, substituting the solution $\rho = \left(\gamma + \frac{1}{R}\right)$ into Riccatti equation, we obtain the linear equation .

$$-\frac{1}{R^2}\frac{dR}{d\theta} = -\frac{1}{R}\left(\frac{1+a+b\sin 2k\theta}{\gamma\left(a+b\sin 2k\theta\right)}\left(\gamma+\frac{1}{R}\right) - \frac{1}{\left(a+b\sin 2k\theta\right)}\right), \tag{13}$$

thus

$$\frac{dR}{d\theta} = \frac{1}{\gamma} + \frac{1}{\gamma\left(a+b\sin 2k\theta\right)} + R. \tag{14}$$

The general solution of linear equation (14) is

$$R\left(\theta,k\right) = e^{\theta}\left(k - \frac{1}{\gamma}\left(e^{-\theta}-1\right) + \frac{1}{\gamma}\int_0^{\theta}\frac{e^{-s}}{a+b\sin 2ks}ds\right),$$

where $k \in \mathbb{R}$. Going back through the changes of variables (12) we obtain

$$\rho\left(\theta,k\right) = \left(\gamma + \gamma\frac{e^{-\theta}}{\gamma k + 1 - e^{-\theta} + \int_0^{\theta}\frac{e^{-s}}{a+b\sin 2ks}ds}\right),$$

if we take $h = \gamma k + 1$, then the general solution of Riccatti equation (11) is

$$\rho\left(\theta,h\right) = \gamma + \gamma\frac{e^{-\theta}}{h - e^{-\theta} + \int_0^{\theta}\frac{e^{-s}}{a+b\sin 2ks}\,ds}.$$

Consequently, the general solution of (9) is

$$r\left(\theta,h\right) = \left(\gamma + \gamma\frac{e^{-\theta}}{h - e^{-\theta} + \int_0^{\theta}\frac{e^{-s}}{a+b\sin 2ks}\,ds}\right)^{\frac{1}{2k}}.$$

By passing to Cartesian coordinates, we deduce the first integral

$$F(x,y) = \frac{\gamma e^{-\arctan\frac{y}{x}}}{\left(x^2+y^2\right)^k - \gamma} + e^{-\arctan\frac{y}{x}} - \int_0^{\arctan\frac{y}{x}}\frac{e^{-s}}{a+b\sin 2ks}ds.$$

The trajectories of system (2) are the level curves $F(x,y) = h, h \in \mathbb{R}$ and since these curves are obviously all non-algebraic (if we exclude of course the curve (Γ_1) corresponding to $k \to +\infty$), thus any other limit cycle, if exists, should also be non-algebraic.
To go a steep further, we remark that the solution such as $r(0,r_0) = r_0 > 0$, corresponds to the value $h = \frac{r_0^{2k}}{r_0^{2k}-\gamma}$ provided a rewriting of the general solution of (8) as

$$r\left(\theta,r_0\right) = \left(\gamma + \gamma\frac{e^{-\theta}}{\frac{r_0^{2k}}{r_0^{2k}-\gamma} - e^{-\theta} + f\left(\theta\right)}\right)^{\frac{1}{2k}}, \tag{15}$$

where $r_0 = r(0)$ and $f(\theta) = \int_0^{\theta}\dfrac{e^{-s}}{a+b\sin 2ks}ds.$
A periodic solution of system (2) must satisfy the condition:

$$r(2\pi,r_0) = r(0,r_0). \tag{16}$$

The equation (16) equivalent to

$$\left(r_0^{2k}-\gamma\right)\left(\frac{\gamma e^{-\theta}}{r_0^{2k} + \left(r_0^{2k}-\gamma\right)\left(-e^{-\theta}+f\left(\theta\right)\right)} - 1\right) = 0. \tag{17}$$

It is easy to check that the equation (17) admits exactly two distinct solution, the first is $r_0 = \gamma^{\frac{1}{2k}}$ corresponding obviously to the algebraic limit cycle (Γ_1), and the second value is

$$r_0 = r_* = \left(\gamma \frac{f(2\pi)}{1 - e^{-2\pi} + f(2\pi)}\right)^{\frac{1}{2k}}, \tag{18}$$

providing the value $r_* > 0$. Indeed, since $b^2 - a^2$, then, $a + b\sin 2k\theta > 0$ for all $\theta \in \mathbb{R}$, so $f(\theta) > 0$ and $1 - e^{-\theta} + f(\theta) > 0$ for all $\theta \in \mathbb{R}$ therefore, $r_* > 0$.
Injecting this value of r_* in (15), we get the candidate solution

$$r(\theta, r_*) = \left(\gamma + \gamma \frac{e^{-\theta}}{\frac{r_*^{2k}}{r_*^{2k} - \gamma} - e^{-\theta} + f(\theta)}\right)^{\frac{1}{2k}}. \tag{19}$$

To show that (19) is a periodic solution of the system (8), we have to show that : To show that it is a periodic solution, we have to show that :
i) the function $x \mapsto g(\theta)$, where in this case

$$g(\theta) = \gamma + \gamma \frac{e^{-\theta}}{\frac{f(2\pi)}{e^{-2\pi} - 1} - e^{-\theta} + f(\theta)},$$

is $2\pi-$periodic.
ii) $g(\theta) > 0$ for all $\theta \in [0, 2\pi[$. The last condition ensures that $r(\theta, r_*)$ is well defined for all $\theta \in [0; 2\pi[$ and the periodic solution do not pass through the unique equilibrium point $(0, 0)$ of system (2).

Periodicity. Let $\theta \in [0, 2\pi[$, then

$$g(\theta + 2\pi) = \gamma + \gamma \frac{e^{-\theta - 2\pi}}{\frac{f(2\pi)}{e^{-2\pi} - 1} - e^{-\theta - 2\pi} + f(\theta + 2\pi)}, \tag{20}$$

but

$$\begin{aligned} f(\theta + 2\pi) &= \int_0^{\theta + 2\pi} \frac{e^{-s}}{a + b\sin 2ks} ds \\ &= \int_0^{2\pi} \frac{e^{-s}}{a + b\sin 2ks} ds + \int_{2\pi}^{\theta + 2\pi} \frac{e^{-s}}{a + b\sin 2ks} ds \\ &= f(2\pi) + \int_{2\pi}^{\theta + 2\pi} \frac{e^{-s}}{a + b\sin 2ks} ds, \end{aligned}$$

we make the change of variable $u = s - 2\pi$ in the integral $\int_{2\pi}^{\theta + 2\pi} \frac{e^{-s}}{a + b\sin 2ks} ds$, we get

$$\begin{aligned} f(\theta + 2\pi) &= f(2\pi) + \int_0^{\theta} \frac{e^{-(u + 2\pi)}}{a + b\sin 2k(u + 2\pi)} du \\ &= f(2\pi) + e^{-2\pi} f(\theta), \end{aligned}$$

we replace $f(\theta+2\pi)$ by $f(2\pi)+e^{-2\pi}f(\theta)$ in (20), we obtain

$$\begin{aligned} g(\theta+2\pi) &= \gamma+\gamma\frac{e^{-\theta-2\pi}}{\frac{f(2\pi)}{e^{-2\pi}-1}-e^{-\theta-2\pi}+f(\theta+2\pi)} \\ &= \gamma+\gamma\frac{e^{-(\theta+2\pi)}}{\frac{f(2\pi)}{e^{-2\pi}-1}-e^{-(\theta+2\pi)}+\left(f(2\pi)+e^{-2\gamma\pi}f(\theta)\right)} \\ &= \gamma+\gamma\frac{e^{-(\theta+2\pi)}}{\frac{e^{-2\pi}}{e^{-2\pi}-1}f(2\pi)-e^{-(\theta+2\pi)}+e^{-2\gamma\pi}f(\theta)} \\ &= \gamma+\gamma\frac{e^{-(\theta+2\pi)}}{e^{-2\pi}\left(\frac{f(2\pi)}{e^{-2\pi\gamma}-1}-e^{-\theta}+f(\theta)\right)} \\ &= g(\theta), \end{aligned} \tag{21}$$

hence g is 2π-periodic.

Strict positivity of $g(\theta)$ **for all** $\theta\in[0,2\pi[$. Since $b^2-a^2<0$ and $a\in\mathbb{R}_+^*$, $b\in\mathbb{R}_+$, then $b<a$, therefore, $0<a-b<a+b\sin 2k\theta<a+b$, so $f(\theta)>0$ for all $\theta\in[0,2\pi[$, moreover we have

$$\begin{aligned} f(2\pi) &= \int_0^{2\pi}\frac{e^{-s}}{a+b\sin 2ks}ds \\ &= f(\theta)+\int_\theta^{2\pi}\frac{e^{-s}}{a+b\sin 2ks}ds, \end{aligned}$$

since $\dfrac{e^{-s}}{a+b\sin 2ks}>0$ then, $\int_\theta^{2\pi}\dfrac{e^{-s}}{a+b\sin 2ks}ds>0$, so

$$f(2\pi)\geq f(\theta)>0 \text{ pour tout } \theta\in[0,2\pi[,$$

and

$$\begin{aligned} g(\theta) &= \gamma+\gamma\frac{e^{-\theta}}{\frac{f(2\pi)}{e^{-2\pi}-1}-e^{-\theta}+f(\theta)} \\ &\geq \gamma+\gamma\frac{e^{-\theta}}{\frac{f(2\pi)}{e^{-2\pi}-1}-e^{-\theta}+f(2\pi)} \\ &= \gamma\frac{f(2\pi)}{f(2\pi)+e^{-\theta}\left(e^{2\pi}-1\right)}>0, \end{aligned}$$

hence $g(\theta)>0$ for all $\theta\in[0,2\pi[$.

Finally $r(\theta,r_*)$ defines through (19) a periodic solution. To show that it is a limit cycle, we consider (19), and introduce the Poincaré return map $r_*\mapsto P(r_*)=r(2\pi,r_*)$. to prove that the periodic solution is an isolated periodic orbit, see [8], it is sufficient for the function of Poincaré first return

$$\left.\frac{dr(2\pi,r_*)}{dr_*}\right|_{r_*=\left(\gamma\frac{f(2\pi)}{1-e^{-2\pi}+f(2\pi)}\right)^{\frac{1}{2k}}}\neq 1,$$

which is already the case because we have

$$\left.\frac{dr(2\pi,r_*)}{dr_*}\right|_{r_*=\left(\gamma\frac{f(2\pi)}{1-e^{-2\pi}+f(2\pi)}\right)^{\frac{1}{2k}}}=e^{2\pi}>1.$$

Consequently the limit cycle of the differential equation (9) is unstable and hyperbolic (see [[8]], section 1.6 for more details). Consequently, this is a stable and hyperbolic limit cycle for the differential system (2). Since the Poincaré return map do not possess other fixed points, the system (2) admit exactly two limit cycles. □

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