U.P.B. Sci. Bull., Series A, Vol. 87, Iss. 2, 2025 ISSN 1223-7027

ALGORITHM ANALYSIS OF SOLVING VARIATIONAL INEQUALITY PROBLEMS BASED ON THE TWO-STEP INERTIAL MANN METHOD

Luhong ZHANG[1], Li-Jun ZHU[2], Claudia MICU[3]

This paper introduces a novel algorithm that blends the two-step inertial method, Tseng's extragradient method, and Mann's method for finding common solutions to quasi-monotone variational inequality problems and fixed-point problems in real Hilbert spaces. Finally, the algorithm's effectiveness is validated using rigorous numerical examples.

Keywords: two-step inertial method, Tseng's extragradient method, Mann method
MSC2020: 53C 05.

1. Introduction

Variational inequality problems (VIP) are one of the core contents of optimization theory and are widely used in fields such as decision-making management, transportation, and operational research. Signorini [1] first proposed the "Signorini problem" in 1933 when studying the frictionless contact between linear and rigid elastomers. In 1964, Stampacchia [2] proposed the initial theory of variational inequalities and established key theorems such as those related to existence and uniqueness. Specifically, the objective is to find $\omega^\ddagger \in \mathbb{C}$ that satisfies condition

$$\langle \mathbb{A}(\omega^\ddagger), \omega^\dagger - \omega^\ddagger \rangle \geq 0, \quad \text{for all } \omega^\dagger \in \mathbb{C},$$

where $\mathbb{H}$ is a real Hilbert space, $\mathbb{C} \subset \mathbb{H}$ is a nonempty, closed, and convex set, and $\mathbb{A}\colon \mathbb{H} \to \mathbb{H}$ is a continuous mapping. Herein, $\|\cdot\|$ denotes the norm, and $\langle \cdot, \cdot \rangle$ represents the inner product.

How to construct a simple and computationally efficient algorithm for solving variational inequality problems has always been a hot topic among scholars. For example, in 1964, Goldstein [3] proposed the projection algorithm, whose iterative format is

$$\omega^{k+1} = P_{\mathbb{C}}(\omega^k - \lambda \mathbb{A}(\omega^k)),$$

where $\lambda > 0$, and $P_{\mathbb{C}}\colon \mathbb{H} \to \mathbb{C}$ is the metric projection operator.

In order to utilize the projection algorithm to solve the VIP, the mapping $\mathbb{A}$ generally satisfies the conditions of being L-Lipschitz continuous and strongly

[1]School of Mathematics and Information Science, North Minzu University, Yinchuan 750021, China. Email: zhang_lu_hong@163.com
[2]The Key Laboratory of Intelligent Information and Big Data Processing of NingXia Province, North Minzu University, Yinchuan 750021, China. Email: zljmath@outlook.com
[3]Department of Mathematics and Informatics, National University of Science and Technology Politehnica Bucharest, 060042 Bucharest, Romania. Email: claudia.aldescu@upb.ro

monotone. It is possible for the projection method to diverge regardless of the step-size λ if the mapping $\mathbb{A}$ is monotone (not strongly monotone).

In 1976, in order to weaken the strong monotonicity of operators, Korpelevich [4] proposed an extragradient (EG) method, which adds one projection after each projection. The operator $\mathbb{A}\colon \mathbb{H} \to \mathbb{H}$ should only be monotone, L-Lipschitz continuous, and the step-size λ in the interval $\left(0, \frac{1}{L}\right)$ for weak convergence in order for this method to work. The iterative format is

$$\begin{cases} v^k = P_{\mathbb{C}}(\omega^k - \lambda\mathbb{A}(\omega^k)) \\ \omega^{k+1} = P_{\mathbb{C}}(\omega^k - \lambda\mathbb{A}(v^k)). \end{cases}$$

To minimize the number of projections and make calculations more convenient, Tseng [5] improved the EG method, which merely requires calculating one projection on the constraint set $\mathbb{C}$. Notably, the constraint conditions remain unchanged. Its iteration format is

$$\begin{cases} v^k = P_{\mathbb{C}}(\omega^k - \lambda\mathbb{A}(\omega^k)) \\ \omega^{k+1} = v^k + \lambda(\mathbb{A}(\omega^k) - \mathbb{A}(v^k)). \end{cases}$$

Similarly, in 2011, the subgradient extragradient algorithm, which employs a half-space construction for the second projection, was introduced by Censor *et al.* [6] with the aim of overcoming the challenge of projection computations. The iterative format is

$$\begin{cases} v^k = P_{\mathbb{C}}(\omega^k - \lambda\mathbb{A}(\omega^k)) \\ \mathbb{T}_k = \{\omega \in \mathbb{H} | \langle \omega^k - \lambda\mathbb{A}(\omega^k) - v^k, \omega - v^k\rangle \leq 0\} \\ \omega^{k+1} = P_{\mathbb{T}_k}\lambda(\omega^k - \lambda\mathbb{A}(v^k)). \end{cases}$$

In general, variational inequalities can be transformed into fixed-point problems (FPP). By introducing new projection algorithms, we can approximate the common solutions of VIP and FPP. Specifically, the common solutions are of the form

$$\omega^{k+1} = \theta^k z^k + (1 - \theta^k)\sum_{i=1}^{m} s(i)\mathbb{T}_{\lambda^i}\omega^k.$$

Recently, many scholars have started to study the iterative algorithms of VIP and FPP; in this respect please, see [7]-[17], and many others.

Inspired by these articles, we present a novel algorithm to find common solutions for the VIP and the FPP by employing the two-step inertial Mann method. Underline that Krasnoselskii-Mann (KM) method [18] is a well-known and traditional method for resolving fixed-point problems, with the iteration format as follows

$$\omega^{k+1} = (1 - \beta^k)\omega^k + \beta^k\mathbb{T}\omega^k,$$

where $\omega^0 \in \mathbb{C}$, $\beta^k \in [0, 1]$, and $\mathbb{T}\colon \mathbb{H} \to \mathbb{H}$ is a nonexpansive mapping.

The Krasnoselskii-Mann methods and Extragradient methods are both iterative algorithms widely used to solve optimization and fixed point problems. The KM methods are mainly employed to estimate the fixed points of nonexpansive operators and control the iterative process through relaxation parameters; the EG methods focuss on variational inequality problems, improving the convergence speed through prediction and correction steps. Moreover, both can introduce inertial terms to accelerate convergence.

We know that the iterative format of Douglas-Rachford splitting (DRS) [19] method is

$$\omega^{k+1} = (1-\beta^k)\omega^k + \beta^k(2\text{prox}_{\gamma R}) \circ (2\text{prox}_{\gamma J})\omega^k,$$

where $\beta^k \in (0,1)$ and $\gamma \in (0,+\infty)$.

When we consider $(2\text{prox}_{\gamma R}) \circ (2\text{prox}_{\gamma J})$ in the DRS methods as the nonexpansive operator $\mathbb{T}$ in the KM methods, the DRS methods is transformed into the KM methods. In other words, the KM method can be regarded as a particular instance of the DRS methods. In fact, the KM methods is quite slow, especially when dealing with large-scale problems. As a result, many scholars consider incorporating inertia or relaxation to accelerate the KM methods; please, see Cortild and Peypouquet [20], Yao *et al.* [21], Iutzeler and Hendrickx [22].

The two-step inertial algorithm converges faster and can find solutions more effectively in certain nonlinear problems compared to the one-step inertial algorithm. It improves algorithm stability and adaptability by utilizing more historical iteration information, thereby providing acceleration effects in specific situations. Therefore, studying the two-step inertial algorithm is of great significance for solving complex nonlinear problems.

Recently, Dong *et al.* [23] introduced the more general KM method, the following iterative format

$$\begin{cases} v^k = \omega^k + \alpha^k(\omega^k - \omega^{k-1}) \\ z^k = \omega^k + \beta^k(\omega^k - \omega^{k-1}) \\ \omega^{k+1} = (1-\lambda^k)v^k + \lambda^k\mathbb{T}(z^k). \end{cases}$$

In this paper, a novel method is constructed by integrating the two-step inertial method, KM method, and the Tseng method. This method is designed to analyze the common solutions of VIP and FPP. We introduce the concept of dynamic strings, where a dynamic string is defined as the linear combination $\sum_{i=1}^{m} s(i)\mathbb{T}_{\lambda^i}$, with the aim of enhancing the computation speed of the method. The advantages of this new approach are demonstrated through numerical examples.

This article is structured as follows. We introduce a few lemmas and properties in Section 2 that will be used in the ensuing parts. We examined the algorithm's weak convergence in Section 3. We verify the effectiveness of the introduced algorithm in Section 4 through several numerical examples.

2. Preliminaries

The following convergence analysis will benefit from some properties and conclusions that we recall in this section.

Definition 2.1. *Let $\mathbb{A}\colon \mathbb{H} \to \mathbb{H}$ be an operator. Then $\mathbb{A}$ is:*

1. *nonexpansive, namely*

$$\|\mathbb{A}(\varrho^\ddagger) - \mathbb{A}(\varrho^\dagger)\| \le \|\varrho^\ddagger - \varrho^\dagger\|, \quad \textit{for all } \varrho^\ddagger, \varrho^\dagger \in \mathbb{H}.$$

2. *monotone, namely*

$$\langle \mathbb{A}(\varrho^\ddagger) - \mathbb{A}(\varrho^\dagger), \varrho^\ddagger - \varrho^\dagger \rangle \ge 0, \quad \textit{for all } \varrho^\ddagger, \varrho^\dagger \in \mathbb{H}.$$

3. *quasimonotone, namely*

$$\langle \mathbb{A}(\varrho^\dagger), \varrho^\ddagger - \varrho^\dagger \rangle > 0 \Rightarrow \langle \mathbb{A}(\varrho^\ddagger), \varrho^\ddagger - \varrho^\dagger \rangle \geq 0, \quad \text{for all } \varrho^\ddagger, \varrho^\dagger \in \mathbb{H}.$$

4. *L-Lipschitz continuous, if exists $L > 0$, such that*

$$\|\mathbb{A}(\varrho^\ddagger) - \mathbb{A}(\varrho^\dagger)\| \leq L\|\varrho^\ddagger - \varrho^\dagger\|, \quad \text{for all } \varrho^\ddagger, \varrho^\dagger \in \mathbb{H}.$$

Remark that all monotone operators are quasi-monotone, but not all quasi-monotone operators are monotone. Therefore, being monotone is a special case of being quasi-monotone.

Definition 2.2. *Let $f\colon \mathbb{H} \rightrightarrows 2^{\mathbb{H}}$ be an extremal mapping. The mapping f is maximal monotone if f is monotone, namely,*

$$\langle \delta^\ddagger - \delta^\dagger, \omega^\ddagger - \omega^\dagger \rangle \geq 0, \quad \text{for all } \delta^\ddagger \in f(\omega^\ddagger), \delta^\dagger \in f(\omega^\dagger),$$

and the graph $D(f)$ of f,

$$D(f) = \{(\omega^\ddagger, \delta^\ddagger) \in \mathbb{H} \times \mathbb{H} | \delta^\ddagger \in f(\omega^\ddagger)\},$$

is not a strict subset of any other monotone operator's graph, where $\times$ denotes the Cartesian product.

Obviously, operator f is maximal if, for every $(\omega^\ddagger, \delta^\ddagger) \in \mathbb{H} \times \mathbb{H}$ for which $\langle \delta^\ddagger - \delta^\dagger, \omega^\ddagger - \omega^\dagger \rangle \geq 0$, for every $(\delta^\dagger, \omega^\dagger) \in D(f)$, then necessarily $\delta^\ddagger \in f(\omega^\ddagger)$.

Lemma 2.1 ([24])**.** Let $\omega^\ddagger \in \mathbb{H}$, the necessary and sufficient condition for z to be the projection of $\omega^\ddagger$ onto C is that for all $\omega^\dagger \in \mathbb{C}$, we have $\langle \omega^\ddagger - z, z - \omega^\dagger \rangle \geq 0$.

Lemma 2.2. Let $\zeta \in \mathbb{R}$. For any $\chi^\ddagger$, $\chi^\dagger \in \mathbb{H}$, we have:

1. $\|\chi^\ddagger \pm \chi^\dagger\|^2 = \|\chi^\ddagger\|^2 \pm 2\langle \chi^\ddagger, \chi^\dagger \rangle + \|\chi^\dagger\|^2$;
2. $\|\zeta\chi^\ddagger + (1-\zeta)\chi^\dagger\|^2 = \zeta\|\chi^\ddagger\|^2 + (1-\zeta)\|\chi^\dagger\|^2 - \zeta(1-\zeta)\|\chi^\ddagger - \chi^\dagger\|^2$;
3. $\|\sum_{i=1}^m s_i\mu_i\|^2 = \sum_{i=1}^m s_i\|\mu_i\|^2 - \frac{1}{2}\sum_{i,j=1}^m s_i s_j \|\mu_i - \mu_j\|^2$,

where $\mu_i \in \mathbb{H}$, $s(i) \in (0,1), i = 1, 2, \dots m$, $\sum_{i=1}^m s_i = 1$.

Lemma 2.3 ([25])**.** Let $\{\alpha^k\}_{k=0}^\infty$ be a real sequence that satisfies $0 < b < \alpha^k \leq d < 1$ for any $k \geq 0$. Also, consider two sequences $\{y^k\}_{k=0}^\infty$ and $\{\rho^k\}_{k=0}^\infty$ in $\mathbb{H}$. There exists $\sigma \geq 0$ such that:

1. $\limsup_{k\to\infty} \|y^k\| \leq \sigma, \limsup_{k\to\infty} \|\rho^k\| \leq \sigma$.
2. $\lim_{k\to\infty} \|\alpha^k y^k + (1-\alpha^k)\rho^k\| = \sigma$.

Furthermore, assume that $\{\lambda^k\}$ is a non-negative real number sequence that satisfies

$$\lambda^{k+1} \leq \varsigma^k \lambda^k + \tau^k, \forall k \in \mathbb{N},$$

where $\{\varsigma^k\}$ and $\{\tau^k\}$ are non-negative real numner sequences, such that $\{\varsigma^k\} \subset [1, +\infty]$, $\sum_{k=1}^\infty (\varsigma^k - 1) < +\infty$, and $\sum_{k=1}^\infty \tau^k < +\infty$. Then,

1. $\lim_{k\to\infty} \|y^k - \rho^k\| = 0$.
2. $\lim_{k\to\infty} \lambda^k$ exists.

Lemma 2.4 ([26])**.** Let $\mathbb{U}\colon \mathbb{C} \to \mathbb{H}$ be a nonexpansive mapping, namely,

$$\|\mathbb{U}(\mu) - \mathbb{U}(v)\| \leq \|\mu - v\|, \quad \text{for all } \mu, v \in \mathbb{C}.$$

Then $I - \mathbb{U}$ is demiclosed at $v \in \mathbb{H}$, which means that for all sequence $\{\mu^k\}_{k=0}^\infty$ in $\mathbb{C}$ such that $\mu^k \rightharpoonup \mu^\ddagger \in \mathbb{C}$ and $(I - \mathbb{U})\mu^k \to v$, we can deduce that $(I - \mathbb{U})\mu^\ddagger = v$.

Lemma 2.5 ([27])**.** Let $\{\mu^k\}_{k=0}^{\infty}$ be an arbitrary sequence in $\mathbb{H}$ that converges weakly to μ, then

$$\liminf_{k\to\infty} \|\mu^k - \mu\| \leq \liminf_{k\to\infty} \|\mu^k - v\|, \text{for all } v \neq \mu.$$

Let the quasi-monotone operator $\mathbb{S}$ satisfy Opial's condition, and $\mathbb{S} - Id$ be demiclosed at zero. It can be seen that for the constant sequence of operators $\mathbb{S}^k = \mathbb{S}, k \geq 0$, Opial's theorem's requirements are all satisfied. Therefore, for an arbitrary $\mu \in \mathbb{C}$, we have $\mathbb{S}^k\mu \rightharpoonup \mu^\ddagger$ and $\mu^\ddagger \in \text{Fix } \mathbb{S}$.

Let the sequence $\{\mu^k\}_{k=0}^{\infty}$ be a sequence in $\mathbb{H}$. Then:

1. $\mu^k \rightharpoonup u$ denotes that $\{\mu^k\}_{k=0}^{\infty}$ converges weakly to μ, i.e., for any $v \in \mathbb{H}$, the sequence $\{\langle v, \mu^k\rangle\}_{k=0}^{\infty}$ converges to $\langle v, \mu\rangle$.

2. $\mu^k \to u$ denotes that $\{\mu^k\}_{k=0}^{\infty}$ converges strongly to μ, i.e., $\lim_{k\to\infty} \|\mu^k - u\| = 0$.

3. **Main results**

We provide a novel method for solving the VIP and FPP under the L-Lipschitz continuity assumption in this section. We first make the following assumptions to guarantee the convergence of the method.

The operator $\mathbb{A} \colon \mathbb{H} \to \mathbb{H}$ is quasimonotone, and $s(i) \in (0,1)$, $i = 1, 2, \ldots, m$ are such that $\sum_{i=1}^{m} s(i) = 1$.

Algorithm 3.1. Step 0. Set parameters $\lambda^1 > 0, \phi \in (0,1), \alpha^k \in [0,1], \beta^k = -\alpha^k$, and any initial point $\omega^0, \omega^1, \omega^2 \in \mathbb{H}$.

Step 1. Calculate

$$\rho^k = \omega^k + \alpha^k(\omega^k - \omega^{k-1}) + \beta^k(\omega^{k-1} - \omega^{k-2}),$$

where

$$\alpha^k = \begin{cases} \min\left\{\dfrac{1}{k^2\|\omega^k - \omega^{k-1}\|^2}, \alpha\right\}, & \|\omega^k - \omega^{k-1}\| \neq 0. \\ \alpha, & \|\omega^k - \omega^{k-1}\| = 0. \end{cases}$$

Step 2. Compute

$$v^k = P_{\mathbb{C}}(\rho^k - \lambda^k \mathbb{A}\rho^k).$$

If $v^k = \rho^k$ or $\mathbb{A}\rho^k = 0$, then the program stops. If not, proceed to step 3.

Step 3. Calculate

$$z^k = v^k - \lambda^k(\mathbb{A}v^k - \mathbb{A}\rho^k),$$

where

$$\lambda^{k+1} = \begin{cases} \min\left\{\dfrac{\phi\|\rho^k - v^k\|}{\|\mathbb{A}\rho^k - \mathbb{A}v^k\|}, \varsigma^k\lambda^k + \tau^k\right\}, & \|\mathbb{A}\rho^k - \mathbb{A}v^k\| \neq 0. \\ \varsigma^k\lambda^k + \tau^k, & \|\mathbb{A}\rho^k - \mathbb{A}v^k\| = 0. \end{cases}$$

Step 4.

$$\omega^{k+1} = \theta^k z^k + (1-\theta^k)\sum_{i=1}^{m} s(i)\mathbb{T}_{\lambda^i}\omega^k.$$

Update k as $k := k+1$ and go back to Step 1.

Lemma 3.1. *If there is $v^k = \rho^k$ or $\mathbb{A}\rho^k = 0$ for a certain k, then ρ^k satisfies the original variational inequality.*

Proof. We analyze the problem by considering two possible scenarios.

Case 1: If $\mathbb{A}\rho^k = 0$, then $\langle \mathbb{A}\rho^k, \omega - \rho^k\rangle = 0$ holds for for all $\omega \in \mathbb{C}$.

Case 2: If $v^k = \rho^k$, due to the properties of projection, we have

$$\langle \rho^k - \lambda^k \mathbb{A}\rho^k - v^k, \omega - v^k\rangle \leq 0,$$

which leads to

$$-\lambda^k \langle \mathbb{A}\rho^k, \omega - v^k\rangle \leq 0,$$

that is

$$\langle \mathbb{A}\rho^k, \omega - v^k\rangle \geq 0.$$

So the inequality $\langle \mathbb{A}\rho^k, \omega - \rho^k\rangle \geq 0$ holds for every $\omega \in \mathbb{C}$. In summary, ρ^k satisfies the original variational inequality. □

Lemma 3.2. *If sequence $\{\lambda^k\}$ is obtained by Algorithm 3.1, and satisfies the following conditions:*

1. The sequence $\{\lambda^k\}$ is monotonically decreasing and bounded below.

2. $\lambda^k \geq \min\left\{\frac{\mu}{L}, \lambda^1\right\}$ holds for any $k > 0$.

Then $\lim_{k\to\infty} \lambda^k = \lambda$ exists, and $\lambda \geq \min\left\{\frac{\mu}{L}, \lambda^1\right\} > 0$, where $\lambda^1 > 0$ is the initial step size.

Proof. Since $\mathbb{A}$ is an L-Lipschitz continuous mapping, then if $\|\mathbb{A}\rho^k - \mathbb{A}v^k\| \neq 0$,

$$\frac{\phi\|\rho^k - v^k\|}{\|\mathbb{A}\rho^k - \mathbb{A}v^k\|} \geq \frac{\phi\|\rho^k - v^k\|}{L\|\rho^k - v^k\|} = \frac{\phi}{L}. \tag{1}$$

This suggests that $\lambda^{k+1} = \min\left\{\frac{\phi\|\rho^k - v^k\|}{\|\mathbb{A}\rho^k - \mathbb{A}v^k\|}, \varsigma^k\lambda^k + \tau^k\right\} \geq \min\left\{\frac{\phi}{L}, \lambda^k\right\}$, where $\varsigma^k \geq 1$ and $\tau^k \geq 0$. By mathematical induction, we have $\inf\{\lambda^k\} \geq \min\{\frac{\phi}{L}, \lambda^1\}$. Moreover, from the definition of $\{\lambda^{k+1}\}$, it is known that

$$\lambda^{k+1} \leq \varsigma^k\lambda^k + \tau^k. \tag{2}$$

According to Lemma 2.3, we have that $\lim_{k\to\infty} \lambda^k$ exists and is denoted by $\lim_{k\to\infty} \lambda^k = \lambda$. Since $\inf\{\lambda^k\} \geq \min\left\{\frac{\phi}{L}, \lambda^1\right\}$, then $\lambda \geq \min\{\frac{\phi}{L}, \lambda^1\} > 0$. □

Lemma 3.3. The sequences $\{\rho^k\}$, $\{v^k\}$, and $\{z^k\}$ generated by Algorithm 3.1 satisfy

$$\|z^k - q^\ddagger\|^2 \leq \|\rho^k - q^\ddagger\|^2 - (1 - \phi^2)\|v^k - \rho^k\|^2,$$

for any $q^\ddagger \in VI(\mathbb{C}, \mathbb{A}) \cap \mathrm{Fix}(\mathbb{T})$.

Proof. As $q^\ddagger \in \mathrm{Fix}(\mathbb{T})$, from the definition of $\{z^k\}$, it can be concluded that

$$\begin{aligned}
\|z^k - q^\ddagger\|^2 &= \|v^k - \lambda^k(\mathbb{A}v^k - \mathbb{A}\rho^k) - q^\ddagger\|^2 \\
&= \|v^k - q^\ddagger\|^2 + (\lambda^k)^2\|\mathbb{A}v^k - \mathbb{A}\rho^k\|^2 \\
&\quad -2\lambda^k\langle v^k - q^\ddagger, \mathbb{A}v^k - \mathbb{A}\rho^k\rangle \\
&= \|v^k - \rho^k\|^2 + \|\rho^k - q^\ddagger\|^2 + 2\langle v^k - \rho^k, \rho^k - q^\ddagger\rangle \\
&\quad +(\lambda^k)^2\|\mathbb{A}v^k - \mathbb{A}\rho^k\|^2 - 2\lambda^k\langle v^k - q^\ddagger, \mathbb{A}v^k - \mathbb{A}\rho^k\rangle \\
&= \|v^k - \rho^k\|^2 + \|\rho^k - q^\ddagger\|^2 + 2\langle v^k - \rho^k, v^k - q^\ddagger\rangle
\end{aligned}$$

$$\begin{aligned}
&-2\langle v^k-\rho^k, v^k-\rho^k\rangle+(\lambda^k)^2\|\mathbb{A}v^k-\mathbb{A}\rho^k\|^2\\
&-2\lambda^k\langle v^k-q^\ddagger, \mathbb{A}v^k-\mathbb{A}\rho^k\rangle\\
=\ &\|\rho^k-q^\ddagger\|^2-\|v^k-\rho^k\|^2+2\langle v^k-\rho^k, v^k-q^\ddagger\rangle\\
&+(\lambda^k)^2\|\mathbb{A}v^k-\mathbb{A}\rho^k\|^2-2\lambda^k\langle v^k-q^\ddagger, \mathbb{A}v^k-\mathbb{A}\rho^k\rangle. \qquad (3)
\end{aligned}$$

According to Lemma 2.1, considering that $v^k=P_{\mathbb{C}}(\rho^k-\lambda^k\mathbb{A}\rho^k)$, it can be inferred that $\langle\rho^k-\lambda^k\mathbb{A}\rho^k-v^k, v^k-q^\ddagger\rangle\geq 0$, for all $v^k\in\mathbb{C}$, therefore

$$\langle v^k-\rho^k, v^k-q^\ddagger\rangle\leq-\lambda^k\langle\mathbb{A}\rho^k, v^k-q^\ddagger\rangle. \qquad (4)$$

From (3) and (4), it follows that

$$\begin{aligned}
\|z^k-q^\ddagger\|^2\ \leq\ &\|\rho^k-q^\ddagger\|^2-\|v^k-\rho^k\|^2-2\lambda^k\langle\mathbb{A}\rho^k, v^k-q^\ddagger\rangle\\
&+(\lambda^k)^2\|\mathbb{A}v^k-\mathbb{A}\rho^k\|^2-2\lambda^k\langle v^k-q^\ddagger, \mathbb{A}v^k-\mathbb{A}\rho^k\rangle\\
=\ &\|\rho^k-q^\ddagger\|^2-\|v^k-\rho^k\|^2+(\lambda^k)^2\|\mathbb{A}v^k-\mathbb{A}\rho^k\|^2\\
&-2\lambda^k\langle v^k-q^\ddagger, \mathbb{A}v^k\rangle.
\end{aligned}$$

By $\lambda^k>0$, $q^\ddagger\in VI(\mathbb{C},\mathbb{A})$ and the fact that $\mathbb{A}$ is quasimonotone, we get $\langle\mathbb{A}q^\ddagger, v^k-q^\ddagger\rangle>0$. Then there is

$$\langle\mathbb{A}v^k, v^k-q^\ddagger\rangle\geq 0.$$

So,

$$\|z^k-q^\ddagger\|^2\leq\|\rho^k-q^\ddagger\|^2-\|v^k-\rho^k\|^2+(\lambda^k)^2\|\mathbb{A}\rho^k-\mathbb{A}v^k\|^2. \qquad (5)$$

According to the definition of $\{\lambda^k\}$, it is clear that

$$(\lambda^k)^2\|\mathbb{A}\rho^k-\mathbb{A}v^k\|^2\leq\phi^2\|\rho^k-v^k\|^2. \qquad (6)$$

Substituting (6) into (5), we get

$$\begin{aligned}
\|z^k-q^\ddagger\|^2&\leq\|\rho^k-q^\ddagger\|^2-\|v^k-\rho^k\|^2+\phi^2\|\rho^k-v^k\|^2\\
&=\|\rho^k-q^\ddagger\|^2-(1-\phi^2)\|v^k-\rho^k\|^2.
\end{aligned} \qquad (7)$$

which concludes the proof. □

Lemma 3.4. The sequence obtained through Algorithm 3.1 weakly converges to a point $q^\ddagger$, where $q^\ddagger$ satisfies $\|q^\ddagger\|=\min\{\|z^\ddagger\|: z^\ddagger\in VI(\mathbb{C},\mathbb{A})\}$.

Proof. According to Lemma 3.3, and $\phi\in(0,1)$,

$$\begin{aligned}
\|z^k-q^\ddagger\|&\leq\|\rho^k-q^\ddagger\|\\
&=\|\omega^k+\alpha^k(\omega^k-\omega^{k-1})+\beta^k(\omega^{k-1}-\omega^{k-2})-q^\ddagger\|\\
&\leq\|\omega^k-q^\ddagger\|+\alpha^k\|\omega^k-\omega^{k-1}\|+|\beta^k|\gamma^k\|\omega^{k-1}-\omega^{k-2}\|\\
&\leq\|\omega^k-q^\ddagger\|+M^k,
\end{aligned} \qquad (8)$$

where $M^k=\alpha^k\|\omega^k-\omega^{k-1}\|+|\beta^k|\|\omega^{k-1}-\omega^{k-2}\|$.

Then, we obtain

$$\|\omega^{k+1}-q^\ddagger\|^2\ =\ \|\theta^k z^k+(1-\theta^k)\sum_{i=1}^{m}s(i)\mathbb{T}_{\lambda^i}\omega^k-q^\ddagger\|^2$$

$$
\begin{aligned}
&\leq \theta^k\|z^k-q^\ddagger\|^2+(1-\theta^k)\|\sum_{i=1}^m s(i)\mathbb{T}_{\lambda^i}\omega^k-q^\ddagger\|^2\\
&= \theta^k\|z^k-q^\ddagger\|^2+(1-\theta^k)\|\sum_{i=1}^m s(i)(\mathbb{T}_{\lambda^i}\omega^k-\mathbb{T}_{\lambda^i}q^\ddagger)\|^2\\
&= \theta^k\|z^k-q^\ddagger\|^2+(1-\theta^k)\big[\sum_{i=1}^m s(i)\|\mathbb{T}_{\lambda^i}\omega^k-\mathbb{T}_{\lambda^i}q^\ddagger\|^2\\
&\quad -\frac{1}{2}\sum_{i,j=1}^m s(i)s(j)\|(\mathbb{T}_{\lambda^i}\omega^i-\mathbb{T}_{\lambda^i}q^\ddagger)-(\mathbb{T}_{\lambda^j}\omega^j-\mathbb{T}_{\lambda^j}q^\ddagger)\|^2\big]\\
&\leq \theta^k\|z^k-q^\ddagger\|^2+(1-\theta^k)\sum_{i=1}^m s(i)\|\mathbb{T}_{\lambda^i}\omega^k-\mathbb{T}_{\lambda^i}q^\ddagger\|^2\\
&\leq \theta^k\|z^k-q^\ddagger\|^2+(1-\theta^k)\|\omega^k-q^\ddagger\|^2\\
&\leq \theta^k\|\omega^k-q^\ddagger\|^2+(1-\theta^k)(\|\omega^k-q^\ddagger\|^2+M^k)\\
&= \|\omega^k-q^\ddagger\|^2+M^k. \qquad (9)
\end{aligned}
$$

There exists $\bar{k}$ such that $\|\omega^k-\omega^{k-1}\|\neq 0$, for all $k\geq\bar{k}$. Since $\alpha^k\|\omega^k-\omega^{k-1}\|\leq\min\{\frac{1}{k^2},\alpha\}$, we have

$$\alpha^k\|\omega^k-\omega^{k-1}\|\leq\frac{1}{k^2},\ \text{for all } k\geq\bar{k}.$$

This leads to

$$\|\omega^{k+1}-q^\ddagger\|\leq\|\omega^k-q^\ddagger\|+\frac{\alpha}{k^2},\ \text{for all } k\geq\bar{k}.$$

We take the range of i from $\bar{k}$ to k and then make the sum. It follows that

$$\|\omega^{k+1}-q^\ddagger\|\leq\|\omega^{\bar{k}}-q^\ddagger\|+\sum_{i=\bar{k}}^k\frac{\alpha}{i^2}<\|\omega^{\bar{k}}-q^\ddagger\|+A,\ \text{for all } k\geq\bar{k}. \qquad (10)$$

So $\|\omega^k-q^\ddagger\|$ is bounded which implies that $\{\omega^k\}$ and $\{z^k\}$ are bounded.

Moreover, note that the sequences $\{\alpha^k\|\omega^k-\omega^{k-1}\|\}$, and $\{beta^k\|\omega^k-\omega^{k-1}\|\}$ are convergent to zero.

From (7) and (9), we have

$$\|\omega^{k+1}-q^\ddagger\|^2\leq\theta^k\|\rho^k-q^\ddagger\|^2-\theta^k(1-\phi^2)\|v^k-\rho^k\|^2+(1-\theta^k)\|\omega^k-q^\ddagger\|^2. \qquad (11)$$

By (8), we get

$$\|\rho^k-q^\ddagger\|^2\leq(\|\omega^k-q^\ddagger\|+M^k)^2. \qquad (12)$$

After substituting (12) into (11), we have

$$
\begin{aligned}
\|\omega^{k+1}-q^\ddagger\|^2 &\leq \theta^k(\|\omega^k-q^\ddagger\|+\gamma^kM_3^k)^2-\theta^k(1-\phi^2)\|v^k-\rho^k\|^2\\
&\quad +(1-\theta^k)\|\omega^k-q^\ddagger\|^2\\
&= \theta^k\|\omega^k-q^\ddagger\|^2+2\theta^kM^k\|\omega^k-q^\ddagger\|+\theta^k(M^k)^2\\
&\quad -\theta^k(1-\phi^2)\|v^k-\rho^k\|^2+(1-\theta^k)\|\omega^k-q^\ddagger\|^2\\
&= \|\omega^k-q^\ddagger\|^2-\theta^k(1-\phi^2)\|v^k-\rho^k\|^2\\
&\quad +\theta^kM^k(2\|\omega^k-q^\ddagger\|+M^k). \qquad (13)
\end{aligned}
$$

From the boundedness of the sequence $\{\|\omega^k - q^\ddagger\|\}$ and (13), we get

$$\lim_{k\to\infty} \|v^k - \rho^k\| = 0.$$

Meanwhile,

$$\|\rho^k - \omega^k\| \le \alpha^k \|\omega^k - \omega^{k-1}\| + |\beta^k| \|\omega^k - \omega^{k-1}\| \to 0.$$

Therefore,

$$\lim_{k\to\infty} \|v^k - \omega^k\| = \lim_{k\to\infty} \|v^k - \rho^k\| + \lim_{k\to\infty} \|\rho^k - \omega^k\| = 0. \tag{14}$$

$\{\omega^k\}$ has at least one weak convergence point since it is bounded. Let $\{\omega^{k_j}\}$ be a subsequence of $\{\omega^k\}$, with $\omega^{k_j} \rightharpoonup q^\ddagger \in \mathbb{H}$.

Then

$$w\text{-}\lim_{j\to\infty} \omega^{k_j} = q^\ddagger,$$

and

$$w\text{-}\lim_{j\to\infty} v^{k_j} = q^\ddagger.$$

Here, w-lim represents the weak limit, indicating that the sequences $\{\omega^{k_j}\}$ and $\{v^{k_j}\}$ converge weakly to $q^\ddagger$.

Let

$$f(\delta^\dagger) = \begin{cases} \mathbb{A}(\delta^\dagger) + N_{\mathbb{C}}(\delta^\dagger), & \delta^\dagger \in \mathbb{C}, \\ \emptyset, & \delta^\dagger \notin \mathbb{C}, \end{cases}$$

where $N_{\mathbb{C}}(\delta^\dagger)$ represents the normal cone of $\mathbb{C}$ at $\delta^\dagger \in \mathbb{C}$. Obviously, f is maximal monotone and $f^{-1}(0) = VI(\mathbb{C}, \mathbb{A})$. If $(\delta^\dagger, \rho) = G(f)$, since $\rho \in f(\delta^\dagger) = \mathbb{A}(\delta^\dagger) + N_{\mathbb{C}}(\delta^\dagger)$, we have $\rho - \mathbb{A}(\delta^\dagger) \in N_{\mathbb{C}}(\delta^\dagger)$. Thus leads us to

$$\langle \rho - \mathbb{A}(\delta^\dagger), \delta^\dagger - v\rangle \ge 0, \text{ for all } v \in \mathbb{C}. \tag{15}$$

By $v^k = P_{\mathbb{C}}(\rho^k - \lambda^k \mathbb{A}\rho^k)$, we get

$$\langle \rho^k - \lambda^k \mathbb{A}\rho^k - v^k, v^k - \delta^\dagger\rangle \ge 0, \text{ for all } \delta^\dagger \ge 0,$$

and

$$\left\langle \frac{v^k - \rho^k}{\lambda^k} + \mathbb{A}\rho^k, \delta^\dagger - v^k \right\rangle \ge 0, \text{ for all } \delta^\dagger \ge 0. \tag{16}$$

Using (14) and applying (15) with $\{v^{k_j}\}_{j=0}^\infty$, we have

$$\langle \rho - \mathbb{A}(\delta^\dagger), \delta^\dagger - v^{k_j}\rangle \ge 0, \text{ for all } \delta^\dagger \ge 0.$$

From (15) and (16), we can get

$$\begin{aligned}\langle \rho, \delta^{\dagger} - v^{k_j} \rangle &\geq \langle \mathbb{A}\delta^{\dagger}, \delta^{\dagger} - v^{k_j} \rangle \\ &\geq \langle \mathbb{A}\delta^{\dagger}, \delta^{\dagger} - v^{k_j} \rangle - \langle \frac{v^{k_j} - \rho^{k_j}}{\lambda^{k_j}} + \mathbb{A}\rho^{k_j}, \delta^{\dagger} - v^{k_j} \rangle \\ &= \langle \mathbb{A}\delta^{\dagger} - \mathbb{A}\rho^{k_j}, \delta^{\dagger} - v^{k_j} \rangle - \langle \frac{v^{k_j} - \rho^{k_j}}{\lambda^{k_j}}, \delta^{\dagger} - v^{k_j} \rangle \\ &= \langle \mathbb{A}\delta^{\dagger} - \mathbb{A}v^{k_j}, \delta^{\dagger} - v^{k_j} \rangle + \langle \mathbb{A}v^{k_j} - \mathbb{A}\rho^{k_j}, \delta^{\dagger} - v^{k_j} \rangle \\ &- \langle \frac{v^{k_j} - \rho^{k_j}}{\lambda^{k_j}}, \delta^{\dagger} - v^{k_j} \rangle \\ &\geq \langle \mathbb{A}v^{k_j} - \mathbb{A}\rho^{k_j}, \delta^{\dagger} - v^{k_j} \rangle - \langle \frac{v^{k_j} - \rho^{k_j}}{\lambda^{k_j}}, \delta^{\dagger} - v^{k_j} \rangle.\end{aligned}$$

Therefore $\langle \rho, \delta^{\dagger} - v^{k_j} \rangle \geq 0$, let $j \to \infty$, we have

$$\langle \rho, \delta^{\dagger} - q^{\ddagger} \rangle \geq 0.$$

Since f is a maximal monotone operator, we obtain $q^{\ddagger} \in f^{-1}(0) = VI(\mathbb{C}, \mathbb{A})$.

To demonstrate the weak convergence of the whole sequence to $q^{\ddagger}$, we postulate the existence of another subsequence $\{\omega^{\bar{k}_j}\}$ of $\{\omega^k\}$, which converges weakly to some $q^{\dagger} \neq q^{\ddagger}$, thus we have

$$\begin{aligned}\lim_{k\to\infty} \|\omega^k - q^{\ddagger}\| &= \liminf_{j\to\infty} \|\omega^{k_j} - q^{\ddagger}\| < \liminf_{j\to\infty} \|\omega^{k_j} - q^{\dagger}\| \\ &= \lim_{k\to\infty} \|\omega^k - q^{\dagger}\| = \liminf_{j\to\infty} \|\omega^{\bar{k}_j} - q^{\dagger}\| \\ &< \liminf_{j\to\infty} \|\omega^{\bar{k}_j} - q^{\ddagger}\| = \lim_{k\to\infty} \|\omega^k - q^{\ddagger}\|.\end{aligned}$$

and this is a contradiction, so $q^{\ddagger} = q^{\dagger}$. Therefore $\{\omega^k\}$ and $\{v^k\}$ converge weakly to $q^{\ddagger} \in VI(\mathbb{C}, \mathbb{A})$. □

Theorem 3.1. *If the sequence* $\{\omega^k\}$ *is obtained by Algorithm* 3.1 *and the operator* $\mathbb{T}\colon \mathbb{C} \to \mathbb{C}$ *is nonexpansive, then* $\{\omega^k\}$ *converges weakly to a point in* $VI(\mathbb{C}, \mathbb{A}) \cap \mathrm{Fix}(\mathbb{T})$.

Proof. Presume that there exists $q^{\ddagger} \in VI(\mathbb{C}, \mathbb{A}) \cap Fix(\mathbb{T})$, since $\mathbb{T}$ is nonexpansive, it follows that

$$\|\mathbb{T}_{\lambda^i}\omega^k - q^{\ddagger}\| = \|\mathbb{T}_{\lambda^i}\omega^k - \mathbb{T}_{\lambda^i}q^{\ddagger}\| \leq \|\omega^k - q^{\ddagger}\|.$$

Since $\|\omega^k - q^{\ddagger}\|$ is bounded, then for a sufficiently large ξ, we have

$$\limsup_{k\to\infty} \|\mathbb{T}_{\lambda^i}\omega^k - q^{\ddagger}\| \leq \xi.$$

Therefore,

$$\begin{aligned}
&\lim_{k\to\infty} \|\theta^k z^k + (1-\theta^k)(\sum_{i=1}^{m} s(i)\mathbb{T}_{\lambda^i}\omega^k) - q^\ddagger\| \\
=&\lim_{k\to\infty} \|\theta^k (z^k - q^\ddagger) + (1-\theta^k)(\sum_{i=1}^{m} s(i)\mathbb{T}_{\lambda^i}\omega^k - q^\ddagger)\| \\
=&\lim_{k\to\infty} \|\omega^{k+1} - q^\ddagger\| \\
=&\xi.
\end{aligned}$$

Lemma 2.3 indicates that

$$\lim_{k\to\infty} \|\mathbb{T}_{\lambda^i}\omega^k - \omega^k\| = 0.$$

Since $\mathbb{T}$ is nonexpansive, and $\{\omega^{k_j}\}$ converges weakly to $q^\ddagger$, it follows that

$$\lim_{j\to\infty} \|(I - \mathbb{T}_{\lambda^i})\omega^{k_j}\| = \lim_{j\to\infty} \|\omega^{k_j} - \mathbb{T}_{\lambda^i}\omega^{k_j}\| = 0.$$

According to Lemma 2.4, we deduce that $(I - \mathbb{T})q^\ddagger = 0$, so $q^\ddagger \in \mathrm{Fix}(\mathbb{T})$. Namely, $q^\ddagger \in VI(\mathbb{C}, \mathbb{A}) \cap \mathrm{Fix}(\mathbb{T})$.

Above all, the proof has been accomplished. □

4. Numerical Examples

Example 4.1 ([28]). Consider the bounded linear operator

$$\mathbb{G}\colon \mathbb{R}^m \to \mathbb{R}^m, \quad \mathbb{G}(u) = Ku + t,$$

where $K = HH^T + I + L$ is positive definite. Here, H, I and L are $m \times m$ matrices over the real number field, where the matrix I is skew-symmetric, and the matrix L is a diagonal matrix with non-negative diagonal entries, and $t \in \mathbb{R}^m$. We analyze Algorithm 3.1 in comparison with Algorithm Tian from [29], which constructed a single-step method by combining Mann and Tseng methods to solve variational inequalities. Using a normally distributed random generation for matrices H, I, and L, we choose the feasible set as

$$\mathbb{C} = \{u \in \mathbb{R}^m : a \le u_i \le b, \forall i = 1, \ldots, m\}.$$

With the choice of parameters $\mu = 0.5$, $\alpha = 0.3$, $\lambda_1 = 0.5$, $\theta_k = \frac{1}{k^2}$, and an initial point $u_0 = (1, 1, \ldots, 1) \in \mathbb{R}^n$, we compare the convergence speed of three algorithms for different interval selections, stopping iterations when $\|u_k - q\| < 1 \times 10^{-8}$.

Table 1. Result of Example 4.1

	$\mathbb{C} = [-5, 5]$		$\mathbb{C} = [-3, 3]$	
	Inter_1	Time (microseconds)	Inter_2	Time (microseconds)
Algo. 3.1	5	910	7	1035
Algo. Tian	23	2989	12	1957
Algo. KM	14	1005	15	2030

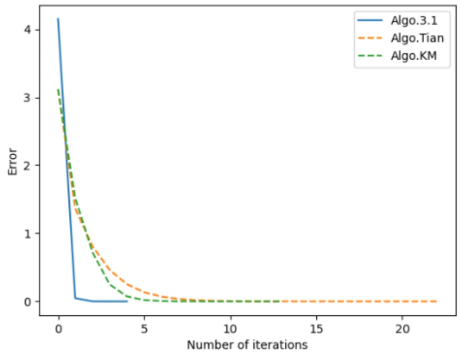

FIGURE 1. $\mathbb{C} = [-5, 5]$

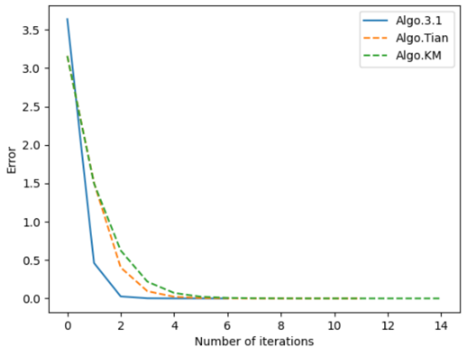

FIGURE 2. $\mathbb{C} = [-3, 3]$

Based on the experimental results of Example 4.1, a clear conclusion can be drawn that Algorithm 3.1 presented in this study outperforms the algorithm proposed by Tian and the DRS algorithm concerning the iteration time and the number of iterations. The model $K = HH^T + I + L$ can be employed to describe diverse situations, especially in machine learning, statistical analysis, or signal processing. Specifically, K may represent a target matrix, while HH^T is a part obtained by multiplying matrix H and its transpose are often used to represent covariance or correlation structure. The terms I and L are additional components that may represent noise, bias, or specific structural elements. This expression is used to optimize models through matrix operations, analyze relationships in data, or build predictive models.

Example 4.2 ([28])**.** Assume that $\varkappa$ is a linear operator from $\mathbb{R}^n$ to $\mathbb{R}^m$ and $u_{ob} \in \mathbb{R}^n$. Consider the minimization problem

$$\min_{u \in \mathbb{R}^n} P(u) + Q(u)$$

where P represents the regularization term and $Q = \iota_C(\cdot)$ represents the indicator function of the set $\mathbb{C} = \{u \in \mathbb{R}^n : \varkappa u = \varkappa u_{ob}\}$. For the tests in this case, $\varkappa$ is obtained by sampling from the standard Gaussian ensemble, with P being the l_1-norm. The parameters are $(m, n) = (48, 128)$ and u has 8 non-zero elements.

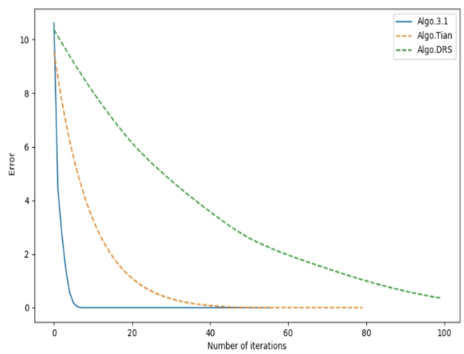

FIGURE 3. l_1-norm

Table 2. Result of Example 4.2

	l_1-norm	
	Inter_1	Time (microseconds)
Algo. 3.1	56	4391
Algo. Tian	80	6928
Algo. DRS	100	7603

From the results of Example 4.2, it can be seen that Algorithm 3.1 presented in this article is superior to the Tian algorithm and DRS algorithm concerning iteration time, and also has fewer iterations, [30].

5. Conclusions

This article presents a novel method by integrating the Mann method, the two-step inertial method and Tseng's extragradient method. The iteratively generated sequence produced by the proposed method can be demonstrated to weakly converge to the solution of FPP and VIP under certain conditions. Finally, we use several numerical examples to validate the strengths of our suggested method.

6. Acknowledgement

Our thanks to Natural Science Foundation of Ningxia province (grant no. NZ17015, NXYLXK2017B09), National Natural Science Foundation of China (grant no. 11861003), Major Scientific and Technological Innovation Projects of Yinchuan (grant no. 2022RKX03), and Ningxia science and technology key project (grant no. 2021BEG03049). All provided funding for this work.

REFERENCES

[1] A. Signorini, *Sopra alcune questioni di elastostatica*, Atti della Societa Italiana per il Progresso delle Scienze, **21**(2), 143-148, 1933.

[2] G. Stampacchia, *Formes bilinéaires coercives sur les ensembles convéxes*, Comptes rendus hebdomadaires des séances de l'Académie des sciences, **258**, 4413-4416, 1964.

[3] A. A. Goldstein, *Convex programming in Hilbert space*, Proc. Nat. Acad. Sci. USA, **51**, 709-710, 1964.

[4] G. M. Korpelevich, *The extragradient method for finding saddle points and other problems*, MATECON, **12**, 747-756, 1976.

[5] P. Tseng, *A modified forward-backward splitting method for maximal monotone mappings*, SIAM J. Control Optim., **38**(2), 431-446, 2000.

[6] Y. Censor, A. Gibali, and S. Reich, *The subgradient extragradient method for solving variational inequalities in Hilbert space*, J. Optim. Theory Appl., **148**(2), 318-335, 2011.

[7] L. Yang, L. J. Zhu, and M. Postolache, *An accelerate Mann-type method for split common fixed point problem in Hilbert spaces*, U. Politeh. Buch. Ser. A, **86**(1), 3-16, 2024.

[8] L. C. Ceng, L. J. Zhu, and T. C. Yin, *On generalized extragradient implicit method for systems of variational inequalities with constraints of variational inclusion and fixed point problems*, Open Mathematics, **20**(1), 1770-1784, 2022.

[9] L. J. Zhu, T. C. Yin, *A reconsideration on convergence of the extra-gradient method for solving quasimonotone variational inequalities*, Optimization, **63**(1), 1-12, 2024.

[10] W. Singh, S. Chandok, *Mann-type extragradient algorithm for solving variational inequality and fixed point problems*, Comput. Appl. Math., **43**(4), 2024.

[11] B. S. Thakur, M. Postolache, *Existence and approximation of solutions for generalized extended nonlinear variational inequalities*, J. Inequal. Appl. **2013**, Art. No. 590, 2013.

[12] Y. Yao, N. Shahzad, M Postolache, and J. C. Yao, *Convergence of self-adaptive Tseng-type algorithms for split variational inequalities and fixed point problems*, Carpathian J. Math. **40**(3), 753-768, 2024.

[13] Y. Yao, M. Postolache, and J. C. Yao, *Strong convergence of an extragradient algorithm for variational inequality and fixed point problems*, U. Politeh. Buch. Ser. A **82**(1), 3-12, 2020.

[14] Y. Yao, M. Postolache, and J. C. Yao, *Iterative algorithms for generalized variational inequalities*, U. Politeh. Buch. Ser. A **81**(2), 3-16, 2019.

[15] Y. Yao, M. Postolache, and J. C. Yao, *An iterative algorithm for solving generalized variational inequalities and fixed points problems*, Mathematics **7**(1), Art. No. 61, 2019.

[16] Y. Yao, M. Postolache, Y. C. Liou, and Z. Yao, *Construction algorithms for a class of monotone variational inequalities*, Optim. Lett. **10**(7), 1519-1528, 2016.

[17] Y. Yao, M. Postolache, *Iterative methods for pseudomonotone variational inequalities and fixed point problems*, J. Optim. Theory Appl. **155**(1), 273-287, 2012.

[18] W. R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc., **4**(3), 506-510, 1953.

[19] J. Douglas, H. H. Rachford, *On the numerical solution of heat conduction problems in two and three space variables*, Trans. Am. Math. Soc., **82**, 421-439, 1956.

[20] D. Cortild, J. Peypouquet, *Krasnoselskii–Mann Iterations: Inertia, Perturbations and Approximation*, J. Optim. Theory Appl., **204**(2), 2025.

[21] Y. Yao, O. S. Iyiola, and Y. Shehu, *Subgradient extragradient method with double inertial steps for variational inequalities*, J. Sci. Comput., **90**, 1-29, 2022.

[22] F. Iutzeler, M. J. Hendrickx, *A generic linear rate acceleration of optimization algorithms via relaxation and inertia*, Optim. Methods Soft., **34**(2), 383-405, 2019.

[23] Q. L. Dong, Y. J. Cho, and T. M. Rassias, *General inertial Mann algorithms and their convergence analysis for nonexpansive mappings*, Appl. Nonlinear Anal., 175-191, 2018.

[24] K. Goebel, S. Reich, *Uniform Convexity, hyperbolic Geometry, and nonexpansive mappings*, 1984.

[25] M. O. Osilike, S. C. Aniagbosor, *Weak and strong convergence theorems for fixed points of asymptotically nonexpansive mappings*, Math. Comput. Modelling, **32**(10), 1181-1191, 2000.

[26] F.E. Browder, *Fixed-point theorems for noncompact mappings in Hilbert space*, Proc. Nat. Acad. Sci., **53**, 1272-1276, 1965.

[27] Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull. Amer. Math. Soc., **73**, 591–597, 1967.

[28] D. V. Thong, N. T. Vinh, and Y. J. Cho, *A strong convergence theorem for Tseng's extragradient method for solving variational inequality problems*, Optim. Lett., **14**(4), 1157-1175, 2020.

[29] M. Tian and G. Xu, *Improved inertial projection and contraction method for solving pseudomonotone variational inequality problems*, J. Inequal. Appl., **2021**(1), 2021.

[30] Q. L. Dong, X. H. Li, and Y. J. Cho, et al., *Multi-step inertial Krasnosel'skii-Mann iteration with new inertial parameters arrays*, J. Fixed Point Theory Appl., **23**, 1-18, 2021.