EXPONENTIAL DECAY OF THE SOLUTION OF A DOUBLE POROUS ELASTIC SYSTEM

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In the present paper we consider a one-dimensional double porous elastic system with two dissipative mechanisms : a viscoelastic dissipation in the displacement field and visco-porous dissipations. We prove the existence and uniqueness of a solution, and that this solution decays exponentially as t tends to infinity.

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1. Introduction

The origin of the theory of double porosity goes back to the works of Barenblatt *et al.* [4, 5]. The authors distinguished the liquid pressure in the pores from the liquid pressure in the fissures and introduced a double porosity structure. This theory is an important generalization of Biot's theory [3] for porous materials with single porosity. Wilson and Aifentis [17] presented a theory of consolidation for elastic materials with double porosity which unifies the earlier models of Barenblatt and Biot. However, the theory proposed by Wilson and Aifentis ignored the cross-coupling effects between the volume change of the pores and fissures in the system. Khalili and Valliappan [10] modified Aifiens' theory and proposed a cross-coupling terms included in the equations of conservation of mass for the pores and fissures fluid. Barryman and coauthors [1, 2] included a cross-coupling in Darcy's law for solids with double porosity.

In [9] Ieşan and Quintanilla derived a double porosity model based on the Nunziato-Cowin theory for materials with voids [8, 13]. According to this theory the porosity structure in the equilibrium case is influenced by the displacement field, which is different from the theory based on Darcy's law.

The basic feature of the Nunziato-Cowin theory is the concept that the mass at each point is the product of the mass density of the material matrix and the volume fraction. In the framework of this theory Quintanilla [15] considered the following system of porous elastic solid

$$\left\{ \begin{array}{ll} \rho_0 u_{tt} = \mu u_{xx} + \beta \varphi_x, & \text{in } (0,\pi) \times (0,+\infty), \\ \rho_0 \kappa \varphi_{tt} = \alpha \varphi_{xx} - \beta u_x - \xi \varphi - \tau \varphi_t, & \text{in } (0,\pi) \times (0,+\infty), \end{array} \right. \tag{1}$$

where u is the transversal displacement, φ is the volume fraction of the porous material, and $\rho_0, \kappa, \mu, \alpha, \xi, \tau$ are positive constitutive coefficients, that satisfy $\beta \neq 0$ and $\xi\mu > \beta^2$. He proved that the dissipation caused by the porous damping $\tau\varphi_t$ is not powerful to produce an exponential stability.

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Various dissipations mechanisms have been added to the system (1) and different types of decay have been obtained. Magaña and Quintanilla [12] added the viscoelastic dissipation $-\gamma u_{txx}$ to the first equation of (1) and established an exponential rate of decay. The same result was obtained by Casas and Quintanilla [7] when they added thermal dissipation to the system (1).

In this paper we consider a double porous elastic solid in the framework of Ieşan and Quintanilla theory [9]. In the one-dimensional case the evolution equations are

$$\begin{aligned} \rho u_{tt} &= \mathbb{T}_x, \\ \kappa_1 \varphi_{tt} &= \sigma_x + \xi, \\ \kappa_2 \psi_{tt} &= \chi_x + \zeta, \end{aligned}$$

where u is the displacement, φ and ψ are the porous variables, ρ, κ_1 and κ_2 are positive constants. $\mathbb{T}$ is the first Piola-Kirchhoff stress tensor, $\sigma,\ \chi$ are equilibrated stress vectors, ξ and ζ are the intrinsic equilibrated body forces that they must be given by constitutives assumptions. We assume

$$\begin{aligned} \mathbb{T} &= \mu u_x + b\varphi + d\psi + \lambda u_{tx}, \\ \sigma &= \alpha\varphi_x + b_1\psi_x,\ \chi = b_1\varphi_x + \gamma\psi_x, \\ \xi &= -bu_x - \alpha_1\varphi - \alpha_3\psi - \tau_1\varphi_t, \\ \zeta &= -du_x - \alpha_3\varphi - \alpha_2\psi - \tau_2\psi_t. \end{aligned}$$

Here $\mu, b, d, \lambda, \alpha, \alpha_1, \alpha_2, \alpha_3, b_1, b_2, \gamma, \tau_1$ and τ_2 are constants.
If we introduce the constitutive equations into the evolution equations we obtain the system

$$\begin{cases} \rho u_{tt} = \mu u_{xx} + b\varphi_x + d\psi_x + \lambda u_{txx}, & \text{in } (0,\infty)\times(0,L), \\ \kappa_1\varphi_{tt} = \alpha\varphi_{xx} + b_1\psi_{xx} - bu_x - \alpha_1\varphi - \alpha_3\psi - \tau_1\varphi_t & \text{in } (0,\infty)\times(0,L), \\ \kappa_2\psi_{tt} = b_1\varphi_{xx} + \gamma\psi_{xx} - du_x - \alpha_3\varphi - \alpha_2\psi - \tau_2\psi_t & \text{in } (0,\infty)\times(0,L). \end{cases} \tag{2}$$

We assume the boundary conditions

$$u(t,0) = u(t,L) = \varphi_x(t,0) = \varphi_x(t,L) = \psi_x(t,0) = \psi_x(t,L) = 0 \quad \text{in } (0,\infty), \tag{3}$$

and the initial conditions

$$\begin{gathered} u(0,x) = u_0(x),\ u_t(0,x) = u_1(x),\ \varphi(0,x) = \varphi_0(x),\ \varphi_t(0,x) = \varphi_1(x), \\ \psi(0,x) = \psi_0(x),\ \psi_t(0,x) = \psi_1(x),\ x \in [0,L]\,. \end{gathered} \tag{4}$$

There are solutions (uniform in the variable x) that do not decay. To avoid this case, we also assume that

$$\int_0^L \varphi_0(x)\,dx = \int_0^L \varphi_1(x)\,dx = \int_0^L \psi_0(x)\,dx = \int_0^L \psi_1(x)\,dx = 0.$$

We introduce the energy associated with the system (2)-(4) as

$$\begin{aligned} E(t) :=& \frac{1}{2}\int_0^L \Big[\rho|u_t|^2 + \kappa_1|\varphi_t|^2 + \kappa_2|\psi_t|^2 + \mu|u_x|^2 \\ &+ \alpha|\varphi_x|^2 + \gamma|\psi_x|^2 + \alpha_1|\varphi|^2 + \alpha_2|\psi|^2\Big] \\ &+ b\int_0^L \operatorname{Re}(u_x\overline{\varphi}) + d\int_0^L \operatorname{Re}\left(u_x\overline{\psi}\right) + \alpha_3\int_0^L \operatorname{Re}\left(\varphi\overline{\psi}\right) + b_1\int_0^L \operatorname{Re}\left(\varphi_x\overline{\psi_x}\right), \end{aligned}$$

which can be written as

$$\begin{aligned} E(t) = \frac{1}{2}\int_0^L \Big[&(u_t,\varphi_t,\psi_t)\,A\left(\overline{u_t},\overline{\varphi_t},\overline{\psi_t}\right)^T + (u_x,\varphi,\psi)\,B\left(\overline{u_x},\overline{\varphi},\overline{\psi}\right)^T \\ &+ (\varphi_x,\psi_x)\,C\left(\overline{\varphi_x},\overline{\psi_x}\right)^T\Big], \end{aligned}$$

where,

$$A=\begin{pmatrix}\rho & 0 & 0\\ 0 & \kappa_1 & 0\\ 0 & 0 & \kappa_2\end{pmatrix}, B=\begin{pmatrix}\mu & b & d\\ b & \alpha_1 & \alpha_3\\ d & \alpha_3 & \alpha_2\end{pmatrix}, C=\begin{pmatrix}\alpha & b_1\\ b_1 & \gamma\end{pmatrix}.$$

It is assumed that the internal mechanical energy density is a positive definite form. Thus the matrix A, B and C must be positive definite.

We have the following result:

Lemma 1.1. *If (u,φ,ψ) is the solution of (2)-(4), then the energy $E(t)$ satisfies the estimate*

$$E'(t)\le -\tau_1\int_0^L|\varphi_t|^2\,dx-\tau_2\int_0^L|\psi_t|^2\,dx-\lambda\int_0^L|u_{xt}|^2\,dx.$$

Proof. Taking the L^2-product of $(2)_1$ by u_t, $(2)_2$ by φ_t and $(2)_3$ by ψ_t and summing up we obtain

$$\begin{aligned}&\rho\int_0^L u_{tt}\overline{u}_t dx+\mu\int_0^L u_x\overline{u}_{xt}dx+\kappa_1\int_0^L\varphi_{tt}\overline{\varphi}_t dx+\alpha\int_0^L\varphi_x\overline{\varphi}_{xt}dx+\alpha_1\int_0^L\varphi\overline{\varphi}_t dx\\ &+\kappa_2\int_0^L\psi_{tt}\overline{\psi}_t dx+\gamma\int_0^L\psi_x\overline{\psi}_{xt}dx+\alpha_2\int_0^L\psi\overline{\psi}_t dx+b\int_0^L(u_x\overline{\varphi}_t+\overline{u}_{xt}\varphi)dx\\ &+d\int_0^L(u_x\overline{\psi_t}+\overline{u}_{xt}\psi)dx+b_1\int_0^L(\psi_x\overline{\varphi}_{xt}+\overline{\psi}_{xt}\varphi_x)dx+\alpha_3\int_0^L(\psi\overline{\varphi}_t+\overline{\psi}_t\varphi)dx\\ &=-\lambda\int_0^L|u_{xt}|^2\,dx-\tau_1\int_0^L|\varphi_t|^2\,dx-\tau_2\int_0^L|\psi_t|^2\,dx.\end{aligned}$$

□

The aim of this paper is to prove that the problem determined by (2)-(4) has a unique solution that decays exponentially in time. For the well-posedness we use the Lumer-Phillips theorem and for the exponential stability we use the method developed by Liu and Zheng [11].

To the best of our knowledge the problem is novel and no study has been done to determine the rate of decay of the solution of problems in double porous elasticity.

2. **Well posedness**

The aim of this section is to prove that the problem (2)-(4) has a unique solution. Our main tools are the two following theorems from the theory of semigroups of operators in Hilbert spaces.

Theorem 2.1. (Lumer-Phillips) [14, 16] *Let $\mathcal{A}: D(\mathcal{A})\subset H\longrightarrow H$ be a densely defined operator. Then $\mathcal{A}$ generates a C_0-semigroup of contractions on H if and only if*
(i) $\mathcal{A}$ is dissipative;
(ii) there exists $\lambda>0$ such that $\lambda I-\mathcal{A}$ is surjective.

Theorem 2.2. [16] *Let $\mathcal{A}: D(\mathcal{A})\subset H\longrightarrow H$ be the infintesimal generator of a C_0-semigroup $\{S(t);t\ge 0\}$. Then, for each $\xi\in D(\mathcal{A})$ and each $t\ge 0$, we have $S(t)\xi\in D(\mathcal{A})$, and the mapping*

$$t\longrightarrow S(t)\xi$$

is of class C^1 on $[0,+\infty)$ and satisfies

$$\frac{d}{dt}(S(t)\xi)=\mathcal{A}S(t)\xi=S(t)\mathcal{A}\xi.$$

In order to rewrite the problem (2)-(4) in the semigroup setting we introduce the Hilbert space

$$\mathcal{H} := H_0^1(0,L) \times L^2(0,L) \times H_*^1(0,L) \times L^2(0,L) \times H_*^1(0,L) \times L^2(0,L),$$

where $H^1(0,L)$, $H^2(0,L)$ are the usual Sobolev spaces, $H_0^1(0,L)$ is the closure of $C_0^\infty(0,L)$ in $H^1(0,L)$ [6, 11] and

$$H_*^1(0,L) := \left\{ \varphi \in H^1(0,L); \int_0^L \varphi(t,x)dx = 0 \right\}.$$

The space $\mathcal{H}$ is endowed with the inner product $\langle \cdot,\cdot \rangle_{\mathcal{H}}$ defined for $U = (u,v,\varphi,\phi,\psi,\omega)^T$ and $U^* = (u^*,v^*,\varphi^*,\phi^*,\psi^*,\omega^*)^T$ by

$$\begin{aligned}\langle U,U^*\rangle_{\mathcal{H}} = \quad & \int_0^L [\rho v\overline{v^*} + \mu u_x \overline{u^*}_x + \kappa_1 \phi\overline{\phi^*} + \alpha\varphi_x\overline{\varphi^*}_x + \alpha_1\varphi\overline{\varphi^*} + \kappa_2\omega\overline{\omega^*} + \gamma\psi_x\overline{\psi^*}_x \\ & +\alpha_2\psi\overline{\psi^*} + b(u_x\overline{\varphi^*} + \varphi\overline{u^*}_x) + d(u_x\overline{\psi^*} + \psi\overline{u^*}_x) + b_1(\psi_x\overline{\varphi^*}_x + \varphi_x\overline{\psi^*}_x) \\ & +\alpha_3(\psi\overline{\varphi^*} + \varphi\overline{\psi^*})]dx.\end{aligned}$$

By introducing the new variables $v = u_t, \phi = \varphi_t$ and $\omega = \psi_t$, system (2) becomes

$$\left\{ \begin{array}{c} U_t = \mathcal{A}U, \\ U(0) = (u_0,u_1,\varphi_0,\varphi_1,\psi_0,\psi_1)^T, \end{array} \right. \tag{5}$$

where $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \longrightarrow \mathcal{H}$ is the operator defined by

$$\mathcal{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{\mu}{\rho}\partial_{xx} & \frac{\lambda}{\rho}\partial_{xx} & \frac{b}{\rho}\partial_x & 0 & \frac{d}{\rho}\partial_x & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -\frac{b}{\kappa_1}\partial_x & 0 & \frac{\alpha}{\kappa_1}\partial_{xx} - \frac{\alpha_1}{\kappa_1} & -\frac{\tau_1}{\kappa_1} & \frac{b_1}{\kappa_1}\partial_{xx} - \frac{\alpha_3}{\kappa_\&} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -\frac{d}{\kappa_2}\partial_x & 0 & \frac{b_1}{\kappa_2}\partial_{xx} - \frac{\alpha_3}{\kappa_2} & 0 & \frac{\gamma}{\kappa_2}\partial_{xx} - \frac{\alpha_2}{\kappa_2} & -\frac{\tau_2}{\kappa_2} \end{pmatrix} \tag{6}$$

with domain

$$D(\mathcal{A}) = \left\{ \begin{array}{c} (u,v,\varphi,\phi,\psi,w) \in \mathcal{H} \; | v \in H_0^1(0,L), \phi,\omega \in H_*^1(0,L), \\ \mu u + \lambda v \in H^2 \cap H_0^1, \varphi,\psi \in H^2, \; \varphi_x(t,x) = \; \psi_x(t,x) = 0, \; x = 0,L \end{array} \right\}.$$

Our existence and uniqueness result reads as follows.

Theorem 2.3. *For any $(u_0,u_1,\varphi_0,\varphi_1,\psi_0,\psi_1) \in \mathcal{H}$ the problem (2)-(4) has a unique weak solution (u,φ,ψ) that satisfies*

$$u \in C\left(0,+\infty; H_0^1(0,L)\right), \; \varphi \in C\left(0,+\infty; H_*^1(0,L)\right), \; \psi \in C\left(0,+\infty; H_*^1(0,L)\right).$$

Moreover, if $(u_0,u_1,\varphi_0,\varphi_1,\psi_0,\psi_1) \in D(\mathcal{A})$ then the solution (u,φ,ψ) satisfies

$$u \in C\left(0,+\infty; H^2 \cap H_0^1(0,L)\right) \cap C^1\left(0,+\infty; H_0^1(0,L)\right),$$

$$\varphi,\psi \in C\left(0,+\infty; H^2 \cap H_*^1(0,L)\right) \cap C^1\left(0,+\infty; H_*^1(0,L)\right).$$

Proof. First, for any $(u,v,\varphi,\phi,\psi,\omega) \in D(\mathcal{A})$ we have

$$\operatorname{Re}\langle \mathcal{A}U,U\rangle_{\mathcal{H}} = -\int_0^L |v_x|^2dx - \tau_1\int_0^L |\phi|^2dx - \tau_2\int_0^L |w|^2dx \le 0.$$

Thus, $\mathcal{A}$ is dissipative.

Secondly, we prove that $0 \in \rho(\mathcal{A})$. Indeed, let $F = (f, g, h, l, q, k) \in \mathcal{H}$ and find $U = (u, v, \varphi, \phi, \psi, \omega) \in D(\mathcal{A})$ such that $\mathcal{A}U = F$, that is

$$\begin{cases} v = f, \\ \mu u_{xx} + b\varphi_x + d\psi_x + \lambda v_{xx} = \rho g, \\ \phi = h, \\ \alpha\varphi_{xx} + b_1\psi_{xx} - bu_x - \alpha_1\varphi - \alpha_3\psi - \tau_1\phi = \kappa_1 k, \\ \omega = l, \\ b_1\varphi_{xx} + \gamma\psi_{xx} - du_x - \alpha_3\varphi - \alpha_2\psi - \tau_2\omega = \kappa_2 k. \end{cases} \tag{7}$$

From $(7)_1$, $(7)_3$ and $(7)_5$ we have $v \in H_0^1(0, L), \phi, \omega \in H_*^1(0, L)$.

Replacing v, ϕ and ω by f, h and l respectively we obtain

$$\begin{cases} \mu u_{xx} + b\varphi_x + d\psi_x = \rho g - \lambda f_{xx} = g_1 \in H^{-1}(0, L), \\ \alpha\varphi_{xx} + b_1\psi_{xx} - bu_x - \alpha_1\varphi - \alpha_3\psi = \kappa_1 k + \tau_1 h = g_2 \in L^2(0, L), \\ b_1\varphi_{xx} + \gamma\psi_{xx} - du_x - \alpha_3\varphi - \alpha_2\psi = \kappa_2 k + \tau_2 l = g_3 \in L^2(0, L). \end{cases} \tag{8}$$

Taking the duality product of $(8)_1$ by u^* and the L^2-product of $(8)_2$ and $(8)_3$ by φ^* and ψ^*, respectively, then summing up we obtain

$$a(U, U^*) = L(U^*), \tag{9}$$

where $U = (u, \varphi, \psi)^T$, $U^* = (u^*, \varphi^*, \psi^*)^T$ and

$$a(U, U^*) = \mu\int_0^L u_x\overline{u_x^*}dx + b\int_0^L \varphi\overline{u_x^*}dx + d\int_0^L \psi\overline{u_x^*}dx$$
$$+\alpha\int_0^L \varphi_x\overline{\varphi_x^*}dx + b_1\int_0^L \psi_x\overline{\varphi_x^*}dx + b\int_0^L u_x\overline{\varphi^*}dx + \alpha_1\int_0^L \varphi\overline{\varphi^*}dx + \alpha_3\int_0^L \psi\overline{\varphi^*}dx$$
$$+b_1\int_0^L \varphi_x\overline{\psi_x^*}dx + \gamma\int_0^L \psi_x\overline{\psi_x^*}dx + d\int_0^L u_x\overline{\psi^*}dx + \alpha_3\int_0^L \varphi\overline{\psi^*}dx + \alpha_2\int_0^L \psi\overline{\psi^*}dx,$$
$$L(U^*) = \langle -g_1, u^*\rangle_{H^{-1}\times H_0^1} - \int_0^L g_2\overline{\varphi^*}dx - \int_0^L g_3\overline{\psi^*}dx,$$

are a bilinear and linear forms over the Hilbert space $\mathcal{W} = H_0^1(0, L) \times H_*^1(0, L) \times H_*^1(0, L)$ respectively,. A straightforward calculation shows that there exists a positive constant C such that

$$|a(U, U^*)| \leq C\|U\|_{\mathcal{W}}\|U^*\|_{\mathcal{W}}$$

and

$$|L(U^*)| \leq C\|U^*\|_{\mathcal{W}}.$$

Thus $a(\cdot, \cdot)$ and L are continuous. Moreover, straightforward calculations show that

$$a(U, U) = \mu\int_0^L |u_x|^2 dx + b\int_0^L \varphi\overline{u}_x dx + d\int_0^L \psi\overline{u}_x dx + d\int_0^L u_x\overline{\psi}dx$$
$$+b\int_0^L u_x\overline{\varphi}dx + \alpha_1\int_0^L |\varphi|^2 dx + \alpha_3\int_0^L \psi\overline{\varphi}dx + \alpha_3\int_0^L \varphi\overline{\psi}dx + \alpha_2\int_0^L |\psi|^2 dx$$
$$+\frac{1}{2}\int_0^L \left[\alpha\left|\varphi_x + \frac{b_1}{\alpha}\psi_x\right|^2 + \gamma\left|\psi_x + \frac{b_1}{\gamma}\varphi_x\right|^2\right]$$
$$+\frac{1}{2}\left(\alpha - \frac{b_1^2}{\gamma}\right)\int_0^L |\varphi_x|^2 dx + \frac{1}{2}\left(\gamma - \frac{b_1^2}{\alpha}\right)\int_0^L |\psi_x|^2 dx.$$

On the other hand, there exists $\eta > 0$ such that the matrix

$$B' = \begin{pmatrix} \mu - \eta & b & d \\ b & \alpha_1 - \eta & \alpha_3 \\ d & \alpha_3 & \alpha_2 - \eta \end{pmatrix}$$

still positive definite. Therefore,

$$a(U,U) = (u_x,\varphi,\psi)\, B'\left(\overline{u}_x,\overline{\varphi},\overline{\psi}\right)^T + \frac{1}{2}\int_0^L \left[\alpha\left|\varphi_x+\frac{b_1}{\alpha}\psi_x\right|^2+\gamma\left|\psi_x+\frac{b_1}{\gamma}\varphi_x\right|^2\right]$$

$$+\eta\left(\|u_x\|^2+\|\varphi\|^2+\|\psi\|^2\right)$$

$$+\frac{1}{2}\left(\alpha-\frac{b_1^2}{\gamma}\right)\int_0^L |\varphi_x|^2\,dx+\frac{1}{2}\left(\gamma-\frac{b_1^2}{\alpha}\right)\int_0^L |\psi_x|^2\,dx \geq \widetilde{\eta}\,\|U\|_{\mathcal{W}}^2,$$

for a positive constant $\widetilde{\eta}>0$, which shows the coercivity of $a(\cdot,\cdot)$.

Thus, Lax-Milgram theorem ensures the existence of unique $(u,\varphi,\psi)\in\mathcal{W}$ satisfying

$$a(U,U^*) = L(U^*),\ \forall U^*\in\mathcal{W}.$$

Now, taking $\varphi^*=\psi^*=0$ in (9) and replacing f by v we get

$$\mu\int_0^L u_x\overline{u^*}_x dx + b\int_0^L \varphi\overline{u^*}_x dx + d\int_0^L \psi\overline{u^*}_x dx = \int_0^L (\lambda v_{xx}-\rho g)\,\overline{u^*}dx,$$

and integration by parts gives

$$\int_0^L (\mu u_x+\lambda v_x)\,\overline{u^*}_x dx = \int_0^L (b\varphi_x+d\psi_x-\rho g)\,\overline{u^*}dx,\quad \forall u^*\in H_0^1(0,L),$$

therefore,

$$\mu u+\lambda v\in H^2(0,L).$$

Next, let $\varphi^*\in H_0^1(0,L)$ and define

$$\varphi_1^*(x) = \varphi^*(x) - \int_0^L \varphi^*(x)\,dx.$$

Observing that $\varphi_1^*\in H_*^1(0,L)$ and taking $u^*=\psi^*=0$ in (9) we obtain

$$\int_0^L (\alpha\varphi_x+b_1\psi_x)\,\overline{\varphi_{1x}^*}dx = -\int_0^L (bu_x+\alpha_1\varphi+\alpha_3\psi+g_2)\,\overline{\varphi_1^*}dx,\ \forall\varphi^*\in H_0^1(0,L),$$

therefore,

$$\alpha\varphi+b_1\psi\in H^2(0,L). \tag{10}$$

Moreover, integration by parts gives

$$(\alpha\varphi_x(L)+b_1\psi_x(L))\,\overline{\varphi_{1x}^*}(L)-(\alpha\varphi_x(0)+b_1\psi_x(0))\,\overline{\varphi_{1x}^*}(0)=0,\ \forall\varphi^*\in H_0^1(0,L).$$

Since φ^* is arbitrary we obtain

$$\alpha\varphi_x(0)+b_1\psi_x(0)=\alpha\varphi_x(L)+b_1\psi_x(L)=0.$$

Similarly,

$$b_1\varphi+\gamma\psi\in H^2(0,L) \tag{11}$$

and

$$b_1\varphi_x(0)+\gamma\psi_x(0)=b_1\varphi_x(L)+\gamma\psi_x(L)=0.$$

Thus,

$$\varphi,\psi\in H^2(0,L)\ \text{ and }\ \varphi_x=\psi_x=0,\qquad \text{for } x=0,L.$$

Therefore, $U\in D(\mathcal{A})$ and $0\in\rho(\mathcal{A})$. Moreover, using a geometric series argument we prove that $\lambda I-\mathcal{A}=\mathcal{A}(\lambda\mathcal{A}^{-1}-I)$ is invertible for $|\lambda|<\|\mathcal{A}^{-1}\|$, then $\lambda\in\rho(\mathcal{A})$, which completes the proof that $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup, then the Lumer-Phillips theorem ensures the existence of unique solution to the problem (2)-(4) satisfying the statements of Theorem 1. □

Remark 2.1. We note that if $U_0 \in D(\mathcal{A})$ then the solution $U(t) = e^{t\mathcal{A}}U_0 \in C((0,\infty); D(\mathcal{A})) \cap C^1((0,\infty); \mathcal{H})$ and (5) is satisfied in $\mathcal{H}$ for every $t > 0$. It turns out that u, φ, ψ satisfy (2) in the strong sense.

If $U_0 \in \mathcal{H}$ there exists a sequence $U_{0n} \in D(\mathcal{A})$ converging to U_0 in $\mathcal{H}$. Accordingly, there exists a sequence of solutions $U_n(t) = e^{t\mathcal{A}}U_{0n}$ such that u_n, φ_n, ψ_n satisfy (2) in L^2 for every $t > 0$, and for any $T > 0$, $u_n \to u$ in $C((0,T), H_0^1) \cap C^1((0,T); L^2)$, $\varphi_n \to \varphi$ and $\psi_n \to \psi$ in $C((0,T), H_*^1) \cap C^1((0,T); L^2)$. Therefore, if we multiply the equations of (2) for u_n, φ_n, ψ_n by $u^* \in H_0^1$ and $\varphi^*, \psi^* \in H_*^1$, respectively, then integrate by parts with respect to x and integrate with respect to t, finally passing to the limit, we find that u, φ and ψ are weak solutions to the variational form of system (2).

3. **Exponential stability**

In this section we establish an exponential decay of the solution of the system (2)-(4). The following theorem, due to Gearhart and Prüss [11], gives the necessary and sufficient conditions of exponential stability of a C$_0$-semigroup generated by an operator $\mathcal{A}$.

Theorem 3.1. *A C_0-semigroup of contractions $S(t) = e^{-\mathcal{A}t}$, generated by an operator $\mathcal{A}$ in a Hilbert space $\mathcal{H}$, is exponentially stable if and only if*

i) $i\mathbb{R} = \{i\lambda, \lambda \in \mathbb{R}\} \subset \rho(\mathcal{A})$, ii) $\overline{\lim\limits_{|\lambda|\to\infty}} \left\|(i\lambda I - \mathcal{A})^{-1}\right\| < \infty.$

The main result of this section is given by the following theorem:

Theorem 3.2. (Gearhart Prüss) *For any $(u_0, u_1, \varphi_0, \varphi_1, \psi_0, \psi_1) \in D(\mathcal{A})$, the energy associated with the solution of the problem (5) satisfies the estimate*

$$E(t) \leq \beta e^{-\omega t}, \qquad \forall t \geq 0,$$

where β, ω are two positive constants.

Proof. The proof of this theorem will be established through the two following lemmas. □

Lemma 3.1. *Let $\mathcal{A}$ be the operator defined by (6). Then,*

$$i\mathbb{R} = \{i\lambda; \lambda \in \mathbb{R}\} \subset \rho(\mathcal{A}),$$

where $\rho(\mathcal{A})$ is the resolvent set of $\mathcal{A}$.

Proof. The proof of the lemma will be established in 3 steps:
(i) Using a geometric series convergence argument and the fact that $0 \in \rho(\mathcal{A})$ it follows that for $|\lambda| < \|\mathcal{A}^{-1}\|^{-1}$, the operator $i\lambda I - \mathcal{A} = \mathcal{A}(i\lambda\mathcal{A}^{-1} - I)$ is invertible. Moreover $\left\|(i\lambda I - \mathcal{A})^{-1}\right\|$ is a continuous function of λ in the interval $(-\left\|\mathcal{A}^{-1}\right\|^{-1}, \left\|\mathcal{A}^{-1}\right\|^{-1})$.
(ii) If there exists a constant $M > 0$, such that

$$\sup\left\{\left\|(i\lambda I - \mathcal{A})^{-1}\right\|, |\lambda| < \left\|\mathcal{A}^{-1}\right\|^{-1}\right\} = M < \infty, \tag{12}$$

then, for $|\lambda_0| < \left\|\mathcal{A}^{-1}\right\|^{-1}$, again the geometric series argument ensures that the operator

$$i\lambda I - \mathcal{A} = (i\lambda_0 I - \mathcal{A})(I + i(\lambda - \lambda_0)(i\lambda_0 I - \mathcal{A})^{-1})$$

is invertible for

$$|\lambda - \lambda_0| < \frac{1}{M} \leq \frac{1}{\|(i\lambda_0 I - \mathcal{A})^{-1}\|}.$$

It turns out that if we choose $|\lambda_0|$ as close as possible to $\left\|\mathcal{A}^{-1}\right\|^{-1}$, we have that $i\lambda I - \mathcal{A}$ is invertible for $|\lambda| < \left\|\mathcal{A}^{-1}\right\|^{-1} + \frac{1}{M}$. Therefore,

$$\left\{i\lambda; |\lambda| < \left\|\mathcal{A}^{-1}\right\|^{-1} + \frac{1}{M}\right\} \subset \rho(\mathcal{A})$$

and $\left\|(i\lambda I-\mathcal{A})^{-1}\right\|$ is a continuous function of λ in the interval $\left(-\left\|\mathcal{A}^{-1}\right\|^{-1}-\frac{1}{M},\left\|\mathcal{A}^{-1}\right\|^{-1}+\frac{1}{M}\right)$. So on

$$\left\{i\lambda;|\lambda|<\left\|\mathcal{A}^{-1}\right\|^{-1}+\frac{1}{M}+\frac{1}{M'}\right\}\subset\rho(\mathcal{A})$$

provided that

$$\sup\left\{\left\|(i\lambda I-\mathcal{A})^{-1}\right\|,|\lambda|<\left\|\mathcal{A}^{-1}\right\|^{-1}+M^{-1}\right\}=M'<\infty.$$

The interval of imaginary axis included in the resolvent set can be extended indefinitely until it coincides with $i\mathbb{R}$.
(iii) If $i\mathbb{R}\not\subset\rho(\mathcal{A})$ then from the argument (ii) above, there exists $\sigma\in\mathbb{R}$ with $\left\|\mathcal{A}^{-1}\right\|^{-1}\leq|\sigma|<\infty$ such that

$$\{i\lambda;|\lambda|<|\sigma|\}\subset\rho(\mathcal{A})$$

and

$$\sup\left\{\left\|(i\lambda I-\mathcal{A})^{-1}\right\|,|\lambda|<|\sigma|\right\}=\infty.$$

Thus, there exists a sequence $(\lambda_n)\subset\mathbb{R}$, $|\lambda_n|<|\sigma|,\lambda_n\longrightarrow\sigma$ and a sequence of unit vectors $U_n=(u_n,v_n,\phi_n,\varphi_n,\theta_n)\in D(\mathcal{A})$, such that

$$\lim_{n\longrightarrow\infty}\|(i\lambda_n I-\mathcal{A})U_n\|=0,$$

that is

$$i\lambda_n u_n-v_n\to 0,\text{ in }H_0^1, \tag{13}$$

$$i\lambda_n\rho v_n-\mu D^2u_n-bD\varphi_n-dD\psi_n-\lambda D^2v_n\longrightarrow 0,\text{ in }L^2, \tag{14}$$

$$i\lambda_n\varphi_n-\phi_n\longrightarrow 0,\text{ in }H_*^1, \tag{15}$$

$$i\lambda_n\kappa_1\phi_n-\alpha D^2\varphi_n-b_1D^2\psi_n+bDu_n+\alpha_1\varphi_n+\alpha_3\psi_n+\tau_1\phi_n\longrightarrow 0,\text{ in }L^2, \tag{16}$$

$$i\lambda_n\psi_n-\omega_n\longrightarrow 0,\text{ in }H_*^1, \tag{17}$$

$$i\lambda_n\kappa_2\omega_n-b_1D^2\varphi_n-\gamma D^2\psi_n+dDu_n+\alpha_3\varphi_n+\alpha_2\psi_n+\tau_2\omega_n\longrightarrow 0,\text{ in }L^2. \tag{18}$$

First we have

$$\mathrm{Re}\langle(i\lambda_nI-\mathcal{A})U_n,U_n\rangle_{\mathcal{H}}\longrightarrow 0.$$

Thus,

$$\begin{aligned}\mathrm{Re}\langle(i\lambda_nI-\mathcal{A})U_n,U_n\rangle_{\mathcal{H}}&=-\mathrm{Re}\langle\mathcal{A}U_n,U_n\rangle_{\mathcal{H}},\\&=\tau_2\int_0^L|\omega_n|^2dx+\tau_1\int_0^L|\phi_n|^2dx+\gamma\int_0^L|Dv_n|^2dx\longrightarrow 0.\end{aligned}$$

Therefore

$$\|\phi_n\|_{L^2}\ \ \|\omega_n\|_{L^2}\longrightarrow 0, \tag{19}$$

and

$$\|Dv_n\|_{L^2}\longrightarrow 0. \tag{20}$$

Moreover, Poincaré's inequality leads to

$$\|v_n\|_{L^2}\to 0. \tag{21}$$

Since λ_n is bounded, the arguments (21), (20) and (13) give

$$\|u_n\|_{L^2}\to 0,\ \|Du_n\|_{L^2}\to 0. \tag{22}$$

Similarly, from (15), (17) and (19) we get

$$\|\varphi_n\|_{L^2}\to 0,\ \|\psi_n\|_{L^2}\to 0. \tag{23}$$

Removing the terms that tend to 0 from (16) and (18), it remains

$$\left\{ \begin{array}{l} \alpha D^2\varphi_n + b_1 D^2\psi_n \longrightarrow 0, \text{ in } L^2, \\ b_1 D^2\varphi_n + \gamma D^2\psi_n \longrightarrow 0, \text{ in } L^2. \end{array} \right. \tag{24}$$

Multiplying $(24)_1$ by $\gamma\varphi_n$, $(24)_2$ by $b_1\varphi_n$ and subtracting we obtain

$$\|D\varphi_n\|_{L^2} \to 0. \tag{25}$$

Similarly,

$$\|D\psi_n\|_{L^2} \to 0. \tag{26}$$

By combining (19),(21),(22),(23),(25) and (26) we obtain that $\|U_n\|_{\mathcal{H}} \to 0$, which contradicts the fact that $\|U_n\|_{\mathcal{H}} = 1$. Thus, the proof of Lemma 3.1 is completed. □

Lemma 3.2. *The operator $\mathcal{A}$ defined by (6) satisfies*

$$\lim_{|\lambda|\to\infty} \sup \left\| (i\lambda I - \mathcal{A})^{-1} \right\|_{\mathcal{L}(\mathcal{H})} < \infty. \tag{27}$$

Proof. To prove the lemma statement we use a contradiction argument. Suppose that (27) does not hold, that is

$$\lim_{|\lambda|\to\infty} \sup \left\| (i\lambda I - \mathcal{A})^{-1} \right\|_{\mathcal{L}(\mathcal{H})} = \infty.$$

Then, there exist a sequence $(\lambda_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ and a sequence of unit vectors $U_n = (u_n, v_n, \varphi_n, \phi_n, \psi_n, \omega_n) \in D(\mathcal{A})$ such that

$$\lim_{|\lambda_n|\longrightarrow+\infty} \|(i\lambda_n I - \mathcal{A})U_n\|_{\mathcal{H}} \longrightarrow 0.$$

As in the proof of the previous lemma, (13)-(18) hold. Consequently,

$$\|\omega_n\|_{L^2} \longrightarrow 0, \|\phi_n\|_{L^2} \longrightarrow 0, \|Dv_n\|_{L^2} \longrightarrow 0, \ \|v_n\|_{L^2} \longrightarrow 0. \tag{28}$$

By dividing (13),(15) and (17) by λ_n we obtain

$$\|u_n\|_{L^2}, \|Du_n\|_{L^2} \longrightarrow 0, \tag{29}$$

$$\|\varphi_n\|_{L^2}, \|\psi_n\|_{L^2} \longrightarrow 0. \tag{30}$$

The L^2 product of (15) by ϕ_n gives $i\lambda_n \langle \varphi_n, \phi_n \rangle - \|\phi_n\|^2 \longrightarrow 0$. Therefore,

$$i\lambda_n \langle \varphi_n, \phi_n \rangle \longrightarrow 0. \tag{31}$$

Taking the inner product of (17) by φ_n we get $i\lambda_n \langle \psi_n, \varphi_n \rangle - \langle \omega_n, \varphi_n \rangle \longrightarrow 0$. The fact that $|\langle \varphi_n, \omega_n \rangle| \le \|\varphi_n\| \|\omega_n\| \longrightarrow 0$ yields

$$i\lambda_n \langle \varphi_n, \psi_n \rangle \longrightarrow 0. \tag{32}$$

Removing the terms that tend to 0 from (16), (18) it remains

$$i\lambda_n\kappa_1\phi_n - \alpha D^2\varphi_n - b_1 D^2\psi_n \longrightarrow 0, \text{ in } L^2. \tag{33}$$

and

$$i\lambda_n\kappa_2\omega_n - b_1 D^2\varphi_n - \gamma D^2\psi_n \longrightarrow 0, \text{ in } L^2. \tag{34}$$

Multiplying (33) by $\gamma\varphi_n$, (34) by $-b_1\varphi_n$, summing up and taking into account (31), (32), we obtain

$$\left(\alpha\gamma - b_1^2\right) \|D\varphi_n\|^2 \longrightarrow 0.$$

Therefore,

$$\|D\varphi_n\| \longrightarrow 0. \tag{35}$$

The L^2-inner product of (17) by ϕ_n gives

$$i\lambda_n \langle \phi_n, \psi_n \rangle - \langle \phi_n, \omega_n \rangle \longrightarrow 0.$$

Recalling that $\|\phi_n\|, \|\omega_n\| \longrightarrow 0$ we arrive at

$$i\lambda_n \langle \phi_n, \psi_n \rangle \longrightarrow 0.$$

Similarly, multiplying (17) by ω_n we get

$$i\lambda_n \langle \omega_n, \psi_n \rangle \longrightarrow 0.$$

At this point we multiply (33) by $b_1\psi_n$, (34) by $-\alpha\psi_n$ summing up and recalling that $\alpha\gamma - b_1^2 > 0$ we obtain

$$\|D\psi_n\| \longrightarrow 0. \tag{36}$$

From (28), (29), (30), (35) and (36) we have $\|U\|_{\mathcal{H}} \longrightarrow 0$ which contradicts the fact that $\|U\|_{\mathcal{H}} = 1$. Therefore, (27) holds and the proof of Lemma 3.2 is completed. □

Combining the results of Lemmas 3.1, 3.2 and Theorem 3.1 the proof of Theorem 3.2 is completed.

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