SOME PROPERTIES OF MINIMAL AND MAXIMAL OPERATORS IN AN ABSTRACT FRAMEWORK

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The main goal of this paper is to give a two-parameter abstract framework in which we build a theory of minimal and maximal operators associated to a linear operator $A : \mathcal{D}(A) \subset X \to X$, with dense domain $\mathcal{D}(A)$, where X is a complex Banach space. We prove an analogue of the Agmon-Douglis-Nirenberg inequality for pseudo-differential operators in our abstract setting. Using this inequality, we show that the minimal and maximal operators of the operator A are equal under suitable hypotheses on the complex Banach space X and on the operator A. As an application, we study the existence and regularity of weak solutions of the linear equations $Au = f$ on the reflexive complex Banach space X. Further, we prove a perturbation result regarding the Agmon-Douglis-Nirenberg estimate when the operator A is perturbed by a potential operator with some suitable properties. Moreover, an application to strongly continuous semigroups of contractions generated by the operator A is given. Finally, we prove that the minimal operator of the operator A is Fredholm under suitable hypotheses.

Keywords: minimal and maximal operators, Agmon-Douglis-Nirenberg inequality, strongly continuous semigroups, dissipative operators, Fredholm operators

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1. Introduction

In this work, following Wong's point of view in the paper [16], we build an abstract framework in which we construct and study the minimal and maximal operators associated to an operator $A : \mathcal{D}(A) \subset X \to X$, with dense domain $\mathcal{D}(A)$, where X is a complex Banach space.

We must emphasize that this abstract framework includes certain concrete cases of Lebesgue and Sobolev spaces and classes of pseudo-differential operators such as M-hypoelliptic pseudo-differential operators (see [6], [7], [12]), SG-pseudo-differential operators (see [4], [5], [8], [13]) or hybrid pseudo-differential operators (see [2]) defined on these spaces.

The paper is organised as follows.

In Section 2, we introduce the weighted Bessel potentials of orders $(s_1, s_2) \in \mathbb{R}^2$ and we define the X-Sobolev spaces by using them. Moreover, we introduce a class of linear operators of orders $(m_1, m_2) \in \mathbb{R}^2$ in connection with the X-Sobolev spaces. This class of linear operators remembers us of the class of hybrid pseudo-differential operators in [2]. Some notations and facts concerning the minimal and maximal operators associated to an operator A are also recalled (see [16]). In Section 3, we state and prove an analogue of the

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Agmon-Douglis-Nirenberg (A-D-N) inequality for pseudo-differential operators in the case of an operator $A : \mathcal{D}(A) \subset X \to X$, with dense domain $\mathcal{D}(A)$, under suitable hypotheses. Using the A-D-N inequality, we prove that the minimal and maximal operators are equal under resonable hypotheses on the complex Banach space X and the operator A. As an application of this fact, we get the existence and regularity of weak solutions of the linear equations $Au = f$ on the Banach space X. Two perturbation results, one regarding the A-D-N inequality and the other result concerning the strongly continuous semigroup of contractions, are given in Section 4. More precisely, we prove a type of the A-D-N inequality in the case when the operator A is perturbed by a potential with some suitable properties. If A is the infinitesimal generator of a semigroup of contractions, then $A_0 + V$ is also the infinitesimal generator of an one parameter strongly semigroup of contractions, where A_0 is the minimal operator associated to the operator A and V is a maximally dissipative operator with some suitable properties. In Section 5, the last one, we prove that under resonable hypotheses on the operator A, its minimal operator A_0 is Fredholm.

2. Preliminaries

Let X be a complex Banach space whose norm is denoted by $\|\|_X$ and let S be a dense subspace of X. We consider that S is a topological vector space of which topology is defined by a countable family of semi-norms $\{|\ |_j : j = 1, 2, \ldots\}$.

We say that a sequence $\{\varphi_k\}$ in S converges to an element φ in S if and only if $|\varphi_k - \varphi|_j \to 0$ as $k \to \infty$ for all $j = 1, 2, \ldots$. We denote by S' the space of all continuous linear functionals on the space S and by (u, φ) the value of a functional u in S' at an element φ in S.

A functional u is continuous if and only if $(u, \varphi_k) \to 0$ as $k \to \infty$ for all sequences $\{\varphi_k\}$ converging to zero in S as $k \to \infty$.

A sequence $\{u_k\}$ in S' is said to converge to an element u in S' if and only if $(u_k, \varphi) \to (u, \varphi)$ as $k \to \infty$ for all φ in S. We assume that the spaces X and X' are continuously embedded in S'.

The definitions and notations used above are similar to the ones used in the theory of distributions and are also used by Wong in [16].

Now, we present the abstract framework in which we will work, framework that is similar to the one in the paper [16] and can concretely be encountered in the theory of distributions.

Let us suppose that there exists a family of reflexive complex Banach spaces $X^{\Lambda}_{s_1,s_2}$ with norms denoted by $\|\|_{s_1,s_2,\Lambda,X}, -\infty < s_1, s_2 < \infty$, where $\Lambda : \mathbb{R}^n \to \mathbb{R}_+$ is a weight positive function and a two-parameter group of continuous linear mappings $J^{\Lambda}_{s_1,s_2} : S' \to S'$, $-\infty < s_1, s_2 < \infty$, satisfying the following conditions:

(i) $J^{\Lambda}_{s_1,s_2}$ maps S into S, $-\infty < s_1, s_2 < \infty$ and $J^{\Lambda}_{\varepsilon,\varepsilon} : X \to X$ is a compact operator for every positive number ε.

(ii) $X^{\Lambda}_{s_1,s_2} = \left\{u \in S' : J^{\Lambda}_{-s_1,-s_2} u \in X\right\}, -\infty < s_1, s_2 < \infty$.

(iii)

$$\|u\|_{s_1,s_2,\Lambda,X} = \left\|J^{\Lambda}_{-s_1,-s_2} u\right\|_X, u \in X^{\Lambda}_{s_1,s_2}, -\infty < s_1, s_2 < \infty. \quad (\mathbf{2}.1)$$

(iv)

Let $s_j \le t_j$, $j = 1, 2$. Then, $X^{\Lambda}_{t_1,t_2} \subseteq X^{\Lambda}_{s_1,s_2}$ and

$$\|u\|_{s_1,s_2,\Lambda,X} \le \|u\|_{t_1,t_2,\Lambda,X}, \quad u \in X^{\Lambda}_{t_1,t_2}. \quad (\mathbf{2}.2)$$

(v) $X^{\Lambda}_{s_1,s_2}$ can be continuously embedded in S', $-\infty < s_1, s_2 < \infty$.

(vi) S can be continuously embedded in $(X^{\Lambda}_{s_1,s_2})'$ and $(X^{\Lambda}_{s_1,s_2})'$ can be continuously embedded in S', $-\infty < s_1, s_2 < \infty$.

(vii)

$$(u,\varphi) = \overline{(\varphi,u)}, u \in X^{\Lambda}_{s_1,s_2}, \varphi \in S, -\infty < s_1, s_2 < \infty. \qquad (\mathbf{2}.3)$$

We call $J^{\Lambda}_{s_1,s_2}$ the weighted Bessel potentials of orders $(s_1,s_2) \in \mathbb{R}^2$ and $X^{\Lambda}_{s_1,s_2}$ the X-Sobolev spaces of orders $(s_1,s_2) \in \mathbb{R}^2$.

Definition 2.1. Let $T : S' \longrightarrow S'$ be a continuous linear mapping.

We assume that there exists a pair of real numbers (m_1,m_2) such that $T : X^{\Lambda}_{s_1,s_2} \to X^{\Lambda}_{s_1-m_1,s_2-m_2}$ is a bounded linear operator for all $(s_1,s_2) \in \mathbb{R}^2$. We say that T is an operator of order (m_1,m_2) if m_1 and m_2 are the least numbers for which $T : X^{\Lambda}_{s_1,s_2} \to X^{\Lambda}_{s_1-m_1,s_2-m_2}$ is a bounded linear operator.

If $m_1 = m_2 = -\infty$, then we call T an infinitely smoothing operator.

Definition 2.2. Let $A : S \subset X \to X$ be a linear operator such that A maps S into S and its formal adjoint A^* maps S into S continuously. We say that A is an operator of order (m_1,m_2) if the extended operator $A : S' \to S'$ is of order (m_1,m_2) (see the relation (**2**.5) for the definition of the extended operator $A : S' \to S'$).

Remark 2.1. The two-parameter family of X-Sobolev spaces $X^{\Lambda}_{s_1,s_2}$, $s_1, s_2 \in \mathbb{R}$, considered previously, define a two-parameter abstract framework which enable us to fit the theory of SG-pseudo-differential operators (see [4], [5], [8]) or the theory of a hybrid class of pseudo-differential operators (see [2]).

A one-parameter abstract framework in which the theory of minimal and maximal operators was studied for the first time was introduced by Wong in [16]. This abstract framework was used later in the joint paper [10] of Wong and Iancu in order to establish some results related to the semi-linear heat equations in Hilbert spaces. In his PhD Thesis (see [11]), Iancu used extensively this abstract framework.

Remark 2.2. It should be mentioned that, in various works, particular cases of the previously considered abstract framework can be found. For example, if we take $X = L^p(\mathbb{R}^n)$, S is a Schwartz space of the rapidly decreasing functions, $1 < p < \infty$, $J^{\Lambda}_{s_1,s_2} = T_{\sigma_{s_1,s_2}}$, where $\sigma_{s_1,s_2}(x,\xi) = \Lambda(x)^{-s_2}\Lambda(\xi)^{-s_1}$, $\infty < s_1, s_2 < \infty$ and $\Lambda \in C^{\infty}(\mathbb{R}^n)$ is a weight function with some suitable properties, then we obtain a concrete two-parameter framework in the paper [2], in which the authors present their results.

In this case, for a fixed $p \in (1,\infty)$, the family of spaces $H^{s_1,s_2}_{\Lambda,p}$ corresponds to spaces $X^{\Lambda}_{s_1,s_2}$, $\infty < s_1, s_2 < \infty$ and satisfies the conditions (i)-(vii).

Proposition 2.1. *Let $s_1, s_2, t_1, t_2 \in (-\infty, \infty)$. Then,*

i) $J^{\Lambda}_{t_1,t_2} : X^{\Lambda}_{s_1,s_2} \to X^{\Lambda}_{s_1+t_1,s_2+t_2}$ is an unitary operator;

ii) S is dense in $X^{\Lambda}_{s_1,s_2}$.

Proof. i) Let $u \in X^{\Lambda}_{s_1,s_2}$. From (**2**.1), we obtain that

$$\left\|J^{\Lambda}_{t_1,t_2}u\right\|_{s_1+t_1,s_2+t_2} = \left\|J^{\Lambda}_{-s_1-t_1,-s_2-t_2}J^{\Lambda}_{t_1,t_2}u\right\|_X = \left\|J^{\Lambda}_{-s_1,-s_2}u\right\|_X = \|u\|_{s_1,s_2,\Lambda}.$$

Hence, $J^{\Lambda}_{t_1,t_2} : X^{\Lambda}_{s_1,s_2} \to X^{\Lambda}_{s_1+t_1,s_2+t_2}$ is an isometry. It remains only to prove that $J^{\Lambda}_{t_1,t_2} : X^{\Lambda}_{s_1,s_2} \to X^{\Lambda}_{s_1+t_1,s_2+t_2}$ is a surjection. For this, let y be in $X^{\Lambda}_{s_1+t_1,s_2+t_2}$. Thus, $J^{\Lambda}_{-t_1,-t_2}y \in X^{\Lambda}_{s_1,s_2}$ and $J^{\Lambda}_{t_1,t_2}\left(J^{\Lambda}_{-t_1,-t_2}y\right) = y$.

ii) Let $u \in X^{\Lambda}_{s_1,s_2}$. Then $J^{\Lambda}_{-s_1,-s_2}u \in X$. Since S is dense in X, there exists a sequence $\{\varphi_k\}$ of elements in S such that $\varphi_k \to J^{\Lambda}_{-s_1,-s_2}u$ in X as $k \to \infty$. Let $\psi_k = J^{\Lambda}_{s_1,s_2}\varphi_k, k = 1,2,\ldots$. Since $J^{\Lambda}_{s_1,s_2}$ maps S into S, it follows that $\psi_k \in S, k = 1,2,\ldots$.

By the definition of $X^{\Lambda}_{s_1,s_2}$, we obtain that

$$\|\psi_k - u\|_{s_1,s_2,\Lambda} = \left\|J^{\Lambda}_{-s_1,-s_2}\psi_k - J^{\Lambda}_{-s_1,-s_2}u\right\|_X = \left\|\varphi_k - J^{\Lambda}_{-s_1,-s_2}u\right\|_X$$

for all $k = 1, 2, \ldots$.Therefore, $\psi_k \to u$ in $X^{\Lambda}_{s_1,s_2}$ as $k \to \infty$. This proves that S is dense in $X^{\Lambda}_{s_1,s_2}$.

Thus, the proof is complete.

□

Remark 2.3. j) From (i), (ii) and Proposition **2**.1, we have that

$$S \subset X \subset X^{\Lambda}_{0,0}$$

and S is dense in $X^{\Lambda}_{0,0}$. Since S in dense in X, it implies that $X = X^{\Lambda}_{0,0}$.

jj) From Proposition **2**.1 i), it follows that $J^{\Lambda}_{t_1,t_2}$ is an operator of orders $(-t_1, -t_2)$.

Now we recall some namings, notations and well-known results concerning the theory of minimal and maximal operators (see [2], [5]-[7], [15], [16]).

Let X be a complex Banach space, S a dense subspace of X and let A be a linear operator from X into X with domain S. We denote by X' the space of all bounded linear functionals on X and by (x', x) the value of a functional x' in X' at an element x in X.

Definition 2.3. Let $\mathcal{D}(A^t)$ be the set of all functionals y' in X' for which there is a functional x' in X' such that

$$(y', Ax) = (x', x), \quad x \in S. \tag{2.4}$$

We can prove that for any y' in X', there exists at most one x' in X' for which (**2**.4) holds. Thus, we can define $A^t y' = x'$, for all y' in $\mathcal{D}(A^t)$. We call A^t the true adjoint of A.

We can prove easily that A^t is a closed linear operator from X' into X' with domain $\mathcal{D}(A^t)$.

Let us observe that if B is a linear extension of A, then A^t is a linear extension of B^t.

Definition 2.4. Let A be a linear operator from X into X with domain S. The operator A is closable if and only if

$$\varphi_k \in S, \varphi_k \to 0 \text{ in } X, A\varphi_k \to x \text{ in } X \Rightarrow x = 0.$$

In the following, we define the minimal operator of the operator A.

Suppose that A is a closable operator. We can construct a closed linear extension A_0 of A.

Definition 2.5. Let $\mathcal{D}(A_0)$ be the set of all x in X for which there exists a sequence $\{\varphi_k\}_{k=1}^{\infty}$ in S such that $\varphi_k \to x$ in X, $A\varphi_k \to y$ for some y in X as $k \to \infty$. We can define $A_0 x = y$, for any $x \in \mathcal{D}(A_0)$.

It can be proved that the definition of A_0 does not depend on the particular choice of the sequence $\{\varphi_k\}_{k=1}^{\infty}$ and it can also be proved that A_0 is the smallest closed linear extension of A (i.e. if B is any closed linear extension of A, then B is also a linear extension of A_0). We call A_0 the minimal operator of A.

We further assume in this work that X is a reflexive complex Banach space.

In order to define the maximal operator, we need to introduce the notion of formal adjoint. We assume that the space X and its dual space X' can be continuously embedded in some topological space Y. Thus, the spaces X and X' will be identified as subspaces of Y. We also assume that there exists a subspace S of Y such that S is a dense subspace of X and X'.

For the following definitions and results, we let A be a linear operator from X into X with domain S.

Definition 2.6. The formal adjoint A^* of the operator A, if it exists, is defined to be the restriction of the true adjoint A^t to the space S.

From Definition **2**.6, we observe that the formal adjoint A^* exists if and only if S is contained in the domain of A^t.

Definition 2.7. We define the linear operator A_1 from X into X by $A_1 = (A^*)^t$.

Let $\varphi \in S$. By Definition **2**.7, we have

$$(\varphi, A_1 x) = (A^* \varphi, x)$$

for all x in $\mathcal{D}(A_1)$.

By the definition of the true adjoint, $\varphi \in \mathcal{D}(A_1^t)$ and $A_1^t \varphi = A^* \varphi$.

Proposition 2.2. *(*[16]*) A_1 is a closed linear operator from X into X with domain $\mathcal{D}(A_1)$ containing the space S.*

Proposition 2.3. *(*[16]*) The domain $\mathcal{D}(A_1^t)$ of the adjoint of A_1 contains the space S.*

Proposition 2.4. *(*[16]*) A_1 is a linear extension of A_0.*

From Proposition **2**.4 we see that $(A_0)^t$ is a linear extension of $(A_1)^t$ and by Proposition **2**.3, the domain of $(A_1)^t$ contains the space S. It follows that the domain of $(A_0)^t$ contains the space S.

Theorem 2.1. *(*[16]*) A_1 is the largest closed linear extension of A with the property that the space S is contained in the domain of its adjoint (i.e. if B is any closed linear extension of A such that $S \subseteq \mathcal{D}(B^t)$, then A_1 is a linear extension of B).*

Definition 2.8. The operator A_1 from Theorem **2**.1 is called the maximal operator of A.

Let A be a linear operator from X into X with domain S. We suppose that A maps S into S and its formal adjoint A^* maps S into S continuously (i.e. if $\{\varphi_k\}$ is any sequence in S such that $\varphi_k \to 0$ in S as $k \to \infty$, then $A\varphi_k \to 0$ and $A^*\varphi_k \to 0$ in S as $k \to \infty$).

The linear operator A can be extended to the space S'.

For any u in S', Au is an element in S' given by the relation

$$(Au, \varphi) = (u, A^* \varphi), \quad \varphi \in S. \tag{2.5}$$

It is easy to show that $A : S' \to S'$ is a continuous linear mapping.

3. Some properties of minimal and maximal operators

In this section, we prove an analogue of the A-D-N inequality for pseudo-differential operators in the case of the operator $A : S \subset X \to X$ with domain S that satisfies some hypotheses. Using this inequality, we will obtain the equality of minimal and maximal operators associated to the operator A when certain suitable hypotheses are satisfied. As an application, we study the existence and regularity of weak solutions of the linear equations $Au = f$ on X.

Theorem 3.1. ***(Agmon-Douglis-Nirenberg inequality*** [1]*) Let $A : S \subset X \to X$ be a linear operator such that A maps S into S and its formal adjoint A^* maps S into S*

continuously. Suppose that A is of positive order (m_1, m_2) and there exists a linear operator B of order $(-m_1, -m_2)$ from X into X with domain S such that

$$BA = I + R, \tag{3.1}$$

where I is the identity operator, and R is an infinitely smoothing operator. Then, there exist two positive constants C_1 and C_2 such that

$$C_1 \|x\|_{m_1,m_2,\Lambda} \leq \|Ax\|_{0,0,\Lambda} + \|x\|_{0,0,\Lambda} \leq C_2 \|x\|_{m_1,m_2,\Lambda}, \quad x \in X^{\Lambda}_{m_1,m_2}. \tag{3.2}$$

Proof. First, we prove the right-hand side of inequality (**3**.2). Since A is of order (m_1, m_2), by (**2**.2) and by boundedness of A it follows that there exist two positive constants K_1 and K_2 such that

$$\|Ax\|_{0,0,\Lambda} \leq K_1 \|x\|_{m_1,m_2,\Lambda} \text{ and}$$
$$\|x\|_{0,0,\Lambda} \leq K_2 \|x\|_{m_1,m_2,\Lambda}, x \in X^{\Lambda}_{m_1,m_2}.$$

Hence, there exists a positive constant $C_2 = max(K_1, K_2)$ such that

$$\|Ax\|_{0,0,\Lambda} + \|x\|_{0,0,\Lambda} \leq C_2 \|x\|_{m_1,m_2,\Lambda}, x \in X^{\Lambda}_{m_1,m_2}.$$

Now, we have to prove the left-hand side of inequality (**3**.2).

By (**3**.1) it follows that

$$x = BAx - Rx, x \in X^{\Lambda}_{m_1,m_2},$$

where B is an operator of order $(-m_1, -m_2)$ and R is an infinitely smoothing operator. Hence,

$$\|x\|_{m_1,m_2,\Lambda} = \|BAx - Rx\|_{m_1,m_2,\Lambda} \leq \|BAx\|_{m_1,m_2,\Lambda} + \|Rx\|_{m_1,m_2,\Lambda}.$$

Since $x \in X^{\Lambda}_{m_1,m_2}$ and A is of order (m_1, m_2), it follows that $Ax \in X^{\Lambda}_{0,0}$. The operator $B : X^{\Lambda}_{0,0} \to X^{\Lambda}_{m_1,m_2}$ is bounded. Therefore, there exists a positive constant K_1' such that

$$\|BAx\|_{m_1,m_2,\Lambda} \leq K_1' \|Ax\|_{0,0,\Lambda}, x \in X^{\Lambda}_{m_1,m_2}.$$

Since $x \in X^{\Lambda}_{m_1,m_2} \subset X^{\Lambda}_{0,0}$, let $x \in X^{\Lambda}_{0,0}$. The operator R is infinitely smoothing, so there exists a positive constant K_2' such that

$$\|Rx\|_{m_1,m_2,\Lambda} \leq K_2' \|x\|_{0,0,\Lambda}, x \in X^{\Lambda}_{m_1,m_2}.$$

Using the last two inequalities above, it follows that

$$\begin{aligned}\|x\|_{m_1,m_2,\Lambda} &\leq K_1' \|Ax\|_{0,0,\Lambda} + K_2' \|x\|_{0,0,\Lambda} \\ &\leq \max(K_1', K_2') \left(\|Ax\|_{0,0,\Lambda} + \|x\|_{0,0,\Lambda}\right), x \in X^{\Lambda}_{m_1,m_2}.\end{aligned}$$

So, taking $C_1 = \frac{1}{\max\left(K_1', K_2'\right)}$, we get

$$C_1 \|x\|_{m_1,m_2,\Lambda} \leq \|Ax\|_{0,0,\Lambda} + \|x\|_{0,0,\Lambda}, x \in X^{\Lambda}_{m_1,m_2}.$$

Thus, the proof is complete. □

Remark 3.1. The estimate (**3**.2) can be seen as an analogue of the Agmon-Douglis-Nirenberg estimate for pseudo-differential operators in the case of an operator A from X into X with dense domain (see [1]). We must specify that some versions of this estimate can be found, for example, in [5], [7] and [8] for the class of pseudo-differential operators with global symbols introduced by Camperi in [4] or for a class of hybrid pseudo-differential operators introduced by Alimohammady and Kalleji in [2].

Theorem 3.2. *Let A be as in Theorem 3.1. Then, $\mathcal{D}(A_0) = X^{\Lambda}_{m_1,m_2}$.*

Proof. " $\supseteq$ " Let $x \in X^{\Lambda}_{m_1,m_2}$. By Proposition **2**.1, S is dense in $X^{\Lambda}_{m_1,m_2}$. Hence, there exists a sequence $\{\varphi_k\}$ of elements in S such that $\varphi_k \to x$ in $X^{\Lambda}_{m_1,m_2}$ as $k \to \infty$. By the right-hand side of inequality (**3**.2), $\{A\varphi_k\}$ and $\{\varphi_k\}$ are Cauchy sequences in X. Therefore, $\varphi_k \to x$ and $A\varphi_k \to f$ in X for some x and f in X as $k \to \infty$. Thus, by the definition of A_0, $x \in \mathcal{D}(A_0)$ and $A_0 x = f$.

" $\subseteq$ " Let $x \in \mathcal{D}(A_0)$. By the definition o f A_0, t here e xists a s equence $\{\varphi_k\}$ of elements in S for which $\varphi_k \to x$ in X and $A\varphi_k \to f$ in X for some f in X as $k \to \infty$. Hence, $\{\varphi_k\}$ and $\{A\varphi_k\}$ are Cauchy sequences in X. Using (**3**.2), $\{\varphi_k\}$ is a Cauchy sequence in $X^{\Lambda}_{m_1,m_2}$. Since $X^{\Lambda}_{m_1,m_2}$ is complete, it follows that $\varphi_k \to u$ in $X^{\Lambda}_{m_1,m_2}$ for some u in $X^{\Lambda}_{m_1,m_2}$ as $k \to \infty$. By (**2**.2), $\varphi_k \to u$ in X as $k \to \infty$, so $x = u$. Thus, $x \in X^{\Lambda}_{m_1,m_2}$.

The proof of the Theorem **3**.2 is complete. □

Now we came to the main result of this section.

Theorem 3.3. *Let A be as in Theorem* **3**.*1. Then,* $A_0 = A_1$.

Proof. Since A_1 is an extension of A_0 and $\mathcal{D}(A_0) = X^{\Lambda}_{m_1,m_2}$, it remains to prove that $\mathcal{D}(A_1) \subseteq X^{\Lambda}_{m_1,m_2}$.

Let $x \in \mathcal{D}(A_1)$.

Using the hypotheses from Theorem **3**.1, it follows that there exists an operator B of order $(-m_1, -m_2)$ such that $x = BAx - Rx$, where R is an infinitely smoothing operator.

Let $u \in \mathcal{D}(A_1)$. By Definition **2**.7,

$$(\varphi, A_1 u) = (A^*\varphi, u), \quad \varphi \in S. \tag{3.3}$$

By (**2**.5), we have

$$(Au, \varphi) = (u, A^*\varphi), \quad \varphi \in S. \tag{3.4}$$

Using (**2**.3) and (**3**.4),

$$(\varphi, Au) = (A^*\varphi, u), \quad \varphi \in S. \tag{3.5}$$

Therefore, by (**2**.3), (**3**.3) and (**3**.5),

$$(A_1 u, \varphi) = (Au, \varphi), \quad \varphi \in S.$$

Hence, $A_1 u = Au$ for all $u \in \mathcal{D}(A_1)$.

Since $Ax = A_1 x \in X = X^{\Lambda}_{0,0}$, we have that $BAx \in X^{\Lambda}_{m_1,m_2}$. Since $x \in X$ and R is an infinitely smoothing operator, we obtain that $Rx \in X^{\Lambda}_{m_1,m_2}$.

Thus, $x \in X^{\Lambda}_{m_1,m_2}$.

The proof of this theorem is complete. □

For more details concerning the minimal and maximal operators corresponding to different types of pseudo-differential operators, see, for example, [2], [5]-[8], [12].

Now, we give an application regarding the existence and regularity of weak solutions of the linear equations on the reflexive complex Banach space X.

Definition 3.1. Let $f \in X$. Then, an element u in X is called a weak solution of the linear equation $Au = f$ if $(A^*\varphi, u) = (\varphi, f)$, for all $\varphi \in S$.

Proposition 3.1. Let $A : S \subset X \to X$ be a linear operator and let $f \in X$. Then $u \in X$ is a weak solution of the linear equation $Au = f$ iff $u \in \mathcal{D}(A_1)$ and $A_1 u = f$.

Proof. The "only if" part follows from the definitions of the maximal operator A_1 and of the weak solutions. Indeed, $u \in \mathcal{D}(A_1)$ and $A_1 u = f$ implies that $(\varphi, A_1 u) = (A^*\varphi, u)$, for all $\varphi \in S$ or equivalently $(\varphi, f) = (A^*\varphi, u)$, for all $\varphi \in S$. Therefore, u is a weak solution of the linear equation $Au = f$.

The "if" part follows from the definition of the weak solutions. Let $u \in X$ be a weak solution of the linear equation. Then, $(A^*\varphi, u) = (\varphi, Au)$, for all $\varphi \in S$. From the definition of the maximal operator A_1, we obtain that $(\varphi, A_1 u) = (A^*\varphi, u)$, for all $\varphi \in S$. Hence, $u \in \mathcal{D}(A_1)$ and $A_1 u = f$. □

Now, we can state and prove the following theorem.

Theorem 3.4. *Let $A : \mathcal{D}(A) \subset X \to X$ be a linear operator as in Theorem* **3**.*1 and let $f \in X$. Then, every weak solution u of the linear equation $Au = f$ is in $X^{\Lambda}_{m_1,m_2}$.*

Proof. Let u be a weak solution of $Au = f$.

Using Proposition **3**.1, we obtain that $u \in \mathcal{D}(A_1)$. By Theorem **3**.3, $A_1 = A_0$. Hence, $u \in \mathcal{D}(A_0)$. By Theorem **3**.2, $\mathcal{D}(A_0) = X^{\Lambda}_{m_1,m_2}$.

Therefore, $u \in X^{\Lambda}_{m_1,m_2}$.

The proof of the theorem is complete. □

Remark 3.2. The previous theorem represents a regularity result because it tells us that every weak solution u of the linear equation $Au = f$ belongs to a more regular space $X^{\Lambda}_{m_1,m_2}$ in the sense that $X^{\Lambda}_{m_1,m_2} \subset X^{\Lambda}_{0,0} = X$ by (**2**.2).

4. Two perturbation results

In this section, we give a perturbation result concerning the A-D-N inequality and another result regarding the strongly continuous semigroup of contractions.

First, we assume that for $0 < s_1 < t_1$ and for every positive number $\varepsilon > 0$, there exists a positive constant C_ε such that

$$\|u\|_{s_1,0,\Lambda} \le \varepsilon \|u\|_{t_1,0,\Lambda} + C_\varepsilon \|u\|_{0,0,\Lambda}, u \in X^{\Lambda}_{t_1,0}. \tag{4.1}$$

Let us observe that when we take $s = (s_1, s_2), t = (t_1, t_2)$ in $\mathbb{R}^2$ such that $0 < s_1 < t_1, s_2 \le 0 < t_2$, we have the estimate

$$\|u\|_{s_1,s_2,\Lambda} \le \varepsilon \|u\|_{t_1,t_2,\Lambda} + C_\varepsilon \|u\|_{0,0,\Lambda}, u \in X^{\Lambda}_{t_1,t_2}, \tag{4.2}$$

using (**2**.2) and (**4**.1).

Remark 4.1. The inequality (**4**.1) is an abstract version of the Erhling inequality related to the pseudo-differential operators on $L^p(\mathbb{R}^n)$ spaces (see [17]) and the inequality (**4**.2) is an almost analogue, in our abstract setting, of this inequality.

Theorem 4.1. *Let A be an operator as in Theorem* **3**.*1 and let $V : \mathcal{D}(V) \subset X \to X$ with $S \subset \mathcal{D}(V)$ be a closed operator such that there exists a positive constant C for which*

$$\|V\varphi\|_{0,0,\Lambda} \le C\|\varphi\|_{s_1,s_2,\Lambda}, \quad \varphi \in S, \tag{4.3}$$

where $0 < s_1 < m_1, s_2 \le 0 < m_2$. Then, there exist positive constants $\tilde{C}_1$ and $\tilde{C}_2$ such that

$$\tilde{C}_1\|\varphi\|_{m_1,m_2,\Lambda} \le \|(A+V)\varphi\|_{0,0,\Lambda} + \|\varphi\|_{0,0,\Lambda} \le \tilde{C}_2\|\varphi\|_{m_1,m_2,\Lambda}, \quad \varphi \in S. \tag{4.4}$$

Proof. Let $\varphi \in S$. By (**4**.3) and the right-hand side of the inequality (**3**.2), we get

$$\begin{aligned}\|(A+V)\varphi\|_{0,0,\Lambda} + \|\varphi\|_{0,0,\Lambda} &\le \|A\varphi\|_{0,0,\Lambda} + \|V\varphi\|_{0,0,\Lambda} + \|\varphi\|_{0,0,\Lambda} \\ &\le \|A\varphi\|_{0,0,\Lambda} + C\|\varphi\|_{s_1,s_2,\Lambda} + \|\varphi\|_{0,0,\Lambda} \\ &\le (C + C_2)\|\varphi\|_{m_1,m_2,\Lambda} = \tilde{C}_2\|\varphi\|_{m_1,m_2,\Lambda}, \quad \varphi \in S.\end{aligned}$$

By (**4**.3) and (**4**.2), for every positive number ε, there exists a positive constant C_ε such that

$$\begin{aligned}\|(A+V)\varphi\|_{0,0,\Lambda} &\ge \|A\varphi\|_{0,0,\Lambda} - \|V\varphi\|_{0,0,\Lambda} \ge \|A\varphi\|_{0,0,\Lambda} - C\|\varphi\|_{s_1,s_2,\Lambda} \\ &\ge \|A\varphi\|_{0,0,\Lambda} - \varepsilon\|\varphi\|_{m_1,m_2,\Lambda} - C_\varepsilon\|\varphi\|_{0,0,\Lambda}, \quad \varphi \in S.\end{aligned}$$

By the left-hand side of the inequality (**3**.2), we get

$$\|(A+V)\,\varphi\|_{0,0,\Lambda} \geq (C_1-\varepsilon)\,\|\varphi\|_{m_1,m_2,\Lambda} - (C_\varepsilon+1)\,\|\varphi\|_{0,0,\Lambda}, \quad \varphi \in S.$$

Choosing $\varepsilon < C_1$, we obtain

$$\begin{aligned}\|\varphi\|_{0,0,\Lambda} + \|(A+V)\,\varphi\|_{0,0,\Lambda} &\geq \|\varphi\|_{0,0,\Lambda} + \frac{1}{C_\varepsilon+1}\,\|(A+V)\,\varphi\|_{0,0,\Lambda}\\ &\geq \frac{C_1-\varepsilon}{C_\varepsilon+1}\|\varphi\|_{m_1,m_2,\Lambda} = \tilde{C}_1\|\varphi\|_{m_1,m_2,\Lambda}, \quad \varphi \in S.\end{aligned}$$

The proof is complete. □

Now, we can state and prove the main result of this section.

Theorem 4.2. *Let A be an operator as in Theorem **3**.1 and let $V : \mathcal{D}(V) \subset X \to X$ with $S \subset \mathcal{D}(V)$ be a closed operator that satisfies the estimate*

$$\|V\varphi\|_{0,0,\Lambda} \leq C\|\varphi\|_{s_1,s_2,\Lambda}, \varphi \in S, \tag{4.5}$$

where $0 < s_1 < m_1, s_2 \leq 0 < m_2$. Then, there exist positive constants C_1, C_2 such that

$$C_1\|u\|_{m_1,m_2,\Lambda} \leq \|(A_0+V)\,u\|_{0,0,\Lambda} + \|u\|_{0,0,\Lambda} \leq C_2\|u\|_{m_1,m_2,\Lambda}, u \in X^{\Lambda}_{m_1,m_2}. \tag{4.6}$$

Proof. Let $u \in X^{\Lambda}_{m_1,m_2}$. There exists a sequence $\{\varphi\}_{j=1}^{\infty}$ of functions in S such that $\varphi_j \to u$ in $X^{\Lambda}_{m_1,m_2}$ as $j \to \infty$.

By the right-hand side of the inequality (**3**.2), we obtain that

$$A\varphi_j \to A_0u \in X \quad \text{as } j \to \infty.$$

By (**4**.5),

$$\|V\varphi_j - V\varphi_k\|_{0,0,\Lambda} \leq C\,\|\varphi_j-\varphi_k\|_{s_1,s_2,\Lambda} \leq C\,\|\varphi_j-\varphi_k\|_{m_1,m_2,\Lambda} \to 0 \quad \text{as } j,k \to \infty.$$

Thus, $V\varphi_j \to v$ for some v in X as $j \to \infty$. Since $V : \mathcal{D}(V) \to X$ is closed, $u \in \mathcal{D}(V)$ and $Vu = v$.

By Theorem **4**.1,

$$\tilde{C}_1\,\|\varphi_j\|_{m_1,m_2,\Lambda} \leq \|(A_0+V)\,\varphi_j\|_{0,0,\Lambda} + \|\varphi_j\|_{0,0,\Lambda} \leq \tilde{C}_2\,\|\varphi_j\|_{m_1,m_2,\Lambda}\,,$$

for $j = 1, 2, \ldots$.

Now, if we let $j \to \infty$, then the proof of the theorem is complete. □

In order to give an application to strongly continuous semigroups of contractions generated by the operator A, we need to recall the following result (see Corollary 3.8 in [9] or Corollary 3.3 in [14]).

Theorem 4.3. *Let X be a complex Banach space and let $\|\cdot\|$ be its norm. Let A be the infinitesimal generator of one-parameter strongly continuous semigroup of contractions on the complex Banach space X. Let B be a dissipative operator which satisfies $\mathcal{D}(A) \subset \mathcal{D}(B)$ and $\|Bx\| \leq a\|Ax\| + C\|x\|$ for $x \in \mathcal{D}(A)$, where $0 < a < 1$ and $C \geq 0$.*

Then $A+B$ is the infinitesimal generator of a one-parameter strongly continuous semigroup of contractions on X.

Now, we can state and prove the following theorem.

Theorem 4.4. *Let A be an operator as in Theorem* **3**.*1 such that A is the infinitesimal generator of a strongly continuous semigroup of contractions on X.*

Let $V : \mathcal{D}(V) \subset X \to X$ be a maximally dissipative operator with $S \subset \mathcal{D}(V)$ such that

$$\|V\varphi\|_{0,0,\Lambda} \leq C\|\varphi\|_{s_1,s_2,\Lambda}, \varphi \in S,$$

where $0 < s_1 < m_1$, $s_2 \leq 0 < m_2$ and C is a positive constant.

Then $A_0 + V$ is the infinitesimal generator of an one-parameter strongly continuous semigroup of contractions on X.

Proof. Let $\varepsilon > 0$ such that $\varepsilon C < 1$.

By (**2**.2), the abstract case of the Erhling inequality (**4**.2) and the left-hand side of the Agmon-Douglis-Nirenberg estimate (**3**.2), we can get a positive constant C_ε such that

$$\begin{aligned} &\|V\varphi\|_{0,0,\Lambda} \leq C\varepsilon\|\varphi\|_{m_1,m_2,\Lambda} + CC_\varepsilon\|\varphi\|_{0,0,\Lambda} \leq \\ &C\varepsilon\left(\|A\varphi\|_{0,0,\Lambda} + \|\varphi\|_{0,0,\Lambda}\right) + CC_\varepsilon\|\varphi\|_{0,0,\Lambda} \\ &= C\varepsilon\,\|A\varphi\|_{0,0,\Lambda} + (C\varepsilon + CC_\varepsilon)\|\varphi\|_{0,0,\Lambda}, \varphi \in S. \end{aligned} \tag{4.7}$$

Let $u \in X^\Lambda_{m_1,m_2}$ and let $\{\varphi_k\}_k$ be a sequence in S such that

$$\varphi_k \to u$$

in $X^\Lambda_{m_1,m_2}$ as $k \to \infty$.

Using (**4**.7), we have

$$\|V\varphi_k\|_{0,0,\Lambda} \leq C\varepsilon\|A\varphi_k\|_{0,0,\Lambda} + (C\varepsilon + CC_\varepsilon)\,\|\varphi_k\|_{0,0,\Lambda},$$

for $k = 1, 2, ...$

Since V is maximally dissipative, if we let $k \to \infty$, we get

$$\|Vu\|_{0,0,\Lambda} \leq C\varepsilon\,\|A_0 u\|_{0,0,\Lambda} + (C\varepsilon + CC_\varepsilon)\,\|u\|_{0,0,\Lambda}$$

for $u \in X^\Lambda_{m_1,m_2} \subset \mathcal{D}(V)$ (because by (iv) $X^\Lambda_{m_1,m_2} \subset X^\Lambda_{s_1,s_2} \subset X^\Lambda_{0,0} = X$).

Now, using Theorem **4**.3 the proof is complete. □

5. Fredholmness of minimal operator A_0

In this section, we prove that the minimal operator A_0 of the operator A is Fredholm when suitable hypotheses are satisfied.

Let us recall that a closed linear operator $A : X \to X$ from a complex Banach space X into a complex Banach space Y with dense domain $D(A)$ is said to be Fredholm if

i) $R(A)$ is a closed subspace of Y;

ii) $N(A)$ and $N(A^t)$ are finite dimensional,

where $R(A)$ is the range of A, $N(A)$ is the null space of A and $N(A^t)$ is the null space of the adjoint A^t.

Now, we recall a result in which we find the necessary and sufficient conditions for an operator to be Fredholm.

Theorem 5.1. (see [3]) *Suppose that A is a closed linear operator from a complex Banach space X into a complex Banach space Y with dense domain $D(A)$. Then, A is Fredholm if and only if one can find a closed linear operator $B : Y \to X$, compact operators $K_1 : X \to X$ and $K_2 : Y \to Y$ such that $BA = I + K_1$ on $D(A)$ and $AB = I + K_2$ on Y.*

In order to prove the main result of this section, we need the following theorem.

Theorem 5.2. *Let $s_1, s_2, t_1, t_2 \in \mathbb{R}$ such that $s_1 < t_1$ and $s_2 < t_2$. Then, the inclusion $i : X^\Lambda_{t_1,t_2} \hookrightarrow X^\Lambda_{s_1,s_2}$ is a compact operator.*

Proof. Since $t_1 - s_1 > 0$, $t_2 - s_2 > 0$ by the hypothesis, then, by a corollary of the Archimedean property, there exists a positive number ε such that $0 < \varepsilon < t_1 - s_1$, $0 < \varepsilon < t_2 - s_2$.

Let us consider the following mappings

$$\left(J^{\Lambda}_{\varepsilon,\varepsilon}\right)^{-1} J^{\Lambda}_{-s_1,-s_2} : X^{\Lambda}_{t_1,t_2} \to X^{\Lambda}_{t_1-s_1-\varepsilon,t_2-s_2-\varepsilon},$$

$i : X^{\Lambda}_{t_1-s_1-\varepsilon,t_2-s_2-\varepsilon} \hookrightarrow X^{\Lambda}_{0,0}$ and $J^{\Lambda}_{\varepsilon,\varepsilon} : X^{\Lambda}_{0,0} \to X^{\Lambda}_{0,0}$.

By the composition of these three mappings, we get

$$J^{\Lambda}_{\varepsilon,\varepsilon} i \left(J^{\Lambda}_{\varepsilon,\varepsilon}\right)^{-1} J^{\Lambda}_{-s_1,-s_2} : X^{\Lambda}_{t_1,t_2} \to X^{\Lambda}_{0,0}.$$

But $\left(J^{\Lambda}_{\varepsilon,\varepsilon}\right)^{-1} J^{\Lambda}_{-s_1,-s_2}$ and i are bounded linear operators, by Proposition **2**.1 and the property (iv) in the definition of the two-parameter abstract framework in Section 2, and $J^{\Lambda}_{\varepsilon,\varepsilon}$ is a compact operator by property (i) in the definition of the same two-parameter abstract framework.

Therefore, $J^{\Lambda}_{\varepsilon,\varepsilon} i \left(J^{\Lambda}_{\varepsilon,\varepsilon}\right)^{-1} J^{\Lambda}_{-s_1,-s_2} : X^{\Lambda}_{t_1,t_2} \to X^{\Lambda}_{0,0}$ is a compact operator.

Let us remark that for u in $X^{\Lambda}_{t_1,t_2}$, it follows that $J^{\Lambda}_{\varepsilon,\varepsilon} i \left(J^{\Lambda}_{\varepsilon,\varepsilon}\right)^{-1} J^{\Lambda}_{-s_1,-s_2} u = J^{\Lambda}_{-s_1,-s_2} u \in X^{\Lambda}_{0,0}$. Hence, $u \in X^{\Lambda}_{s_1,s_2}$ and the inclusion $i : X^{\Lambda}_{t_1,t_2} \hookrightarrow X^{\Lambda}_{s_1,s_2}$ is a compact operator. □

The main result in this section reads as follows.

Theorem 5.3. *Let $A : S \subset X \to X$ be an operator as in Theorem* **3**.*1 such that it satisfies the equality $AB = I + L$, where B is the operator in Theorem* **3**.*1 and L is an infinitely smoothing operator. Then, the bounded linear operator $A_0 : X^{\Lambda}_{m_1,m_2} \subset X \to X$ is Fredholm.*

Proof. Since A satisfies the hypothesis of Theorem **3**.1, there exists a linear operator B of order $(-m_1, -m_2)$ from X to X with domain S such that $BA = I + R$ and $AB = I + L$, where I is the identity operator and R, L are infinitely smoothing operators.

For all positive numbers t_1, t_2, the linear operator $R : X \to X^{\Lambda}_{t_1,t_2}$ is bounded by the definition of the smoothing operator and $i : X^{\Lambda}_{t_1,t_2} \hookrightarrow X$ is compact by Theorem **5**.2. Thus, $R : X \to X$ is a compact operator.

Similarly, $L : X \to X$ is also a compact operator.

By Theorem **5**.1, we obtain that $A_0 : X^{\Lambda}_{m_1,m_2} \subset X \to X$ is Fredholm.

Thus, the proof is complete. □

REFERENCES

[1] *S. Agmon, A. Douglis and L. Nirenberg*, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions I, Comm. Pure Appl. Math. 12 (1959), 623-727.

[2] *M. Alimohammady, M. K. Kalleji*, Spectral theory of a hybrid class of pseudo-differential operators, Complex Variables and Elliptic Equations 59(12), 2014.

[3] *F.V. Atikinson*, The normal solubility of linear equations in normed spaces, Mat. Sbornik N.S. 28 (70) (1951), 3-14.

[4] *I. Camperi*, Global hypoellipticity and Sobolev estimates for generalized SG pseudo-differential operators, Rend. Sem. Mat. Univ. Pol. Torino, vol. 66, 2(2008), 99-112.

[5] *V. Catană, H-G. Georgescu*, Essential spectra and semigroups of perturbations of generalized SG-hypoelliptic pseudo-differential operators on $L^p(\mathbb{R}^n)$. J. PseudoDiffer. Oper. Appl. 13, 25 (2022).

[6] *V. Catană*, M-hypoelliptic pseudo-differential operators on $L^p(\mathbb{R}^n)$. Appl. Anal. 87(6), 657-666 (2008).

[7] *V. Catană*, Essential spectra and semigroups of perturbations of M-hypoelliptic pseudo-differential operators on $L^p(\mathbb{R}^n)$. Complex Var. Elliptic Equ. 54(8), 731-744 (2009).

[8] *A. Dasgupta, M.W. Wong*, Spectral Theory of SG Pseudo-Differential Operators on $L^p(\mathbb{R}^n)$, Studia Math, 187 (2008), 186-197.

[9] *B. Davies B*, One Parameter Semigroups, Academic Press, London (1980).

[10] *G. Mihai Iancu, M.W. Wong*, Global solutions of semiliniar heat equations in Hilbert spaces, Abstr. Appl. Anal. 1(3): 263-276 (1996).

[11] *G. Mihai Iancu*, Dynamical systems generated by pseudo-differential operators, PhD Thesis, York University, Toronto, 1999.

[12] *O. Milatovic*, Minimal and maximal extensions of M-hypoelliptic proper uniform pseudo-differential operators in L^p-spaces on non-compact manifolds, J. PseudoDiffer. Oper. Appl. 12, 16(2021).

[13] *F. Nicola, L.Rodino*, Global Pseudo-Differential Calculus on Euclidean Spaces, Pseudo-differential operators, theory and applications, Vol. 4, Springer, 2010.

[14] *A. Pazy*, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer, Berlin (1983).

[15] *M.W. Wong*, An Introduction to Pseudo-Differential Operators, Second Edition, World Scientific, 1999.

[16] *M.W. Wong*, Minimal and maximal operator theory with applications, Can. J. Math. Vol 43 (3), 1991 pp. 617-627.

[17] *M.W. Wong*, Erhling's inequality and pseudo-differential operators on $L^p(\mathbb{R}^n)$, Cubo 8 (2006), 97-108.