COMBESCURE RELATED PSEUDO NULL CURVES AND THEIR APPLICATIONS TO DA RIOS VORTEX FILAMENT EQUATION

Osman KEÇİLİOĞLU[1], Kazım İLARSLAN[2]

Space curve pairs related through the Combescure transformation constitute a class of curves arising from specific relationships between the Frenet vectors of the curves. In other words, they are pairs of curves whose tangent vectors are parallel at corresponding points. In this study, pseudo null curves connected via the Combescure transformation are investigated within the framework of Minkowski 3-space, and the necessary conditions for such connections are derived. Furthermore, the relationships between the Frenet vectors and curvatures of these curve pairs are established, and illustrative examples are provided.

Another section of the study focuses on applications of pseudo null curves related through the Combescure transformation. Initially, conditions under which a pseudo null curve associated with a pseudo null biharmonic curve via the Combescure transformation also becomes biharmonic are derived and supported with examples. As another application, in the context of pseudo null curves related through a Combescure transformation, necessary and sufficient conditions have been obtained for the ruled surface generated by the conjugate curve to be a solution of the Da Rios vortex filament equation, assuming that the ruled surface generated by the main curve is already a solution. Examples supporting these results have also been provided.

Keywords: Combescure transformation, Minkowski 3-space, pseudo-null space curves, biharmonic curves, Da Rios vortex filament equation.

MSC2020: 53C50, 53C40, 53C22.

1. Introduction

The differential geometry of spatial curves in Euclidean 3-space has been widely studied, with many results extended to non-Euclidean settings using alternative metrics and moving frames. One notable framework is Minkowski 3-space (Lorentz-Minkowski space), which plays a key role in relativity theory [9, 31].

In this space, curves are classified based on the causal character of their velocity vectors: spacelike, timelike, or null (lightlike). Unlike spacelike or timelike cases [24, 31], null curves involve degenerate metrics, introducing additional challenges. A particularly interesting class is pseudo null curves, which have spacelike tangent vectors and null acceleration vectors.

Pseudo null curves offer insight into the geometry of both null and spacelike curves, helping reveal the interplay between curvature, torsion, and metric structure [5]. They are also useful in modeling geometric behavior in Lorentzian settings, representing simplified forms of relativistic motion. Many studies have explored special properties of pseudo null

[1]Professor, Department of Mathematics, Kırıkkale University, Türkiye, e-mail: okecilioglu@yahoo.com

[2]Professor, Department of Mathematics, Kırıkkale University, Türkiye, e-mail: kilarslan@yahoo.com

curves in Minkowski 3-space, such as being helices or slant helices, and have used Frenet frames to classify them as Bertrand, Mannheim, or Involute-Evolute curves [7, 25, 27, 28].

A key concept in such studies is the Combescure transformation, which links two curves by aligning their tangent vectors at corresponding points [8, 16, 32]. This often leads to parallelism in their normal and binormal vectors as well [26]. The transformation has also been extended to Riemannian manifolds, where it's known that only flat manifolds satisfy this property [19].

Recent research uses the Combescure transformation to generate specific curve types, such as Bertrand, Mannheim, and Salkowski curves, from a given base curve [3, 8]. Related investigations on non-null curves in Minkowski space have also been carried out by the authors [22]. Moreover, this transformation is relevant not only in geometry but also in mathematical physics [4, 29, 33].

This paper is structured as follows: Section 2 reviews key concepts in Minkowski 3-space, focusing on the Frenet apparatus of pseudo null curves. Section 3 introduces pseudo null curves associated via the Combescure transformation and derives conditions for such associations, supported by a central theorem and illustrative example. Section 4 examines cases where the principal curve is biharmonic and identifies when the associated curve also becomes biharmonic. It also analyzes when ruled surfaces generated by these curves satisfy the Da Rios vortex filament equation, providing conditions and graphical examples.

2. Preliminaries

Minkowski space $\mathbb{E}_1^3$ is a three-dimensional affine space endowed with an indefinite flat metric $\langle\ ,\ \rangle$ with signature $(-,+,+)$. This means that metric bilinear form can be written as

$$\langle u, v\rangle = -u_1v_1 + u_2v_2 + u_3v_3,$$

for any two vectors $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ in $\mathbb{E}_1^3$. Recall that a vector $u \in \mathbb{E}_1^3\backslash\{0\}$ can be *spacelike* if $g(u, u) > 0$, *timelike* if $g(u, u) < 0$ and *null (lightlike)* if $g(u, u) = 0$ and $u \neq 0$. In particular, the vector $u = 0$ is a spacelike. The norm of a vector u is given by $||u|| = \sqrt{|g(u, u)|}$, and two vectors u and v are said to be orthogonal, if $g(u, v) = 0$. An arbitrary curve $\varphi(s)$ in $\mathbb{E}_1^3$, can locally be *spacelike*, *timelike* or *null* (*lightlike*), if all its velocity vectors $\varphi'(s)$ are respectively spacelike, timelike or null. A null curve φ is parameterized by pseudo-arc s if $g(\varphi''(s), \varphi''(s)) = 1$. A spacelike or a timelike curve $\varphi(s)$ has unit speed, if $g(\varphi'(s), \varphi'(s)) = \pm 1$ [24], [31]. The Lorentzian vector product of two vectors u and v is given by

$$u \times v = (u_3v_2 - u_2v_3, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1).$$

Let $\{T, N, B\}$ denote the moving Frenet frame along a curve φ in $\mathbb{E}_1^3$, T, N and B represent the tangent, principal normal, and binormal vector fields, respectively. The form of the Frenet equations varies depending on the causal character of the curve φ.

If φ is a pseudo null curve, the Frenet equations are given by

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & \tau & 0 \\ -1 & 0 & -\tau \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{1}$$

where the first curvature $k_1 = 0$ if φ is straight line, or $k_1 = 1$ in all other cases. In particular, the following conditions hold:

$$g(N,N) = g(B,B) = g(T,N) = g(T,B) = 0, g(T,T) = g(N,B) = 1$$

and

$$T \times N = N, \ N \times B = T, \ B \times T = B. \tag{2}$$

3. Pseudo Null Curves Related by a Combescure Transformation in Minkowski 3-space

In this section, pseudo null curves in the 3-dimensional Minkowski space that possess parallel tangent vector fields, or equivalently, are related by a transformation of Combescure, have been studied.

Definition 3.1. *Let $\varphi : I \subseteq \mathbb{R} \to \mathbb{E}_1^3$ and $\varphi^* : I^* \subseteq \mathbb{R} \to \mathbb{E}_1^3$ be null curves in $\mathbb{E}_1^3$ with Frenet apparatus $\{T, N, B, \kappa, \tau\}$ and $\{T^*, N^*, B^*, \kappa^*, \tau^*\}$, respectively. If the tangent vectors at the corresponding points of φ and φ^* are equal, these curves are called curves related by a transformation of Combescure.*

Theorem 3.1. *Let $\varphi : I \subseteq \mathbb{R} \to \mathbb{E}_1^3$ and $\varphi^* : I^* \subseteq \mathbb{R} \to \mathbb{E}_1^3$ be unit speed pseudo null curves in $\mathbb{E}_1^3$ with Frenet apparatus $\{T, N, B, \kappa, \tau, s\}$ and $\{T^*, N^*, B^*, \kappa^*, \tau^*, s^*\}$, respectively. Then the tangent vector T of φ is equal to the tangent vector T^* of φ^* at the corresponding points if and only if the parametrization of the curve $\varphi^*$is given one of the followings*

$$\begin{aligned} \varphi^*(s^*) &= \varphi^*(h_1(s)) \\ &= \varphi(s) + C(s)T(s) + e^{-\int \tau(s)ds}\left[\int -C(s)e^{\int \tau(s)ds}ds + c_0\right]N(s) \\ &\quad + e^{\int \tau(s)ds}B(s) \end{aligned} \tag{3}$$

or

$$\begin{aligned} \varphi^*(s^*) &= \varphi^*(h_2(s)) \\ &= \varphi(s) - \left(\frac{dC(s)}{ds} + \tau C(s)\right)T(s) + C(s)N(s) + e^{\int \tau(s)ds}B(s) \end{aligned} \tag{4}$$

where $h_1, h_2 : I \to I^*$ are diffeomorphism and $C : I \subseteq \mathbb{R} \to \mathbb{R}$ is a differentiable function.

Proof. Let $\varphi : I \subseteq \mathbb{R} \to \mathbb{E}_1^3$ and $\varphi^* : I^* \subseteq \mathbb{R} \to \mathbb{E}_1^3$ be pseudo null curves in $\mathbb{E}_1^3$ with Frenet apparatus $\{T, N, B, \kappa, \tau\}$ and $\{T^*, N^*, B^*, \kappa^*, \tau^*\}$, respectively. Assume that $T = T^*$ and

$$\varphi^*(s^*) = \varphi^*(h_1(s)) = \varphi(s) + u(s)T(s) + v(s)N(s) + w(s)B(s) \tag{5}$$

where u, v and w are differentiable functions on $I \subseteq \mathbb{R}$ and $h_1 : I \to I^*$ is a diffeomorphism. Differentiating this equation with respect to s and using (1), we obtain

$$\frac{d\varphi^*}{ds^*}h_1'(s) = (1 + u' - w)T + (u + v' + v\tau)N + (w' - \tau w)B.$$

Since $T = T^*$, it follows that

$$\left.\begin{aligned} 1 + u' - w = \tfrac{ds^*}{ds} = h_1'(s) \\ u + v' + v\tau = 0 \\ w' - \tau w = 0 \end{aligned}\right\} \tag{6}$$

The last equality represents a first order linear differential equation. Solving this equation yields the solution $w = e^{\int \tau(s)ds}$. Let $C : I \to \mathbb{R}$ be a differentiable function. If we take

$u = C(s)$, expression (6) leads to the linear differential equation $v' + \tau v = -C(s)$. Solving this equation gives

$$v = e^{-\int \tau(s)ds}\left[\int -C(s)^{\int \tau(s)ds}\, ds + c_0\right],$$

where $c_0 \in \mathbb{R}$. As a result, we obtain

$$\begin{aligned} \varphi^*(s^*) &= \varphi^*(h_1(s)) \\ &= \varphi(s) + C(s)T(s) + e^{-\int \tau(s)ds}\left[\int -C(s)e^{\int \tau(s)ds}ds + c_0\right]N(s) \\ &\quad + e^{\int \tau(s)ds}B(s). \end{aligned}$$

If we take $v = C(s)$, expression (6) leads to $u = -\left(\frac{dC(s)}{ds} + \tau C(s)\right)$, thus, we get

$$\varphi^*(s^*) = \varphi^*(h_2(s))\varphi(s) - \left(\frac{dC(s)}{ds} + \tau C(s)\right)T(s) + C(s)N(s) + e^{\int \tau(s)ds}B(s)$$

where $h_2 : I \to I^*$ is a diffeomorphism. Conversely, assume that $\varphi : I \subseteq \mathbb{R} \to \mathbb{E}_1^3$ be a pseudo null curve in $\mathbb{E}_1^3$ with Frenet apparatus $\{T, N, B, \kappa, \tau\}$, $C : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function and the parametrization of φ^* given by

$$\begin{aligned} \varphi^*(s^*) &= \varphi^*(h_1(s)) \qquad (7)\\ &= \varphi(s) + C(s)T(s) + e^{-\int \tau(s)ds}\left[\int -C(s)e^{\int \tau(s)ds}ds + c_0\right]N(s) \\ &\quad + e^{\int \tau(s)ds}B(s). \end{aligned}$$

Differentiating (7) with respect to s, we find $\frac{d\varphi^*}{ds^*}\frac{ds^*}{ds} = \left(1 + \frac{dC(s)}{ds} - e^{\int \tau ds}\right)T$. Since T^* and T are unit vectors, we get $T^* = T$. Thus φ^* and φ are pseudo null space curves are related by a transformation of Combescure. If the parametrization of φ^* given by

$$\begin{aligned} \varphi^*(s^*) &= \varphi^*(h_2(s)) = \varphi(s) - \left(\frac{dC(s)}{ds} + \tau C(s)\right)T(s) + C(s)N(s) \\ &\quad + e^{\int \tau(s)ds}B(s). \end{aligned}$$

We easily show that $T^* = T$. So φ^* and φ are pseudo null space curves are related by a transformation of Combescure. This complete the proof. □

Theorem 3.2. *Let $\varphi : I \subseteq \mathbb{R} \to \mathbb{E}_1^3$ and $\varphi^* : I^* \subseteq \mathbb{R} \to \mathbb{E}_1^3$ be pseudo null curves related by transformation of Combescure with Frenet apparatus $\{T, N, B, \kappa, \tau, s\}$ and $\{T^*, N^*, B^*, \kappa^*, \tau^*, s^*\}$, respectively. The curvatures and Frenet vector of φ and φ^* related as follows:*

(1) *If the parameterization of φ^* is given by* (3)*, then we have*

$$T^* = T, N^* = \frac{1}{h_1'}N, B^* = h_1'B \text{ and } \kappa^* = \kappa = 1, \tau^* = \left(\frac{1}{h_1'}\right)' + \frac{\tau}{h_1'}$$

where $h_1'(s) = 1 + \frac{dC(s)}{ds} - e^{\int \tau(s)ds}$.

(2) *If the parameterization of φ^* is given by* (4)*, then, the same equalities as above are obtained for $h_2'(s) = 1 - \frac{d^2C(s)}{ds^2} - \tau'C(s) - \tau(s)\frac{dC(s)}{ds} - e^{\int \tau(s)ds}$.*

Proof. Assume that, φ and φ^* be pseudo null curves related by transformation of Combescure and the parameterization of φ^* is given by (3). Differentiating (3) with respect to s and using Frenet equations, we get $\frac{d\varphi^*}{ds^*}\frac{ds^*}{ds} = \left(1 + \frac{dC(s)}{ds} - e^{\int \tau(s)ds}\right)T$. If we denote $h_1'(s) = \frac{ds^*}{ds}$ and using $T^* = T$, we get

$$h_1'(s) = 1 + \frac{dC(s)}{ds} - e^{\int \tau(s)ds}. \tag{8}$$

Taking the derivative of the equation $T^* = T$ with respect to s and using the Frenet formulas yields

$$\tau^* = \left(\frac{1}{h_1'}\right)' + \frac{\tau}{h_1'}.$$

Now assume that there exist differentiable functions a, b, c such that the binormal vector B^* can be expressed as

$$B^* = aT + bN + cB. \tag{9}$$

Since $T^* = T$, $N^* = \frac{1}{h_1'}N$ and using equations (1) and (2), we obtain $a = b = 0$ and $c = h_1'$. So, we have $B^* = h_1'B$. Thus we complete the proof. □

Example 3.1. *Consider the curve* $\mathbb{E}_1^3$ *given by*

$$\varphi(s) = \left(\frac{1}{3}s^3, \frac{1}{6}\sqrt{3}s^3 + \frac{1}{2}s, \frac{1}{6}s^3 - \frac{1}{2}\sqrt{3}s\right)$$

with the curvatures $\kappa = 1, \tau = \frac{1}{s}$. *By using Theorem 3.1 and taking* $C(s) = 3s$, *we obtain related by transformation of Combescure curve* φ_1^* *and* φ_2^* *as follows*

$$\varphi_1^*(h_1(s)) = \begin{pmatrix} -\frac{1}{4}s^4 + \frac{4}{3}s^3 - \frac{1}{4}, \\ -\frac{1}{8}\sqrt{3}s^4 + \frac{2}{3}\sqrt{3}s^3 - \frac{1}{4}s^2 + 2s + \frac{1}{8}\sqrt{3}, \\ -\frac{1}{8}s^4 + \frac{2}{3}s^3 + \frac{1}{4}\sqrt{3}s^2 - 2\sqrt{3}s + \frac{1}{8} \end{pmatrix}$$

with curvatures $\kappa_1^* = 1$ *and* $\tau_1^* = \frac{4}{s(s-4)^2}$ *and*

$$\varphi_2^*(h_2(s)) = \begin{pmatrix} -\frac{1}{4}s^4 + \frac{1}{3}s^3 - \frac{1}{4}, \\ -\frac{1}{8}\sqrt{3}s^4 + \frac{1}{6}\sqrt{3}s^3 - \frac{1}{4}s^2 + \frac{1}{2}s + \frac{1}{8}\sqrt{3} - 3, \\ -\frac{1}{8}s^4 + \frac{1}{6}s^3 + \frac{1}{4}\sqrt{3}s^2 - \frac{1}{2}\sqrt{3}s + 3\sqrt{3} + \frac{1}{8} \end{pmatrix}$$

with curvatures $\kappa_2^* = 1$ *and* $\tau_2^* = \frac{1}{s(s-1)^2}$. *It can be easily verified that* $T(s) = T_1^*(h_1(s)) = T_2^*(h_2(s))$. *Accordingly, the curve pairs* (φ, φ_1^*), (φ, φ_2^*) *and* $(\varphi_1^*, \varphi_2^*)$ *are related by a transformation of Combescure.*

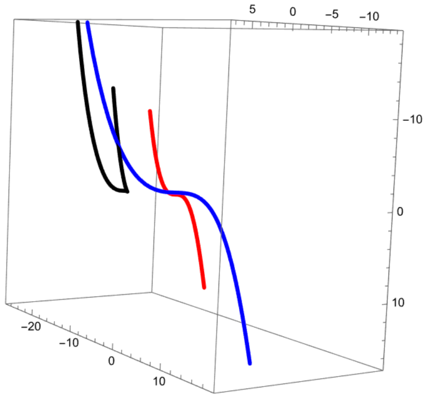

FIGURE 1. The figure illustrates the main curve φ (red) together with the associated curves φ_1^* (blue) and φ_2^* (black), which are connected to φ via a Combescure transformation in Example 3.1.

4. Some Applications of Combescure Related Pseudo Null Curves

In this section, we investigate key problems involving pseudo null curves in Minkowski 3-space related by the Combescure transformation. First, given a biharmonic pseudo null curve $\varphi : I \to \mathbb{E}_1^3$, we determine the necessary and sufficient conditions for its Combescure-related curve $\varphi^* : I^* \to \mathbb{E}_1^3$ to also be biharmonic. Second, assuming that the ruled surface generated by φ satisfies the Da Rios vortex filament equation, we establish the conditions under which the ruled surface generated by φ^* likewise solves this equation.

4.1. Biharmonic Curves and Combescure Related Space Curves

In this subsection, we examine the conditions under which the Combescure transform of a non-null biharmonic curve remains biharmonic. Before doing so, we briefly review biharmonic curves in Minkowski 3-space and summarize relevant previous results.

A unit-speed curve is biharmonic if the Laplacian of its mean curvature vector vanishes. In semi-Euclidean 3-space, this is equivalent to the biharmonic equation on the curve itself. Chen and Ishikawa [10] showed that all biharmonic curves in semi-Euclidean spaces lie within three-dimensional totally geodesic subspaces. In Minkowski 3-space, Inoguchi classified biharmonic curves as helices with curvature and torsion satisfying a specific relation [20]. Studying biharmonic curves sheds light on the behavior of elastic curves in Lorentzian geometry, which differ from Euclidean cases due to the indefinite metric. The characterization of a pseudo-null curve as a biharmonic curve in Minkowski 3-space is given by Inoguchi with Theorem 3.1 in [21].

The following theorem provides the conditions under which the associated curve of a pseudo null biharmonic curve in Minkowski 3-space, related through the Combescure transformation, is also a biharmonic curve.

Theorem 4.1. *Let $\varphi : I \subseteq \mathbb{R} \to \mathbb{E}_1^3$ and $\varphi^* : I^* \subseteq \mathbb{R} \to \mathbb{E}_1^3$ be Combescure related unit speed pseudo null curves in $\mathbb{E}_1^3$ with Frenet apparatus $\{T, N, B, \kappa, \tau, s\}$ and $\{T^*, N^*, B^*, \kappa^*, \tau^*, s^*\}$, respectively. If φ is a biharmonic curve, a necessary and sufficient condition for φ^* to also be a biharmonic curve is that the differentiable function C in the parametrization of φ^* given by (3) is one of the following:*

(1) *If the torsion of the biharmonic curve φ is $\tau(s) = 0$, then for real constants d, λ, μ_1, μ_2, the function $C(s)$ is either*

$$C(s) = \lambda s + \int e^{\int \tau(s)ds} ds \tag{10}$$

or

$$C(s) = -d \mp \sqrt{d^2 + 2\mu_1 s + 2\mu_1\mu_2} - s + \int e^{\int \tau(s)ds} ds \tag{11}$$

(2) *If the torsion of the biharmonic curve φ is $\tau(s) = \frac{1}{s+c}$, where $c \in \mathbb{R}$, then for real constants $d, \delta_1, \delta_2, \sigma_1, \sigma_2$, the function $C(s)$ is either*

$$C(s) = \frac{\delta_1}{2}s^2 + (\delta_1 c - 1)s + \delta_2 + \int e^{\int \tau(s)ds} ds \tag{12}$$

or

$$C(s) = -d \mp \sqrt{d^2 + \left(-1 + (s+c)^2\right) e^{\sigma_1} + \sigma_2} - s + \int e^{\int \tau(s)ds} ds. \tag{13}$$

Proof. Let assume that φ and φ^* are Combescure related pseudo null curves in Minkowski 3-space, and suppose φ is a biharmonic curve. We will determine the function $C(s)$ in the parametrization (3) of φ^* that ensures φ^* is also a biharmonic curve. For this, the following two cases arise:

Case 1. Suppose that φ is a biharmonic curve with torsion function $\tau(s) = 0$. Then, for φ^* to also be a biharmonic curve, the torsion function of φ^* must satisfy $\tau^*(s^*) = 0$ or $\tau^*(s^*) = \frac{1}{s^*+d}$, where d is a real constant (see Theorem 3.1 in [21]).

Case 2. Suppose that φ is a biharmonic curve with torsion function $\tau(s) = \frac{1}{s+c}$ where c is a real constant. In this case, φ^* is also biharmonic if and only if its torsion function is either $\tau^*(s^*) = 0$ or $\tau^*(s^*) = \frac{1}{s^*+d}$, for some real constant d.

We now proceed to analyze these cases in detail. Assume that the parametrization of φ^* is given by (3). From Theorem 3.2, we get

$$\tau^*(s^*) = \frac{1}{(h_1')^2}(h_1'\tau - h_1''). \tag{14}$$

If we take $\tau(s) = 0$ and $\tau^*(s^*) = 0$ in (14), we obtain the differential equation $\frac{h_1''}{(h_1')^2} = 0$. The general solution of this differential equation is

$$h_1(s) = c_1 s + c_2 \tag{15}$$

where $c_1, c_2 \in \mathbb{R}$. Since $h_1(s)$ is a diffeomorphism, it must satisfy $h_1'(s) = c_1 \neq 0$. Substituting equation (15) into equation (8), we obtain

$$\frac{dC(s)}{ds} = c_1 - 1 + e^{\int \tau(s)ds} \tag{16}$$

If we denote $c_1 - 1 = \lambda$, it is clear that $\lambda \neq -1$. Then integrating equation (16) gives $C(s) = \lambda s + \int e^{\int \tau(s)ds} ds$. Next, consider equation (14) with $\tau(s) = 0$ and $\tau^*(s^*) = \frac{1}{s^*+d}$ where d is a real constant. Then, we get

$$\frac{1}{s^* + d} = -\frac{h_1''}{(h_1')^2}. \tag{17}$$

Since $\frac{ds^*}{ds} = h_1'$, it follows that $s^* = h_1(s)$ and substituting into equation (17) gives $h_1'' + \frac{(h_1')^2}{h_1(s)+d} = 0$. The general solution of this differential equation is

$$h_1(s) = -d \mp \sqrt{d^2 + 2\mu_1 s + 2\mu_1\mu_2} \tag{18}$$

for some real constants μ_1 and μ_2. Substituting (18) in (8) yields $\frac{dC}{ds} = h_1'(s) - 1 + e^{\int \tau(s)ds}$, which integrates to $C(s) = -d \mp \sqrt{d^2 + 2\mu_1 s + 2\mu_1\mu_2} - s + \int e^{\int \tau(s)ds}ds$. Now, if in equation (14) we take $\tau(s) = \frac{1}{s+c}$ and $\tau^*(s^*) = 0$ for a real constant c . With the above similar steps are repeated for this case $C(s) = \frac{\delta_1}{2}s^2 + (\delta_1 c - 1)s + \delta_2 + \int e^{\int \tau(s)ds}ds$ can be readily obtained. Finally, in equation (14) if we take $\tau(s) = \frac{1}{s+c}$, $\tau^*(s^*) = \frac{1}{s^*+d}$ and for real constants c and d. A straightforward repetition of the previous steps for this situation gives $C(s) = -d \mp \sqrt{d^2 + \left(-1 + (s+c)^2\right)e^{\sigma_1} + \sigma_2} - s + \int e^{\int \tau(s)ds}ds$.

Conversely, suppose that in the parametrization of the curve φ^* given by (3) the function $C(s)$ is given in the form $C(s) = -d + \sqrt{d^2 + \left(-1 + (s+c)^2\right)e^{\sigma_1} + \sigma_2} - s + \int e^{\int \tau(s)ds}ds$ where d, σ_1, σ_2 are real constants. Then from equation (8) we obtain $h_1'(s) = \frac{(s+c)e^{\sigma_1}}{\sqrt{d^2+\left(-1+(s+c)^2\right)e^{\sigma_1}+\sigma_2}}$. From equation (14) it follows that

$$\tau^*(s^*) \quad = \quad \tau^*(h_1(s)) = \frac{1}{h_1(s) + d}$$

Since $s^* = h_1(s)$, we have $\tau^*(s^*) = \frac{1}{s^*+d}$. Thus, the pseudo-null curve φ^* is a biharmonic curve. The proof can be carried out in a similar manner for the remaining cases of the function $C(s)$ as well. □

Example 4.1. *Consider the pseudo null biharmonic curve in Minkowski 3-space given by*

$$\varphi(s) = \left(\frac{1}{6}s^3 + \frac{1}{2}s^2, s, \frac{1}{6}s^3 + \frac{1}{2}s^2\right)$$

with the curvatures $\kappa(s) = 1, \tau(s) = \frac{1}{s+1}$. *If* $\delta_1 = 2$ *and* $\delta_2 = 0$ *are assumed in part (2) of Theorem 4.1, then it follows that* $C_1(s) = \frac{3s^2+4s+1}{2}$ *and using in (3), the curve* φ_1^*, *which is Combescure related to* φ, *is obtained as follows*

$$\varphi_1^*(h_1(s)) = \left(\frac{-2 + s^2(2+s)^2}{4}, \frac{1}{2} + s(2+s), \frac{2 + s^2(2+s)^2}{4}\right),$$

where $h_1'(s) = 2 + 2s$. *It is easily calculated that* $\tau_1^*(s^*) = 0$. *Therefore, the curve* φ_1^* *is a pseudo null biharmonic curve. If* $d = 2$, $c = 1$ $\sigma_1 = 0$ *and* $\sigma_2 = -2$ *are assumed in part (2) of Theorem 4.1, then it follows that* $C_2(s) = -2 + \sqrt{1 + (s+1)^2} + \frac{s^2+1}{2}$ *and using in (3), the curve* φ_2^*, *which is Combescure related to* φ, *is obtained as follows*

$$\varphi_2^*(h_2(s)) = \begin{pmatrix} \frac{-3+(s^2+2s-4)\sqrt{1+(1+s)^2}}{6}, \\ \\ \frac{-3+2\sqrt{1+(1+s)^2}}{2}, \\ \\ \frac{3+(s^2+2s-4)\sqrt{1+(1+s)^2}}{6} \end{pmatrix},$$

where $h_2'(s) = \frac{1+s}{\sqrt{1+(1+s)^2}}$. *When the torsion function of* φ_2^* *is calculated with respect to* s^*, *we obtain* $\tau_2^*(s^*) = \frac{1}{s^*+2}$. *This implies that* φ_2^* *is a pseudo null biharmonic curve.*

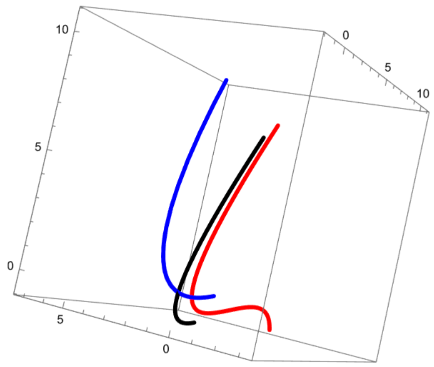

FIGURE 2. The figure illustrates the main pseudo null biharmonic curve φ (red) together with the associated pseudo null biharmonic curves φ_1^* (blue) and φ_2^* (black), which are connected to φ via a Combescure transformation in Example 4.1

4.2. Da Rios Vortex Filament Equation and Combescure Related Pseudo Null Curves

The vortex filament equation, also known as the smoke ring equation or localized induction approximation, models the evolution of a one-dimensional vortex filament in an incompressible, inviscid fluid, originally introduced by L.S. Da Rios [11]. For a curve $x(s,t)$ parametrized by arc length s and time t, its evolution is governed by

$$x_t = x_s \times x_{ss}. \tag{19}$$

This equation was later rediscovered by Betchov and independently by Arms and Hama [1, 2, 6], as an asymptotic form considering only local Biot-Savart contributions.

Beyond fluid mechanics, the equation admits a geometric interpretation as a dynamical system on curves in Minkowski 3-space [30, 34]. Special solutions preserving filament shape correspond to traveling wave solutions of the nonlinear Schrödinger (NLS) equation [18], with related soliton surfaces known as Hasimoto surfaces [14].

Specifically:

- If $x(s,t)$ is spacelike with a timelike normal vector, the motion generates a spacelike Hasimoto surface.
- If the binormal vector is timelike, the resulting surface is timelike.

These cases connect to nonlinear heat-type systems [17]. For timelike curves $x(s,t)$, the vortex filament flow yields a timelike Hasimoto surface governed by a repulsive NLS equation:

$$iq_t = -q_{ss} + 2|q|^2 q,$$

with extensive studies on this correspondence [13, 23].

In Minkowski 3-space, conditions under which ruled surfaces generated by pseudo-null curves or their Frenet vectors satisfy the Da Rios equation have been analyzed with respect to the curve's torsion, yielding important results [12, 15, 27]. When we consider the Theorem 6.1-6.2 in [27] and Theorem 3.1 in [21], we get the following result.

Corollary 4.1. *A ruled surface generated by a pseudo null biharmonic curve in Minkowski 3-space is a solution to the Da Rios vortex filament equation.*

Let φ and φ^* be pseudo-null curves related by a Combescure transformation. When the ruled surface S, generated by the curve φ, is a solution to the Da Rios equation, the conditions under which the ruled surface S^*, generated by the curve φ^*, is also a solution to the Da Rios equation are provided in the following theorem.

Theorem 4.2. *Let $\varphi : I \subseteq \mathbb{R} \to \mathbb{E}_1^3$ and $\varphi^* : I^* \subseteq \mathbb{R} \to \mathbb{E}_1^3$ be Combescure related unit speed pseudo null curves in $\mathbb{E}_1^3$ with Frenet apparatus $\{T, N, B, \kappa, \tau, s\}$ and $\{T^*, N^*, B^*, \kappa^*, \tau^*, s^*\}$, respectively. If the ruled surface S generated by the curve φ satisfies the Da Rios vortex filament equation, then a necessary and sufficient condition for the Combescure related curve φ^* to also satisfy the Da Rios equation is that the differentiable function $C(s)$ in the parametrization of φ^* given by equation (3) is one of the following:*

(1) *If φ is a pseudo-null curve with torsion function $\tau(s) = 0$. In this case:*
   (a) *If $C(s) = c_1 s + \int e^{\int \tau(s)ds} ds$, then the torsion function of φ^* is $\tau^*(s^*) = 0$.*
   (b) *If $C(s) = c_1 + \frac{1}{d}\ln(ds - c_2) - s + \int e^{\int \tau(s)ds} ds$, then the torsion function of φ^* is $\tau^*(s^*) = d \neq 0$.*
   (c) *If $C(s) = -d \mp \sqrt{d^2 + \left(-1 + (s+c)^2\right) e^{\sigma_1} + \sigma_2} - s + \int e^{\int \tau(s)ds} ds$, then the torsion function of φ^* is $\tau^*(s) = \frac{1}{s^*+d}$.*
   (d) *If $C(s) = \frac{-1-c_1 ds - c_2 d}{c_1 s + c_2} - s + \int e^{\int \tau(s)ds} ds$, then the torsion function of φ^* is $\tau^*(s) = \frac{-2}{s+d}$.*
   (e) *If $C(s) = -2d + 2\arctan(c_1 s + c_2) - s + \int e^{\int \tau(s)ds} ds$, then the torsion function of φ^* is $\tan(\frac{s^*}{2} + d)$.*

(2) *If φ is a pseudo-null curve with torsion function $\tau(s) = c$. In this case:*
   (a) *If $C(s) = c_1 e^{cs} + c_2 - s + \int e^{\int \tau(s)ds} ds$, then the torsion function of φ^* is $\tau^*(s^*) = 0$.*
   (b) *If $C(s) = C(s) = \frac{1}{d}\ln\left(1 + de^{c(s+c_1)}\right) + c_2 - s + \int e^{\int \tau(s)ds} ds$, then the torsion function of φ^* is $\tau^*(s^*) = d \neq 0$.*
   (c) *If $C(s) = \frac{-cd \mp \sqrt{c^2 d^2 + ce^{c(s+c_2)} - 2cc_1}}{c} - s + \int e^{\int \tau(s)ds} ds$, then the torsion function of φ^* is $\tau^*(s) = \frac{1}{s^*+d}$.*
   (d) *If $C(s) = \frac{c - de^{c(s+c_2)} - dc_1}{e^{c(s+c_2)} + c_1} - s + \int e^{\int \tau(s)ds} ds$, then the torsion function of φ^* is $\tau^*(s) = \frac{-2}{s+d}$.*
   (e) *If $C(s) = 2\arccos\left(\frac{2c\cos d + \left(e^{c(s+c_2)} - c_1\right)\sin d}{\sqrt{4c^2 + \left(e^{c(s+c_2)} - c_1\right)^2}}\right) - s + \int e^{\int \tau(s)ds} ds$, then the torsion function of φ^* is $\tau^*(s) = \tan(\frac{s^*}{2} + d)$.*

(3) *If φ is a pseudo-null curve with torsion function $\tau(s) = \frac{1}{s+c}$. In this case:*
   (a) *If $C(s) = c_1\left(cs + \frac{s^2}{2}\right) + c_2 - s + \int e^{\int \tau(s)ds} ds$, then the torsion function of φ^* is $\tau^*(s^*) = 0$.*
   (b) *If $C(s) = \frac{1}{d}\ln\left(2cds + ds^2 + 2c_1\right) + c_2 - s + \int e^{\int \tau(s)ds} ds$, then the torsion function of φ^* is $\tau^*(s^*) = d \neq 0$.*
   (c) *If $C(s) = -d \mp \sqrt{d^2 - e^{2c_2} + c^2 e^{2c_2} + 2ce^{2c_2}s + e^{2c_2}s^2 + c_1} - s + \int e^{\int \tau(s)ds} ds$, then the torsion function of φ^* is $\tau^*(s) = \frac{1}{s^*+d}$.*
   (d) *If $C(s) = \frac{-d\left(-1+c^2+2cs+s^2\right) + (1-dc_1)\cos(c_2) + (1+dc_1)\sin(c_2)}{-1+c^2+2cs+s^2+c_1\cos(c_2) - c_1\sin(c_2)} - s + \int e^{\int \tau(s)ds} ds$, then the torsion function of φ^* is $\tau^*(s) = \frac{-2}{s+d}$.*

(e) $C(s) = 2\arccos\left(\frac{\ln(c+s)\sin d - e^{c_2}(2\cos d + c_1 \sin d}{\sqrt{e^{2c_1}(4+c_1^2) - 2c_1 e^{c_2}\ln(c+s) + \ln^2(c+s)}}\right) - s + \int e^{\int \tau(s)ds} ds$, *then* $\tan(\frac{s^*}{2} + d)$.

(4) *If* φ *is a pseudo-null curve with torsion function* $\tau(s) = \frac{-2}{s+c}$*. In this case:*

(a) *If* $C(s) = -\frac{c_1}{s+c} + c_2 - s + \int e^{\int \tau(s)ds} ds$, *then the torsion function of* φ^* *is* $\tau^*(s^*) = 0$.

(b) *If* $C(s) = \frac{1}{d}\ln\left(\frac{d}{c+s} + c_1\right) + c_2 - s + \int e^{\int \tau(s)ds} ds$, *then the torsion function of* φ^* *is* $\tau^*(s^*) = d \neq 0$.

(c) *If* $C(s) = -d \mp \sqrt{d^2 + \frac{2c_1}{s+c} + 2c_2} - s + \int e^{\int \tau(s)ds} ds$, *then the torsion function of* φ^* *is* $\tau^*(s) = \frac{1}{s^*+d}$.

(d) *If* $C(s) = \frac{-c-s-dc_1-cdc_2-dsc_2}{c_1+cc_2+sc_2} - s + \int e^{\int \tau(s)ds} ds$, *then the torsion function of* φ^* *is* $\tau^*(s) = \frac{-2}{s+d}$.

(e) *If* $C(s) = -2d + 2\arctan\left(\frac{c_1+cc_2+sc_2}{2c+2s}\right) - s + \int e^{\int \tau(s)ds} ds$, *then* $\tan(\frac{s^*}{2} + d)$.

(5) *If* φ *is a pseudo-null curve with torsion function* $\tau(s) = \tan\left(\frac{s}{2} + c\right)$*. In this case:*

(a) *If* $C(s) = 2c_1\tan\left(\frac{s}{2} + c\right) + c_2 - s + \int e^{\int \tau(s)ds} ds$, *then the torsion function of* φ^* *is* $\tau^*(s^*) = 0$.

(b) *If* $C(s) = \frac{1}{d}\ln\left(c_1 + 2d\tan\left(\frac{s}{2} + c\right)\right) + c_2 - s + \int e^{\int \tau(s)ds} ds$, *then the torsion function of* φ^* *is* $\tau^*(s^*) = d \neq 0$.

(c) *If* $C(s) = -d \mp \sqrt{d^2 + 2c_2 - 4c_1\tan\left(\frac{s}{2} + c\right)} - s + \int e^{\int \tau(s)ds} ds$, *then the torsion function of* φ^* *is* $\tau^*(s) = \frac{1}{s^*+d}$.

(d) *If* $C(s) = \frac{-1}{c_2 - 2c_1\tan\left(\frac{s}{2}+c\right)} - d - s + \int e^{\int \tau(s)ds} ds$, *then the torsion function of* φ^* *is* $\tau^*(s) = \frac{-2}{s+d}$.

(e) *If* $C(s) = -2d + 2\arctan\left(\frac{c_2}{2} - c_1\tan\left(\frac{s}{2} + c\right)\right) - s + \int e^{\int \tau(s)ds} d$, *then the torsion function of* φ^* *is* $\tan(\frac{s^*}{2} + d)$.

Proof. Let us assume that φ and φ^* are pseudo null curves in Minkowski 3-space, connected via a Combescure transformation. When we consider the Theorem 6.1-6.2 in [27], the possible values of the torsion function for a pseudo null curve whose associated ruled surface is a solution to the Da Rios vortex filament equation are stated. Among these values, it is known that in the cases $\tau(s) = 0$ and $\tau(s) = \frac{1}{s+c}$, the curve is also a biharmonic curve. For these specific cases, Theorem 4.1 proves how the differentiable function $C(s)$, which appears in the parametrization of the associated curve φ^*, should be chosen so that φ^* is also a biharmonic curve. These cases correspond to the subcases 1-a, 1-c, 3-a, and 3-c of our main theorem. Therefore, the remaining cases can similarly be verified using the proof technique provided in Theorem 4.1. □

Example 4.2. *Consider the pseudo null curve in Minkowski 3-space given by*

$$\varphi(s) = \left(4\ln\left(\cos(\frac{s}{2} + 1)\right), s, -4\ln\left(\cos(\frac{s}{2} + 1)\right)\right)$$

with the curvatures $\kappa(s) = 1, \tau(s) = \tan(\frac{s}{2} + 1)$*. If we take* $d = -2, c_1 = -1$ *and* $c_2 = 2$ *in part (5-e) of Theorem 4.2, then it follows that* $C(s) = 4 + 2\arctan\left(1 + \tan\left(\frac{s}{2} + 1\right)\right) - s + 2\tan\left(\frac{s}{2} + 1\right)$ *and using in (3), the curve* φ^**, which is Combescure related to* φ*, is obtained* *as follows*

$$\varphi^*(h(s)) = \begin{pmatrix} \frac{5}{2} + 4\arctan\left(1+\tan(\frac{s}{2}+1)\right) - 2\ln\left(2+\tan(\frac{s}{2}+1)\left(2+\tan(\frac{s}{2}+1)\right)\right), \\ 4 + 2\arctan\left(1+\tan(\frac{s}{2}+1)\right), \\ -\frac{3}{2} - 4\arctan\left(1+\tan(\frac{s}{2}+1)\right) + 2\ln\left(2+\tan(\frac{s}{2}+1)\left(2+\tan(\frac{s}{2}+1)\right)\right) \end{pmatrix},$$

where $s^* = h(s) = 4 + 2\arctan\left(1+\tan\left(\frac{s}{2}+1\right)\right)$. *The ruled surfaces* S *and* S^**, generated by the curves* φ *and* φ^**, respectively, and also satisfying the vortex filament equation, are given by*

$$\begin{aligned} \phi(s,t) &= B(s) + t\left(-\tan\left(\frac{s}{2}+1\right)T(s) + N(s) - \frac{1}{2\cos^2\left(\frac{s}{2}+1\right)}B(s)\right) \\ &= \left(\frac{1}{4}(5-5t-3\cos(2+s)), -\sin(2+s), \frac{1}{4}(-3+3t+5\cos(2+s))\right) \end{aligned}$$

and

$$\begin{aligned} \phi^*(s^*,t) &= \phi^*(h(s),t) \\ &= \begin{pmatrix} -\frac{9t}{4} + \frac{5-3\cos(2+s)}{6+2\cos(2+s)+4\sin(2+s)}, \\ -t - \frac{2}{2\cot(2+s)+3\csc(2+s)}, \\ \frac{7t}{4} + \frac{-3+5\cos(2+s)}{6+2\cos(2+s)+4\sin(2+s)} \end{pmatrix}. \end{aligned}$$

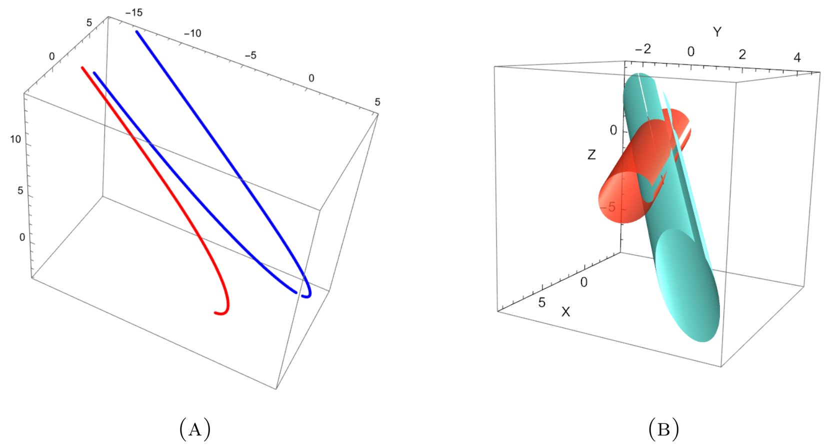

FIGURE 3. Figure A illustrates the curves φ and φ^*, which are pseudo null curves connected via a Combescure transformation, as given in Example 4.2. Figure B presents the ruled surfaces S and S^* , generated by the curves φ and φ^*, respectively. These surfaces are solutions to the Da Rios vortex filament equation.

Remark 4.1. *All the theorems, proofs, results, and examples above have been carried out based on the parametrization of the conjugate curve* φ^* *given by equation* (3) *in Theorem*

3.1. Similar results can be obtained by using the parametrization given in equation (4) *in Theorem 3.1.*

5. Conclusion

This study examines pseudo null curves in Minkowski 3-space that are related through the Combescure transformation, meaning their tangent vectors are parallel. This condition also implies the parallelism of other Frenet vectors, placing them among Bertrand curve pairs. As an application, we consider biharmonic curves and ruled surfaces satisfying the Da Rios vortex filament equation. The results can be extended to other geometric settings, such as Minkowski space-time and (pseudo-)Galilean spaces.

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