U.P.B. Sci. Bull., Series A, Vol. 82, Iss. 4, 2020 ISSN 1223-7027

SOME PROPERTIES OF APPROXIMATELY DUAL CONTINUOUS g-FRAMES IN HILBERT SPACES

Azam YOUSEFZADEHEYNI[1], Mohammad Reza ABDOLLAHPOUR[2]

In this paper, we introduce the notion of approximately dual continuous g-frames in Hilbert spaces and investigate some of their properties. Furthermore, we study relations between approximately dual continuous g-frames and dual continuous g-frames, and between approximately dual continuous g-frames and g-duals. Also, we introduce the concepts of Γ-approximate dual continuous g-frames and $(\Gamma, \| \Gamma \|)$-approximate dual continuous g-frames, where $\Gamma \in B(H)$. Finally, we discuss Q-duals and Q-approximate dual continuous g-frame, where $Q \in B(\widehat{K})$.

Keywords: continuous g-frame, approximately dual continuous g-frame, dual continuous g-frame, g-dual, Q-dual.
MSC(2010): primary: 42C15; secondary: 47A05, 42C40.

1 Introduction

The concept of frame for a Hilbert space was introduced by Duffin and Schaeffer [8] in 1952. They used frames as tools in the study of nonharmonic Fourier analysis. In 2006, the g-frame as a generalization of frame was introduced and investigated by Sun [19]. The notion of continuous frames was introduced by Kaiser in [12] and independently by Ali, Antoine and Cazeau [3]. Gabardo and Han in [9] defined the concept of dual frames for the continuous frames. In 2008, the notion of continuous g-frame was introduced by Abdollahpour and Faroughi [1]. Approximately dual frames were defined by Christensen and Laugensen in [5].

Throughout this paper, H is a complex Hilbert space, (Ω, μ) is a measure space with positive measure μ and $\{K_w\}_{w\in\Omega}$ is a family of Hilbert spaces. We denote the space of all bounded linear operators from H into K_w by $B(H, K_w)$ and we denote $B(H, H)$ by $B(H)$.

Definition 1.1. *Let $F \in \prod_{w\in\Omega} K_w$. We say that F is strongly measurable, if F as a mapping of Ω to $\bigoplus_{w\in\Omega} K_w$ is measurable, where*

[1] Department of Mathematics, Faculty of Sciences, University of Mohaghegh Ardabili, Ardabil 56199-11367, Iran, E-mail: azamyousefzadeh@gmail.com; azam_yousefzadeh@yahoo.com

[2] Department of Mathematics, Faculty of Sciences, University of Mohaghegh Ardabili, Ardabil 56199-11367, Iran, E-mail: mrabdollahpour@yahoo.com; m.abdollah@uma.ac.ir

$$\prod_{w\in\Omega} K_w = \{f:\Omega \longrightarrow \bigcup_{w\in\Omega} K_w : f(w) \in K_w\}.$$

Definition 1.2. *We say that* $\Lambda = \{\Lambda_w \in B(H, K_w): w \in \Omega\}$ *is a continuous* g-*frame for* H *with respect to* $\{K_w\}_{w\in\Omega}$ *if*
(1) for each $f \in H$, $\{\Lambda_w f\}_{w\in\Omega}$ is strongly measurable,
(2) there are two constants $0 < A_\Lambda \leq B_\Lambda < \infty$ such that

$$A_\Lambda ||f||^2 \leq \int_\Omega ||\Lambda_w f||^2 d\mu(w) \leq B_\Lambda ||f||^2, \quad f \in H. \tag{1.1}$$

We call A_Λ and B_Λ the lower and upper continuous g-frame bounds, respectively. Λ is called an A_Λ-tight continuous g-frame if $A_\Lambda = B_\Lambda$ and a Parseval continuous g-frame if $A_\Lambda = B_\Lambda = 1$. If the right hand inequality in (1.1) holds for all $f \in H$, we say that Λ is a continuous g-Bessel family for H with respect to $\{K_w\}_{w\in\Omega}$ with the bound B_Λ.

Proposition 1.3. *[1] Let* $\Lambda = \{\Lambda_w \in B(H, K_w): w \in \Omega\}$ *be a continuous* g-*frame for* H *with respect to* $\{K_w\}_{w\in\Omega}$. *Then, there exists a unique positive and invertible operator* $S_\Lambda: H \longrightarrow H$ *such that for each* $f, g \in H$,

$$\langle S_\Lambda f, g\rangle = \int_\Omega \langle \Lambda_w^* \Lambda_w f, g\rangle d\mu(w),$$

and $A_\Lambda I_H \leq S_\Lambda \leq B_\Lambda I_H$, where I_H is the identity operator on H.

The operator S_Λ is called the continuous g-frame operator of Λ. Also, we have

$$\langle f, g\rangle = \int_\Omega \langle S_\Lambda^{-1} f, \Lambda_w^* \Lambda_w g\rangle d\mu(w) = \int_\Omega \langle f, \Lambda_w^* \Lambda_w S_\Lambda^{-1} g\rangle d\mu(w), \quad f, g \in H. \tag{1.2}$$

Let the space

$$\widehat{K} = \{F \in \prod_{w\in\Omega} K_w : F \text{ is strongly measurable}, \int_\Omega ||F(w)||^2 d\mu(w) < \infty\}.$$

Obviously, $\widehat{K}$ is a Hilbert space with pointwise operations and the inner product given by

$$\langle F, G\rangle = \int_\Omega \langle F(w), G(w)\rangle d\mu(w), \quad F, G \in \widehat{K}.$$

Proposition 1.4. *[1] Let* $\Lambda = \{\Lambda_w \in B(H, K_w): w \in \Omega\}$ *be a continuous* g-*Bessel family for* H *with respect to* $\{K_w\}_{w\in\Omega}$. *Then, the mapping* $T_\Lambda: \widehat{K} \longrightarrow H$ *defined by*

$$\langle T_\Lambda F, g\rangle = \int_\Omega \langle \Lambda_w^* F(w), g\rangle d\mu(w), \quad F \in \widehat{K}, g \in H,$$

is a linear and bounded operator with $||T_\Lambda|| \leq \sqrt{B_\Lambda}$. Moreover, for any $g \in H$ and $w \in \Omega$,

$$T_\Lambda^*(g)(w) = \Lambda_w g.$$

The operators T_Λ and T_Λ^* in Proposition 1.4 are called the synthesis and analysis operators of Λ, respectively.

Definition 1.5. *If $\Lambda = \{\Lambda_w \in B(H, K_w): w \in \Omega\}$ and $\Theta = \{\Theta_w \in B(H, K_w): w \in \Omega\}$ are two continuous g-Bessel families for H with respect to $\{K_w\}_{w\in\Omega}$, such that*

$$\langle f, g\rangle = \int_\Omega \langle \Theta_w f, \Lambda_w g\rangle d\mu(w), \quad f, g \in H,$$

then Θ is called a dual continuous g-frame of Λ.

Let $\Lambda = \{\Lambda_w \in B(H, K_w): w \in \Omega\}$be a continuous$g$-frame for H with respect to $\{K_w\}_{w\in\Omega}$. Then $\widetilde{\Lambda} = \{\Lambda_w S_\Lambda^{-1} \in B(H, K_w): w \in \Omega\}$ is a continuous g-frame for H with respect to $\{K_w\}_{w\in\Omega}$ and by (1.2), $\widetilde{\Lambda}$ is a dual continuous g-frame of Λ. We call $\widetilde{\Lambda}$ the canonical dual of Λ. Also, if $\Lambda = \{\Lambda_w \in B(H, K_w): w \in \Omega\}$ is a continuous g-frame for H with respect to $\{K_w\}_{w\in\Omega}$, then $\Lambda S_\Lambda^{-\frac{1}{2}} = \{\Lambda_w S_\Lambda^{-\frac{1}{2}} \in B(H, K_w): w \in \Omega\}$ is a Parseval continuous g-frame for H with respect to $\{K_w\}_{w\in\Omega}$.

Theorem 1.6. *[6] For $T, U \in B(H)$ the following conditions are equivalent:*
(1) $Range(T) \subset Range(U)$,
(2) $TT^* \leq \lambda^2 UU^*$ for some $\lambda \geq 0$,
(3) there exists $K \in B(H)$ such that $T = UK$.

2 Approximately dual continuous g-frames

In this section we introduce the notion of approximately dual continuous g-frames in Hilbert spaces and we extend some results of [5], [11], [13], [14], [15], [16] and [18] to the continuous g-frames.

Definition 2.1. *Suppose that $\Lambda = \{\Lambda_w \in B(H, K_w): w \in \Omega\}$ and $\Theta = \{\Theta_w \in B(H, K_w): w \in \Omega\}$ are continuous g-Bessel families for H with respect to $\{K_w\}_{w\in\Omega}$. We say that Λ and Θ are approximately dual continuous g-frames for H, if*

$$\| I_H - T_\Lambda T_\Theta^* \| < 1 \quad or \quad \| I_H - T_\Theta T_\Lambda^* \| < 1.$$

In this case, we call Θ (resp. Λ) an approximate dual continuous g-frame of Λ (resp. Θ).

Theorem 2.2. *If $\Lambda = \{\Lambda_w \in B(H, K_w): w \in \Omega\}$ and $\Theta = \{\Theta_w \in B(H, K_w): w \in \Omega\}$ are approximately dual continuous g-frames for H, then both Λ and Θ are continuous g-frames for H with respect to $\{K_w\}_{w\in\Omega}$ with lower bounds $B_\Lambda^{-1}(1 - \| I_H - T_\Lambda T_\Theta^* \|)^2$ and $B_\Theta^{-1}(1 - \| I_H - T_\Theta T_\Lambda^* \|)^2$, respectively.*

Proof. Since $\| I_H - T_\Lambda T_\Theta^* \| < 1$, by [4, Theorem A.5.3], the operator $T_\Lambda T_\Theta^*$ is an invertible operator on H and

$$\| (T_\Lambda T_\Theta^*)^{-1} \| \leq \frac{1}{1 - \| I_H - T_\Lambda T_\Theta^* \|}.$$

For all $f \in H$, we have

$$\| f \| = \| (T_\Lambda T_\Theta^*)^{-1} (T_\Lambda T_\Theta^*) f \| \leq \frac{\sqrt{B_\Lambda}}{1 - \| I_H - T_\Lambda T_\Theta^* \|} \left(\int_\Omega \| \Theta_w f \|^2 \, d\mu(w) \right)^{\frac{1}{2}},$$

hence Θ is a continuous g-frame for H with respect to $\{K_w\}_{w\in\Omega}$ with the lower bound $B_\Lambda^{-1}(1 - \| I_H - T_\Lambda T_\Theta^* \|)^2$. Similarly, Λ is a continuous g-frame for H with respect to $\{K_w\}_{w\in\Omega}$.

□

We mention that the sum of two continuous g-frames is not necessarily a continuous g-frame. Here, we show that the sum of two approximately dual continuous g-frames is a continuous g-frame.

Theorem 2.3. *If $\Lambda = \{\Lambda_w \in B(H, K_w): w \in \Omega\}$ and $\Theta = \{\Theta_w \in B(H, K_w): w \in \Omega\}$ are approximately dual continuous g-frames for H, then $\Lambda + \Theta = \{\Lambda_w + \Theta_w \in B(H, K_w): w \in \Omega\}$ is a continuous g-frame for H with respect to $\{K_w\}_{w\in\Omega}$.*

Proof. It is clear that $\Lambda + \Theta$ is a continuous g-Bessel family for H with respect to $\{K_w\}_{w\in\Omega}$ with the bound $2B_\Lambda + 2B_\Theta$. On the other hand, if

$$\| I_H - T_\Theta T_\Lambda^* \| < 1 \quad or \quad \| I_H - T_\Lambda T_\Theta^* \| < 1,$$

then

$$\| 2I_H - (T_\Theta T_\Lambda^* + T_\Lambda T_\Theta^*) \| < 2. \tag{2.1}$$

The operator $T_\Theta T_\Lambda^* + T_\Lambda T_\Theta^*$ is a self-adjoint operator, thus by [17, Lemma 2.2.2] and (2.1), the operator $T_\Theta T_\Lambda^* + T_\Lambda T_\Theta^*$ is a positive operator. We have

$$\begin{aligned}
\int_\Omega \| (\Lambda_w + \Theta_w) f \|^2 \, d\mu(w) &= \int_\Omega \langle (\Lambda_w + \Theta_w) f, (\Lambda_w + \Theta_w) f \rangle d\mu(w) \\
&= \int_\Omega \| \Lambda_w f \|^2 \, d\mu(w) + \langle (T_\Theta T_\Lambda^* + T_\Lambda T_\Theta^*) f, f \rangle \\
&\quad + \int_\Omega \| \Theta_w f \|^2 \, d\mu(w) \\
&\geq \int_\Omega \| \Lambda_w f \|^2 \, d\mu(w) + \int_\Omega \| \Theta_w f \|^2 \, d\mu(w) \\
&\geq (A_\Lambda + A_\Theta) \| f \|^2, \quad f \in H,
\end{aligned}$$

so the lower bound condition holds.

□

Theorem 2.4. *Let $\Lambda = \{\Lambda_w \in B(H, K_w): w \in \Omega\}$ be a continuous g-frame and $\Theta = \{\Theta_w \in B(H, K_w): w \in \Omega\}$ be a continuous g-Bessel family for H with respect to $\{K_w\}_{w\in\Omega}$. Then, Λ and Θ are approximately dual continuous g-frames for H if and only if there exists $D \in B(H)$ such that $T_\Lambda T_\Theta^* = S_\Lambda^{\frac{1}{2}} D$ and $\|$*

$I_H - S_\Lambda^{\frac{1}{2}} D \| < 1$.

Proof. First, suppose that Λ and Θ are approximately dual continuous g-frames for H. For each $f \in H$,

$$\| T_\Theta T_\Lambda^* f \| = \sup_{\|g\|=1} |\langle T_\Theta T_\Lambda^* f, g\rangle| \leq \sqrt{B_\Theta} (\int_\Omega \| \Lambda_w f \|^2 \, d\mu(w))^{\frac{1}{2}}.$$

Therefore

$$\langle (T_\Lambda T_\Theta^*)(T_\Lambda T_\Theta^*)^* f, f\rangle \leq B_\Theta \langle S_\Lambda f, f\rangle, \quad f \in H.$$

So,

$$(T_\Lambda T_\Theta^*)(T_\Lambda T_\Theta^*)^* \leq B_\Theta S_\Lambda^{\frac{1}{2}} S_\Lambda^{\frac{1}{2}}. \tag{2.2}$$

By (2.2) and Theorem 1.6, there exists $D \in B(H)$ such that $T_\Lambda T_\Theta^* = S_\Lambda^{\frac{1}{2}} D$. Conversely, since

$$\| I_H - T_\Lambda T_\Theta^* \| = \| I_H - S_\Lambda^{\frac{1}{2}} D \| < 1,$$

Λ and Θ are approximately dual continuous g-frames for H. □

Theorem 2.5. *Let $\Lambda = \{\Lambda_w \in B(H, K_w): w \in \Omega\}$ be a continuous g-frame and $\Theta = \{\Theta_w \in B(H, K_w): w \in \Omega\}$ be a continuous g-Bessel family for H with respect to $\{K_w\}_{w\in\Omega}$. Then, Λ and Θ are approximately dual continuous g-frames for H if and only if $\Theta = \Lambda S_\Lambda^{-\frac{1}{2}} D + \Gamma$, where $D \in B(H)$ is such that $\| I_H - S_\Lambda^{\frac{1}{2}} D \| < 1$ and $\Gamma = \{\Gamma_w \in B(H, K_w): w \in \Omega\}$ is a continuous g-Bessel family for H with respect to $\{K_w\}_{w\in\Omega}$ such that $T_\Lambda T_\Gamma^* = 0$.*

Proof. First, we consider $\Theta = \Lambda S_\Lambda^{-\frac{1}{2}} D + \Gamma$, $D \in B(H)$, $\| I_H - S_\Lambda^{\frac{1}{2}} D \| < 1$ and Γ is a continuous g-Bessel family for H with respect to $\{K_w\}_{w\in\Omega}$ such that $T_\Lambda T_\Gamma^* = 0$. Then

$$T_\Theta^* f = T_\Lambda^* (S_\Lambda^{-\frac{1}{2}} D f) + T_\Gamma^* f, \quad f \in H,$$

and

$$T_\Lambda T_\Theta^* f = T_\Lambda T_\Lambda^* (S_\Lambda^{-\frac{1}{2}} D f) + T_\Lambda T_\Gamma^* f = S_\Lambda^{\frac{1}{2}} D f, \quad f \in H.$$

Thus $T_\Lambda T_\Theta^* = S_\Lambda^{\frac{1}{2}} D$, and hence

$$\| I_H - T_\Lambda T_\Theta^* \| = \| I_H - S_\Lambda^{\frac{1}{2}} D \| < 1,$$

that is, Λ and Θ are approximately dual continuous g-frames for H.

Conversely, let Λ and Θ be approximately dual continuous g-frames for H. By Theorem 2.4, there exists $D \in B(H)$ such that $T_\Lambda T_\Theta^* = S_\Lambda^{\frac{1}{2}} D$ and $\| I_H - S_\Lambda^{\frac{1}{2}} D \| < 1$. Put $\Gamma = \Theta - \Lambda S_\Lambda^{-\frac{1}{2}} D$. Γ is a continuous g-Bessel family for H with respect to

$\{K_w\}_{w\in\Omega}$, since

$$\int_{\Omega} \| \Gamma_w f \|^2 \, d\mu(w) \leq 2(B_\Theta + \| D \|^2) \| f \|^2, \quad f \in H.$$

For any $f \in H$, we have

$$T_\Gamma^* f = T_\Theta^* f - T_\Lambda^* S_\Lambda^{-\frac{1}{2}} D f.$$

Therefore

$$T_\Lambda T_\Gamma^* f = T_\Lambda T_\Theta^* f - T_\Lambda T_\Lambda^* S_\Lambda^{-\frac{1}{2}} D f = 0, \quad f \in H,$$

and so, $T_\Lambda T_\Gamma^* = 0$. Moreover, $\Theta = \Lambda S_\Lambda^{-\frac{1}{2}} D + \Gamma$.

□

Corollary 2.6. *Let $\Lambda = \{\Lambda_w \in B(H, K_w) : w \in \Omega\}$ be a continuous g-frame and $\Theta = \{\Theta_w \in B(H, K_w) : w \in \Omega\}$ be a continuous g-Bessel family for H with respect to $\{K_w\}_{w\in\Omega}$. Then Λ and Θ are approximately dual continuous g-frames for H if and only if*

$$\Theta = \Lambda S_\Lambda^{-\frac{1}{2}} D + \Gamma,$$

where $D \in B(H)$ for which $\| S_\Lambda^{-\frac{1}{2}} - D \| < \frac{1}{\sqrt{B_\Lambda}}$ and $\Gamma = \{\Gamma_w \in B(H, K_w) : w \in \Omega\}$ is a continuous g-Bessel family for H with respect to $\{K_w\}_{w\in\Omega}$ such that $T_\Lambda T_\Gamma^* = 0$.

Theorem 2.7. *Let $\Lambda = \{\Lambda_w \in B(H, K_w) : w \in \Omega\}$ be a continuous g-frame and $\Theta = \{\Theta_w \in B(H, K_w) : w \in \Omega\}$ be a g-Bessel family for H with respect to $\{K_w\}_{w\in\Omega}$. Then Λ and Θ are approximately dual continuous g-frames for H if and only if*

$$\Theta = \Lambda S_\Lambda^{-\frac{1}{2}} D - \Lambda + \Gamma S_\Lambda,$$

where $D \in B(H)$ for which $\| I_H - S_\Lambda^{\frac{1}{2}} D \| < 1$ and $\Gamma = \{\Gamma_w \in B(H, K_w) : w \in \Omega\}$ is a continuous g-Bessel family for H with respect to $\{K_w\}_{w\in\Omega}$ such that Γ is a dual continuous g-frame of Λ.

Proof. First, we consider $\Theta = \Lambda S_\Lambda^{-\frac{1}{2}} D - \Lambda + \Gamma S_\Lambda$, $D \in B(H)$, $\| I_H - S_\Lambda^{\frac{1}{2}} D \| < 1$ and Γ is a continuous g-Bessel family for H with respect to $\{K_w\}_{w\in\Omega}$ such that Γ is a dual continuous g-frame of Λ. We have

$$T_\Theta^* f = T_\Lambda^* S_\Lambda^{-\frac{1}{2}} D f - T_\Lambda^* f + T_\Gamma^* S_\Lambda f, \quad f \in H.$$

Then for each $f \in H$,

$$T_\Lambda T_\Theta^* f = T_\Lambda T_\Lambda^* S_\Lambda^{-\frac{1}{2}} D f - T_\Lambda T_\Lambda^* f + T_\Lambda T_\Gamma^* S_\Lambda f = S_\Lambda^{\frac{1}{2}} D f.$$

So, $T_\Lambda T_\Theta^* = S_\Lambda^{\frac{1}{2}} D$. Therefore, by Theorem 2.4, Λ and Θ are approximately dual continuous g-frames for H.
Conversely, let Λ and Θ be approximately dual continuous g-frames for H. By Theorem 2.4, there exists $D \in B(H)$ such that $T_\Lambda T_\Theta^* = S_\Lambda^{\frac{1}{2}} D$ and $\| I_H - S_\Lambda^{\frac{1}{2}} D \| < 1$. Put

$$\Gamma = \Theta S_\Lambda^{-1} - \Lambda S_\Lambda^{-\frac{1}{2}} D S_\Lambda^{-1} + \Lambda S_\Lambda^{-1}.$$

Γ is a continuous g-Bessel family for H with respect to $\{K_w\}_{w\in\Omega}$, since

$$\int_\Omega \| \Gamma_w f \|^2 \, d\mu(w) \le 3(B_\Theta + \| D \|^2 + B_\Lambda) \| S_\Lambda^{-1} \|^2 \| f \|^2, \quad f \in H.$$

We have

$$T_\Gamma^* f = T_\Theta^* S_\Lambda^{-1} f - T_\Lambda^* S_\Lambda^{-\frac{1}{2}} D S_\Lambda^{-1} f + T_\Lambda^* S_\Lambda^{-1} f, \quad f \in H,$$

then

$$T_\Lambda T_\Gamma^* f = S_\Lambda^{\frac{1}{2}} D S_\Lambda^{-1} f - S_\Lambda^{\frac{1}{2}} D S_\Lambda^{-1} f + I_H f = f, \quad f \in H.$$

Thus $T_\Lambda T_\Gamma^* = I_H$. That is, Γ is a dual continuous g-frame of Λ. Furthermore,
$\Theta = \Lambda S_\Lambda^{-\frac{1}{2}} D - \Lambda + \Gamma S_\Lambda$. □

Theorem 2.8. *Suppose that $\Lambda = \{\Lambda_w \in B(H, K_w) : w \in \Omega\}$ is an approximate dual continuous g-frame of $\Theta = \{\Theta_w \in B(H, K_w) : w \in \Omega\}$.*
(1) If U is an isometric operator on H, then $\Lambda U = \{\Lambda_w U \in B(H, K_w) : w \in \Omega\}$ is an approximate dual continuous g-frame of $\Theta U = \{\Theta_w U \in B(H, K_w) : w \in \Omega\}$.
(2) If U is a co-isometric operator on H, then $\Lambda U^* = \{\Lambda_w U^* \in B(H, K_w) : w \in \Omega\}$ is an approximate dual continuous g-frame of $\Theta U^* = \{\Theta_w U^* \in B(H, K_w) : w \in \Omega\}$.
Proof. The proof is easy and we omit it. □

Proposition 2.9. *Assume that $\Lambda = \{\Lambda_w \in B(H, K_w) : w \in \Omega\}$ and $\Theta = \{\Theta_w \in B(H, K_w) : w \in \Omega\}$ are approximately dual continuous g-frames for H. For fixed $N \in \mathbb{N}$, consider the corresponding partial sum,*

$$\Gamma_w^{(N)} = \Theta_w + \sum_{n=1}^{N} \Theta_w (I_H - T_\Lambda T_\Theta^*)^n.$$

Then $\Gamma^{(N)} = \{\Gamma_w^{(N)} \in B(H, K_w) : w \in \Omega\}$ is an approximate dual continuous g-frame of Λ and

$$\| I_H - L_N T_\Lambda^* \| \le \| I_H - T_\Theta T_\Lambda^* \|^{N+1} < 1,$$

where L_N is the synthesis operator of $\Gamma^{(N)}$.

Proof. $\Gamma^{(N)}$ is a continuous g-Bessel family for H with respect to $\{K_w\}_{w\in\Omega}$, since

$$\int_{\Omega} \| \Gamma_w^{(N)} f \|^2 \, d\mu(w) \leq (N+1)B_\Theta \sum_{n=0}^{N} (1+\sqrt{B_\Lambda B_\Theta})^{2n} \| f \|^2 , \quad f \in H.$$

Moreover,

$$\langle L_N T_\Lambda^* f, g\rangle = \langle T_\Lambda^* f, L_N^* g\rangle = \langle \sum_{n=0}^{N} (I_H - T_\Theta T_\Lambda^*)^n T_\Theta T_\Lambda^* f, g\rangle, \quad f, g \in H.$$

So,

$$\begin{aligned} L_N T_\Lambda^* f &= \textstyle\sum_{n=0}^{N} (I_H - T_\Theta T_\Lambda^*)^n T_\Theta T_\Lambda^* f \\ &= \textstyle\sum_{n=0}^{N} (I_H - T_\Theta T_\Lambda^*)^n (I_H - (I_H - T_\Theta T_\Lambda^*)) f \\ &= f - (I_H - T_\Theta T_\Lambda^*)^{N+1} f, \quad f \in H. \end{aligned}$$

Therefore,

$$\| I_H - L_N T_\Lambda^* \| = \| (I_H - T_\Theta T_\Lambda^*)^{N+1} \| \leq \| I_H - T_\Theta T_\Lambda^* \|^{N+1} < 1. \qquad \square$$

Proposition 2.10. *Let $\Lambda = \{\Lambda_w \in B(H, K_w) : w \in \Omega\}$ be a continuous g-frame for H with respect to $\{K_w\}_{w\in\Omega}$. Then Λ and $\frac{2}{A_\Lambda + B_\Lambda}\Lambda = \{\frac{2}{A_\Lambda + B_\Lambda}\Lambda_w \in B(H, K_w) : w \in \Omega\}$ are approximately dual continuous g-frames for H.*

Proof. By assumption, we have $A_\Lambda I_H \leq S_\Lambda \leq B_\Lambda I_H$. Therefore,

$$-\frac{B_\Lambda - A_\Lambda}{A_\Lambda + B_\Lambda} I_H \leq I_H - \frac{2}{A_\Lambda + B_\Lambda} S_\Lambda \leq \frac{B_\Lambda - A_\Lambda}{A_\Lambda + B_\Lambda} I_H,$$

which implies

$$\| I_H - \frac{2}{A_\Lambda + B_\Lambda} S_\Lambda \| = \sup_{\|f\|=1} |\langle (I_H - \frac{2}{A_\Lambda + B_\Lambda} S_\Lambda) f, f\rangle| \leq \frac{B_\Lambda - A_\Lambda}{A_\Lambda + B_\Lambda} < 1.$$

It means that Λ and $\frac{2}{A_\Lambda + B_\Lambda}\Lambda$ are approximately dual continuous g-frames for H. □

Proposition 2.11. *Let $\Lambda = \{\Lambda_w \in B(H, K_w) : w \in \Omega\}$ be a continuous g-frame for H with respect to $\{K_w\}_{w\in\Omega}$. Then Λ and $B_\Lambda^{-1}\Lambda = \{B_\Lambda^{-1}\Lambda_w \in B(H, K_w) : w \in \Omega\}$ are approximately dual continuous g-frames for H.*

Proof. We have $A_\Lambda I_H \leq S_\Lambda \leq B_\Lambda I_H$. Therefore,

$$0 \leq I_H - B_\Lambda^{-1} S_\Lambda \leq \frac{B_\Lambda - A_\Lambda}{B_\Lambda} I_H.$$

Hence

$$\| I_H - B_\Lambda^{-1} S_\Lambda \| = \sup_{\|f\|=1} |\langle (I_H - B_\Lambda^{-1} S_\Lambda) f, f\rangle| \leq \frac{B_\Lambda - A_\Lambda}{B_\Lambda} < 1,$$

and so Λ and $B_\Lambda^{-1}\Lambda$ are approximately dual continuous g-frames for H. □

In [7], Dehghan and Hasankhani Fard defined the concept of g-duals of a frame in the Hilbert space. Recently, Abdollahpour and Khedmati in [2] extended this concept to the continuous g-frames, as follows:

Definition 2.12. *Let* $\Lambda = \{\Lambda_w \in B(H, K_w): w \in \Omega\}$ *and* $\Theta = \{\Theta_w \in B(H, K_w): w \in \Omega\}$ *be two continuous* g*-Bessel families for* H *with respect to* $\{K_w\}_{w\in\Omega}$. *The family* Θ *is called a* g*-dual of* Λ, *whenever* $T_\Theta T_\Lambda^*$ *is an invertible operator.*

Proposition 2.13. *If the continuous* g*-Bessel families* $\Lambda = \{\Lambda_w \in B(H, K_w): w \in \Omega\}$ *and* $\Theta = \{\Theta_w \in B(H, K_w): w \in \Omega\}$ *are approximately dual continuous* g*-frames for* H, *then* Λ *and* Θ *are* g*-duals.*

Proof. It is easy and we remove it. □

The following examples show that the converse of the statement in Proposition 2.13 is not true.

Example 2.14. *Let* $\Lambda = \{\Lambda_w \in B(H, K_w): w \in \Omega\}$ *and* $\Theta = \{\Theta_w \in B(H, K_w): w \in \Omega\}$ *be dual continuous* g*-frames for* H. *Then* 3Λ *is a* g*-dual of* Θ. *But* 3Λ *and* Θ *are not approximately dual continuous* g*-frames for* H.

In fact

$$\langle T_{3\Lambda} T_\Theta^* f, g\rangle = \int_\Omega \langle 3\Lambda_w^* \Theta_w f, g\rangle d\mu(w) = 3\langle T_\Lambda T_\Theta^* f, g\rangle = \langle 3f, g\rangle, \quad f, g \in H.$$

So $T_{3\Lambda} T_\Theta^* = 3T_\Lambda T_\Theta^* = 3I_H$, therefore $T_{3\Lambda} T_\Theta^*$ is an invertible operator. But

$$\| f - T_{3\Lambda} T_\Theta^* f \| = 2 \| f \|, \quad f \in H.$$

Thus, 3Λ and Θ are not approximately dual continuous g-frames for H.

Example 2.15. *Let* $\Lambda = \{\Lambda_w \in B(H, K_w): w \in \Omega\}$ *be a continuous* g*-frame for* H *with respect to* $\{K_w\}_{w\in\Omega}$. *Then,*

(1) Every dual continuous g-frame of Λ is an approximate dual continuous g-frame of Λ.

(2) If Λ is a Parseval continuous g-frame for H with respect to $\{K_w\}_{w\in\Omega}$, then Λ is an approximate dual continuous g-frame of itself.

(3) Suppose that $\Theta_i = \{\Theta_w^i \in B(H, K_w): w \in \Omega\}$ is a continuous g-Bessel family for H with respect to $\{K_w\}_{w\in\Omega}$ and Λ is a dual continuous g-frame of Θ_i for $i \in \mathbb{I}$, where $\mathbb{I}$ is a finite set of natural numbers. Suppose that $\{c_i\}_{i\in\mathbb{I}}$ is a sequence of complex numbers such that $\sum_{i\in\mathbb{I}} c_i \neq 0$. It was proved in [2], that Λ and $\Gamma = \left\{\Gamma_w = \sum_{i\in\mathbb{I}} c_i \Theta_w^i \in B(H, K_w): w \in \Omega\right\}$ are g-duals and $T_\Lambda T_\Gamma^* f = \sum_{i\in\mathbb{I}} c_i f$ for any $f \in H$. Here we show that Λ and Γ are not approximately dual continuous g-frames for H. In fact,

$$\| f - T_\Lambda T_\Gamma^* f \| = \| f - \sum_{i\in\mathbb{I}} c_i f \| \leq (1 + \sum_{i\in\mathbb{I}} |c_i|) \| f \|, \quad f \in H.$$

The following proposition gives a sufficient condition for approximately dual continuous g-frames to be dual continuous g-frames.

Proposition 2.16. *Let $\Lambda = \{\Lambda_w \in B(H, K_w): w \in \Omega\}$ and $\Theta = \{\Theta_w \in B(H, K_w): w \in \Omega\}$ be two continuous g-Bessel families for H with respect to $\{K_w\}_{w\in\Omega}$. Then Λ and Θ are approximately dual continuous g-frames for H if and only if there exists an invertible operator U on H with $\| I_H - U \| < 1$ such that Λ and $\Theta U^{-1} = \{\Theta_w U^{-1} \in B(H, K_w): w \in \Omega\}$ are dual continuous g-frames.*

Proof. Since Λ and Θ are approximately dual continuous g-frames for H, we have
$\| I_H - T_\Lambda T_\Theta^* \| < 1$, and hence the operator $T_\Lambda T_\Theta^*$ is an invertible operator on H. It is sufficient to put $U = T_\Lambda T_\Theta^*$. For each $f, g \in H$, we have

$$\langle f, g\rangle = \langle (T_\Lambda T_\Theta^*)(T_\Lambda T_\Theta^*)^{-1} f, g\rangle = \int_\Omega \langle \Theta_w U^{-1} f, \Lambda_w g\rangle d\mu(w),$$

hence Λ and ΘU^{-1} are dual continuous g-frames.
For the converse, suppose that there exists an invertible operator U on H with $\| I_H - U \| < 1$ such that Λ and ΘU^{-1} are dual continuous g-frames. So,

$$\| I_H - T_\Lambda T_\Theta^* \| = \| I_H - (T_\Lambda T_\Theta^* U^{-1}) U \| = \| I_H - U \| < 1,$$

therefore, Λ and Θ are approximately dual continuous g-frames for H. □

In [15], Mirzaee Azandaryani introduced the notion of Γ-approximate dual g-frames and $(\Gamma, \| \Gamma \|)$-approximate dual g-frames, where $\Gamma \in B(H)$. In the following, we generalize these concepts to the continuous g-frames.

Definition 2.17. *Let $\Lambda = \{\Lambda_w \in B(H, K_w): w \in \Omega\}$ and $\Theta = \{\Theta_w \in B(H, K_w): w \in \Omega\}$ be two continuous g-Bessel families for H with respect to $\{K_w\}_{w\in\Omega}$ and $\Gamma \in B(H)$. Then*
(1) Θ is called a Γ-approximate dual continuous g-frame of Λ, if $\| \Gamma - T_\Lambda T_\Theta^* \| < 1$.
(2) Θ is called a $(\Gamma, \| \Gamma \|)$-approximate dual continuous g-frame of Λ, if $\| \Gamma - T_\Lambda T_\Theta^* \| < \| \Gamma \|$,
where $\| \Gamma \| < 1$.

Proposition 2.18. *Let $\Lambda = \{\Lambda_w \in B(H, K_w): w \in \Omega\}$ be a continuous g-frame for H with respect to $\{K_w\}_{w\in\Omega}$. Let $\Gamma \in B(H)$ be a positive operator. Then*
(1) If Λ is a $\| \Gamma \|$-tight continuous g-frame for H with respect to $\{K_w\}_{w\in\Omega}$ and $\| \Gamma \| < 1$,
then Λ is a $(\Gamma, \| \Gamma \|)$-approximate dual continuous g-frame of itself.
(2) If Λ is an A_Λ-tight continuous g-frame for H with respect to $\{K_w\}_{w\in\Omega}$ and $\|$

$\Gamma \| < A_\Lambda <\ \ 1$, then Λ is a Γ-approximate dual continuous g-frame of itself.
Proof. The proof is similar to the proof of [15, Proposition 4.2], so we remove it here. □

3 Q-approximate dual continuous g-frames

In 2014, the concept of Q-duals for fusion frames were defined by Heineken and et al. [10]. Also, Q-duals and Q-approximate duals for g-frames were introduced by Mirzaee Azandaryani [15]. Here, we generalize these notions to the continuous g-frames as follows:

Definition 3.1. *Suppose that $\Lambda = \{\Lambda_w \in B(H, K_w): w \in \Omega\}$ and $\Theta = \{\Theta_w \in B(H, K_w): w \in \Omega\}$ are two continuous g-Bessel families for H with respect to $\{K_w\}_{w\in\Omega}$.*
(1) If there exists an operator $Q \in B(\widehat{K})$ such that $T_\Lambda Q T_\Theta^* = I_H$, we say that Λ is a Q-dual of
Θ.
(2) If there exists an operator $Q \in B(\widehat{K})$ such that $\| I_H - T_\Lambda Q T_\Theta^* \| < 1$, we say that Λ is a
Q-approximate dual continuous g-frame of Θ.

Clearly, if $\Lambda = \{\Lambda_w \in B(H, K_w): w \in \Omega\}$ is an approximate dual continuous g-frame (resp. dual continuous g-frame) of $\Theta = \{\Theta_w \in B(H, K_w): w \in \Omega\}$, then Λ is a Q-approximate dual continuous g-frame (resp. Q-dual) of Θ, by considering $Q = I_{\widehat{K}}$.

Theorem 3.2. *Suppose that $\Lambda = \{\Lambda_w \in B(H, K_w): w \in \Omega\}$ and $\Theta = \{\Theta_w \in B(H, K_w): w \in \Omega\}$ are two continuous g-Bessel families for H with respect to $\{K_w\}_{w\in\Omega}$. If Λ is a Q-approximate dual continuous g-frame of Θ, then*
(1) Θ is a Q^*-approximate dual continuous g-frame of Λ.
(2) Λ and Θ are continuous g-frames for H with respect to $\{K_w\}_{w\in\Omega}$.
Proof. (1) By assumption, there exists $Q \in B(\widehat{K})$ such that $\| I_H - T_\Lambda Q T_\Theta^* \| < 1$, thus

$$\| I_H - T_\Theta Q^* T_\Lambda^* \| = \| (I_H - T_\Lambda Q T_\Theta^*)^* \| = \| I_H - T_\Lambda Q T_\Theta^* \| < 1,$$

and so, Θ is a Q^*-approximate dual continuous g-frame of Λ.
(2) By (1) and [4, Theorem A.5.3], the operator $T_\Theta Q^* T_\Lambda^*$ is an invertible operator on H and

$$\| (T_\Theta Q^* T_\Lambda^*)^{-1} \| \leq \frac{1}{1 - \| I_H - T_\Theta Q^* T_\Lambda^* \|}.$$

For each $f \in H$,

$$\| f \| = \| (T_\Theta Q^* T_\Lambda^*)^{-1} (T_\Theta Q^* T_\Lambda^*) f \| \leq \| (T_\Theta Q^* T_\Lambda^*)^{-1} \| \| T_\Theta Q^* T_\Lambda^* f \|$$

$$\leq \frac{\| Q \| \sqrt{B_\Theta}}{1-\| I_H - T_\Theta Q^* T_\Lambda^* \|} (\int_\Omega \| \Lambda_w f \|^2 \, d\mu(w))^{\frac{1}{2}},$$

so

$$\| Q \|^{-2} B_\Theta^{-1} (1-\| I_H - T_\Theta Q^* T_\Lambda^* \|)^2 \| f \|^2 \leq \int_\Omega \| \Lambda_w f \|^2 \, d\mu(w), \quad f \in H.$$

Hence Λ is a continuous g-frame for H with respect to $\{K_w\}_{w\in\Omega}$. Similarly, Θ is a continuous g-frame for H with respect to $\{K_w\}_{w\in\Omega}$ with the lower bound

$$\| Q \|^{-2} B_\Lambda^{-1} (1-\| I_H - T_\Lambda Q T_\Theta^* \|)^2.$$

□

REFERENCES

[1] *M.R. Abdollahpour and M.H. Faroughi*, Continuous g-frames in Hilbert spaces, Southeast Asian Bull. Math., vol. **32**, no. 1, 2008, pp. 1-19.

[2] *M.R. Abdollahpour and Y. Khedmati*, G-duals of continuous g-frames and thei perturbations, Results Math., **vol. 73**, no. 4, 2018, 152.

[3] *S.T. Ali, J.P. Antoine and J.P. Cazeau*, Continuous frames in Hilbert spaces, Ann. Physics, **vol. 222**, no. 1, 1993, pp.1-37.

[4] *O. Christensen*, An Introduction to Frames and Riesz Bases, Birkhäuser Boston, 2003.

[5] *O. Christensen and R.S. Laugesen*, Approximately dual frame in Hilbert spaces and applications to Gabor
frames, Sampl. Theory Signal Image Process, **vol. 9**, 2011, pp. 77-90.

[6] *R.G. Douglas*, On majorization, factorization and range inclusion of operators on Hilbert spaces, Proc. Amer. Math. Soc., **vol. 17**, no. 2, 1966, pp. 413-415.

[7] *M.A. Dehghan and M.A. Hasankhani Fard*, G-dual frames in Hilbert spaces, U.P.B. Sci. Bull., Series A, **vol. 75**, no. 1, 2013, pp.129-140.

[8] *R.J. Duffin and A.C. Schaeffer*, A class of nonharmonic Fourier series, Trans. Amer. Math. Soc., **vol. 72**, no. 2, 1952, pp. 341-366.

[9] *J.P. Gabardo and D. Han*, Frames associated with measurable spaces, Adv. Comp. Math., **vol. 18**, no. 2, 2003, pp. 127-147.

[10] *S.B. Heineken, P.M. Morillas, A.M. Benavente and M.I. Zakowicz*, Dual fusion frames, Arch. Math., **vol. 103**, 2014, pp. 355-365.

[11] *H. Javanshiri*, Some properties of generalized and approximately dual frames in Hilbert spaces, Results Math., **vol. 70**, no. 3, 2016, pp. 475-485.

[12] *G. Kaiser*, A Friendly Guide to Wavelets, Birkhäuser Boston, 1995.

[13] *A. Khosravi and M. Mirzaee Azandaryani*, Approximate duality of g-frames in Hilbert spaces, Acta Math. Sci. Ser. B Engle. Ed., **vol. 34**, no. 3, 2014, pp. 639-652.

[14] *Y. Ling Fu and W. Zhang*, Characterization and stability of approximately dual of g-frames in Hilbert spaces, J. Inequal. Appl., **vol. 192**, 2018, 13.

[15] *M. Mirzaee Azandaryani*, On the approximate duality of g-frames and fusion frames, U.P.B. Sci. Bull., Series A, **vol. 79**, no. 2, 2017, pp. 83-94.

[16] *M. Mirzaee Azandaryani and A. Khosravi*, Duals and approximate duals of g-frames in Hilbert spaces, J. Linear and Topol. Algebra, **vol. 04**, no. 04, 2015, pp. 259-262.

[17] *G.J. Murphy*, C^*-Algebras and Operator Theory, Academic Press, San Diego, California, 1990.

[18] *A. Rahimi, Z. Darvishi and B. Daraby*, Dual pair and approximate dual for continuous frames in Hilbert spaces, Math. Rep., **vol. 21,** no. 2, 2019, pp. 173-191.

[19] *W. Sun*, G-frames and g-Riesz bases, J. Math. Anal. Appl., **vol. 322**, no. 1, 2006, pp. 437-452.