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ON A METHOD BASED ON BERNSTEIN OPERATORS FOR $2D$ NONLINEAR FREDHOLM-HAMMERSTEIN INTEGRAL EQUATIONS

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In this research, a new iterative method based on Bernstein polynomials for solving two-dimensional nonlinear Hammerstein-Fredholm integral equations is proposed. By using the posteriori error estimate, a practical stopping criterion of the iterative method is obtained. The convergence analysis and the numerical stability of the method are proved. Finally, three numerical examples approved the theoretical results and illustrate the efficiency and accuracy of the method.

Keywords: Two-dimensional integral equations; Quadrature formula; Iterative method; Bernstein polynomials.

MSC2010: 47H09, 47H10.

1. Introduction

Many problems in applied mathematics, engineering, mechanics, mathematical physics and many other fields can be transformed into the second-kind two-dimensional integral equations [3, 2, 1, 4, 5]. Integral equations also arise as representation formulas in the solutions of differential equations. Some other applications of these equations can be found in [6, 7]. The numerical methods for integral equations involve various techniques and some of them can be extended for solving two-dimensional integral equations. The method of successive approximations and other iterative techniques are applied in [9, 11, 8, 10]. In [12, 13, 14, 15], authors proposed analytic methods, analytic-numeric methods like Adomian decomposition method, the regularization-homotopy method and homotopy perturbation method for solving one and two dimensional integral equations. Other techniques used in the construction of the numerical methods for integral equations are: the well-known collocations and Galerkin methods [18, 16, 17], Bernoulli operational matrix method [19], the method based on the piecewise approximation by Chebyshev polynomials [20], wavelet method [21], multi-step methods [22], Legendre functions

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[23], expansion method [24], triangular function[25], radial basis functions [26], Runge-Kutta method [27], block-pulse functions [28], rationalized Haar functions [29], hat basis function [30] and the Nyström type methods [31]. The theorems on the existence and uniqueness of the solution for the multidimentional integral equations can be found in researches, [32, 33, 34, 35]. In this paper, we propose an iterative method in order to approximate the solution of the following nonlinear Hammerstein integral equations in two dimensions:

$$X(s,t) = r(s,t) + \lambda \int_c^d \int_a^b H(s,t,x,y)\psi(x,y,X(x,y))dxdy,\ (s,t) \in I, \quad (1)$$

where $a, b, c, d \in \mathbb{R},\ r : I = [a,b] \times [c,d] \to \mathbb{R}, H : I \times I \to \mathbb{R}$ and r, H are continuous. In this work, we present a new iterative method based on the two-dimensional Bernsteins approximation. There are some works that used Bernestein polynomials as basis for numerically solving integral equations such as [36, 37, 38, 39, 40, 41], the main characteristic of these techniques is that it reduces these problems to those of solving a system of algebraic equations, but present method is different from them. In this method, we introduce a numerical iterative procedure using successive approximations method to approximate the solution of Eq. (1). The advantages of the presented method are as follows:

(1) The method is very effective and has simple structure for application.
(2) Most of the numerical methods to solve integral equations including Galerkin methods, using quadrature rules, using interpolation polynomials, applying Haar wavelets, finite and divided differences methods, block pulse functions and some hybrid methods, finally end in linear or nonlinear system of algebraic equations whose singularity of these systems might be hard to investigate. However, the proposed iterative method does not have such as problem and can be very useful.
(3) To prove the convergence and the numerical stability of the presented successive approximations method, only Lipschits conditions are required, smoothness conditions being not necessary.

This paper is composed of introduction in Section 1. In Section 2, we give basic definitions, assumptions and mathematical preliminaries of the Bernstein polynomials and their properties. In Section 3, we study the existence and uniqueness of the solution of Eq. (1) and apply Bernstein's approximation to solve these equations. In addition, the convergence analysis and numerical stability of the method are proved in this section. In Section 4, some numerical examples have been solved using the present method and compared with the exact solutions, and the conclusion in Section 5 accomplishes the paper.

2. Preliminaries

In this section, we review some necessary and basic definitions and results which will be further needed.

Definition 2.1. *Let $f : [0,1] \to \mathbb{R}$. The classical Bernstein operators $(B_n f)$ of degree $n \in \mathbb{N}$, are defined as follows* [42]

$$(B_n f)(x) = \sum_{i=0}^{n} f(\frac{i}{n}) P_{n,i}(x),$$

where $P_{n,i}(x) = \binom{n}{i} x^i (1-x)^{n-i}, i = 0, 1, 2, 3, ..., n,$ are called the Bernstein basis polynomialas.

These polynomials were introduced by Bernstein [42]. There is obvious that

$$\sum_{i=0}^{n} P_{n,i}(x) = 1 \tag{2}$$

Corollary 2.1. [43] *Let $x \in [0,1]$. Then $\sum_{i=0}^{n} |x - \frac{i}{n}| P_{n,i}(x) \leq \frac{1}{2\sqrt{n}}$.*

Definition 2.2. *Let $f : [0,1] \times [0,1] \to \mathbb{R}$. Then the two-dimensional Bernstein operators, of degree (n,m), corresponding to the function $f(x,y)$ on the square $\{(x,y) | 0 \leq x \leq 1, 0 \leq y \leq 1\}$ as follows* [44]

$$(B_{n,m}) f(x,y) = \sum_{i=0}^{n} \sum_{j=0}^{m} f(\frac{i}{n}, \frac{j}{m}) P_{n,i}(x) P_{m,j}(y), i = 0, 1, 2, 3, ..., n, j = 0, 1, 2, 3, ..., n \tag{3}$$

where $P_{n,i}(s) = \binom{n}{i} s^i (1-s)^{n-i}, i = 0, 1, 2, 3, ..., n$

Definition 2.3. *A function $f : I \to \mathbb{R}$ is said to be L-Lipschitz if*

$$|f(x_1, y_1) - f(x_2, y_2)| \leq L\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}, \quad \forall x_1, x_2 \in [a,b], y_1, y_2 \in [c,d]$$

Theorem 2.1. *Let $f : [0,1] \times [0,1] \to \mathbb{R}$, be a L-Lipschitz function. Then, we have*

$$\big|(B_{n,m}) f(x,y) - f(x,y)\big| \leq \frac{L}{2}(\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{m}}) \tag{4}$$

Proof.

$$
\begin{aligned}
\big|(B_{n,m})f(x,y)-f(x,y)\big| &= |\sum_{i=0}^{n}\sum_{j=0}^{m} f(\frac{i}{n},\frac{j}{m})P_{n,i}(x)P_{m,j}(y)-f(x,y)| \\
&\leq \sum_{i=0}^{n}\sum_{j=0}^{m}|P_{n,i}(x)||P_{m,j}(y)||f(\frac{i}{n},\frac{j}{m})-f(x,y)| \\
&\leq \sum_{i=0}^{n}\sum_{j=0}^{m}|P_{n,i}(x)||P_{m,j}(y)|L\sqrt{(x-\frac{i}{n})^2+(y-\frac{j}{m})^2} \\
&\leq \sum_{i=0}^{n}\sum_{j=0}^{m}|P_{n,i}(x)||P_{m,j}(y)|L(|x-\frac{i}{n}|+|y-\frac{j}{m}|) \\
&\leq L\sum_{i=0}^{n}|P_{n,i}(x)|\sum_{j=0}^{m}(|x-\frac{i}{n}|+|y-\frac{j}{m}|)|P_{m,j}(y)| \\
&\leq L\sum_{i=0}^{n}|P_{n,i}(x)|\Big(\sum_{j=0}^{m}|x-\frac{i}{n}||P_{m,j}(y)|+\sum_{j=0}^{m}|y-\frac{j}{m}||P_{m,j}(y)|\Big)
\end{aligned}
$$

According to Corollary (2.1) and (2), we have

$$
\begin{aligned}
\big|(B_{n,m})f(x,y)-f(x,y)\big| &\leq L\sum_{i=0}^{n}|P_{n,i}(x)|\big(|x-\frac{i}{n}|+\frac{1}{2\sqrt{m}}\big) \\
&\leq L\sum_{i=0}^{n}P_{n,i}(x)|x-\frac{i}{n}|+L\frac{1}{2\sqrt{m}}\sum_{i=0}^{n}|P_{n,i}(x)| \\
&\leq \frac{L}{2}(\frac{1}{\sqrt{n}}+\frac{1}{\sqrt{m}})
\end{aligned}
$$

□

Remark 2.1. Since $\big|(B_{n,m})f(x,y)-f(x,y)\big|$ tends to zero as $n,m\to+\infty$, we infer that the sequence $(B_{n,m})f(x,y)$ converges uniformly to $f(x,y)$ with respect to $(x,y)\in[0,1]\times[0,1]$.

3. Main results

3.1. The existence result

Consider the space of the two-dimensional functions $\Omega=\{f:I\to\mathbb{R};f\ \textit{is continuous}\}$ with the norm $\| f-g \|=\sup_{\substack{a\leq s\leq b\\ c\leq t\leq d}} | f(s,t)-g(s,t) |$, for all f , $g\in\Omega$. Also, consider the following condition:

(i) $r\in C(I,\mathbb{R}), H\in C(I\times I,\mathbb{R}), \psi\in C(I\times\mathbb{R},\mathbb{R})$,

(ii) there exist $\alpha,\beta>0$, such that $| \psi(x,y,u)-\psi(x',y',u') |\leq \alpha(| x-x' | + | y-y' |)+\beta | u-u' |$, $\forall(x,y)\in I,\forall u,u'\in\mathbb{R}$.

(iii) $\beta\lambda M_H(b-a)(d-c) < 1$, where $M_H > 0$ is such that $\mid H(s,t,x,y) \mid\leq M_H$, $\forall s,x \in [a,b]$,$t,y \in [c,d]$, according continuity of H,

(iv) there exist $\zeta, \eta > 0$, such that
$\mid (H(s,t,x,y) - H(s',t',x',y') \mid\leq \zeta(\mid s-s' \mid + \mid t-t' \mid)$
$+ \eta(\mid x-x' \mid + \mid y-y' \mid)$

(v) there exists $\theta > 0$, such that
$\mid r(s,t) - r(s',t') \mid\leq \theta(\mid s-s' \mid + \mid t-t' \mid) \qquad \forall (s,t),(s',t') \in I.$

Let $\{\psi_k\}_{k\in N}$ be a the sequence of function $\psi_k : [a,b]\times[c,d] \to \mathbb{R}$, defined by $\Psi_k(x,y) = \psi(x,y,X_k(x,y))$. Now, we will prove the existence and uniqueness of the solution of Eq. (1) using the Banach's fixed point principle and method of successive approximations.

Theorem 3.1. *(a) Let the conditions (i)-(iv) are satisfied. then the Eq. (1) has a unique solution $X^* \in \Omega$, and the sequence of successive approximations $\{X_k\}_{k\in N}$*

$$X_0(s,t) = r(s,t),$$

$$X_k(s,t) = r(s,t) + \lambda \int_c^d \int_a^b H(s,t,x,y)\psi(x,y,X_{k-1}(x,y))dxdy, \ k \geq 1, \quad (5)$$

converges to the solution $X^* \in \Omega$. Furthermore, the following a priori and a posteriori error estimates hold

$$\|X^* - X_k\| \leq \frac{(\beta\lambda M_H(b-a)(d-c))^k}{1-\beta\lambda M_H(b-a)(d-c)}\|X_0 - X_1\|, \quad (6)$$

$$\|X^* - X_k\| \leq \frac{\beta\lambda M_H(b-a)(d-c)}{1-\beta\lambda M_H(b-a)(d-c)}\|X_{k-1} - X_k\|, \quad (7)$$

Moreover, the sequence of successive approximations is uniformaly bounded, that is, there exists a constant $\rho \geq 0$ such that $|X_k(s,t)| \leq \rho$.
(b) If all the condition (i)-(vi) are satifed, then the sequence $\{X_k\}_{k\in N}$ and $\{\Psi_k\}_{k\in N}$ are uniformly Lipschitz with constants $L_0 = \theta + \lambda(b-a)(d-c)M\zeta$ and $L' = \alpha + \beta(\theta + \lambda(b-a)(d-c)M\zeta)$, respectively. where M is given in (11).

Proof. (a) Consider the iterative scheme

$$X_{k+1}(s,t) = r(s,t) + \lambda \int_c^d \int_a^b H(s,t,x,y)\psi(x,y,X_k(x,y))dxdy, \ k \geq 1, \quad (8)$$

we have

$$\begin{aligned}
&\big|X_{k+1}(s,t) - X_k(s,t)\big| \\
&\quad\leq \lambda\big|H(s,t,x,y)\big| \int_c^d \int_a^b \big|\psi(x,y,X_k(x,y)) - \psi(x,y,X_{k-1}(x,y))\big|dxdy \\
&\quad\leq \beta\lambda M_H \int_c^d \int_a^b \big|\psi(X_k(x,y)) - \psi(X_{k-1}(x,y))\big|dxdy \\
&\quad\leq (\beta\lambda M_H(b-a)(d-c))d(X_k, X_{k-1})
\end{aligned}$$

Therefore, we obtained $d(X_{k+1}, X_k) \leq (\beta\lambda M_H(b-a)(d-c))d(X_k, X_{k-1})$. Hence $d(X_{k+1}, X_k) \leq (\beta\lambda M_H(b-a)(d-c))^{k-1}d(X_2, X_1)$. Since Ω is a complete metric space, and $\beta\lambda M_H(b-a)(d-c) < 1$, then we conclude by using the Weierstrass M-test that the series

$$\sum_{k=1}^{\infty}(X_{k+1}(s,t) - X_k(s,t)), \tag{9}$$

is absolutely and uniformly convergent on $[a,b] \times [c,d]$. On the other hand, $X_k(s,t)$ can be written as

$$X_k(s,t) = X_1(s,t) + \sum_{m=1}^{k-1}(X_{m+1}(s,t) - X_m(s,t)),$$

therefore from uniform convergence of the series (9), we conclude that $\lim_{k\to\infty} X_k(s,t)$ exists for all $(s,t) \in [a,b] \times [c,d]$, that is, there exists a unique solution $X^* \in \mathbf{X}$ such that $\lim_{k\to\infty} \|X_k - X^*\| = 0$. Taking limit of both sides of Eq. (8), we obtain

$$\begin{aligned}\lim_{k\to\infty} X_{k+1}(x,y) &= \lim_{k\to\infty} r(s,t) + \lambda \int_c^d \int_a^b H(s,t,x,y)\psi(x,y,\lim_{k\to\infty} X_k(x,y))dxdy \\ &= r(s,t) + \lambda \int_c^d \int_a^b H(s,t,x,y)\psi(x,y,X(x,y))dxdy = X(s,t)\end{aligned}$$

that is, $X^*(s,t)$ is the unique solution of (1). Moreover, by the Banach's fixed point principle we obtain the estimates (6) and (7). Let $\Psi_0 : [a,b] \times [c,d] \to \mathbb{R}, \Psi_0(x,y) = \psi(x,y,r(x,y))$. Since ψ, r are continuous, we infer that Ψ_0 is continuous on the compact set $[a,b] \times [c,d]$ and therefore $M_0 \geq 0$ exist, such that

$$|\psi_0(x,y)| \leq M_0 \qquad \forall (x,y) \in [a,b] \times [c,d]. \tag{10}$$

For $(s,t) \in [a,b] \times [c,d]$, it follows that

$$\begin{aligned}\big|X_k(s,t) - X_{k-1}(s,t)\big| &\leq \lambda M_H \int_c^d \int_a^b \big|\psi(x,y,X_k(x,y)) - \psi(x,y,X_{k-1}(x,y))\big|dxdy \\ &\leq \beta\lambda M_H(b-a)(d-c) \max_{\substack{a\leq x\leq b \\ c\leq y\leq d}} |X_k(x,y) - X_{k-1}(x,y)| \\ &= \beta\lambda M_H(b-a)(d-c)\big\|X_k - X_{k-1}\big\|\end{aligned}$$

and by induction,

$$\big|X_k(s,t) - X_{k-1}(s,t)\big| \leq (\beta\lambda M_H(b-a)(d-c))^{k-1}\big\|X_1 - X_0\big\|.$$

So,

$$\begin{aligned}\big|X_k(s,t)-X_0(s,t)\big| &\leq \big|X_k(s,t)-X_{k-1}(s,t)\big|+...+\big|X_1(s,t)-X_0(s,t)\big|\\ &\leq ((\beta\lambda M_H(b-a)(d-c))^{k-1}+...+\beta\lambda M_H(b-a)(d-c)+1)\big\|X_1-X_0\big\|\\ &=\frac{1-(\beta\lambda M_H(b-a)(d-c))^k}{1-\beta\lambda M_H(b-a)(d-c)}.\big\|X_1-X_0\big\|\\ &\leq \frac{\beta\lambda M_H(b-a)(d-c)M_0}{\beta(1-\beta\lambda M_H(b-a)(d-c))}\quad \forall(s,t)\in[a,b]\times[c,d].\end{aligned}$$

Let $M_r \geq 0$ such that $|r(s,t)| \leq M_r$ for all $(s,t)\in[a,b]\times[c,d]$. Then

$$\big|X_k(s,t)\big| \leq \big|X_k(s,t)-X_0(s,t)\big|+\big|X_0(s,t)\big| \leq \frac{\beta\lambda M_H(b-a)(d-c)M_0}{\beta(1-\beta\lambda M_H(b-a)(d-c))}+M_r=\rho$$

for all $(s,t)\in[a,b]\times[c,d]$.
(b) cosidering

$$M=max\big(M_0, max\{|\psi(s,t,u)| : (s,t)\in[a,b]\times[c,d], u\in[-\rho,\rho]\}\big) \qquad (11)$$

we get $|\Psi_k(s,t)| = |\psi(s,t,X_k(s,t))| \leq M$ for all $(s,t)\in[a,b]\times[c,d]$ and $k\in\mathbf{N}$. Let $(s,t),(s',t')\in[a,b]\times[c,d]$, we obtain $\big|X_0(s,t)-X_0(s',t')\big| \leq \theta(\big|s-s'\big|+\big|t-t'\big|)$ and

$$\begin{aligned}\big|X_k(s,t)-X_k(s',t')\big| &\leq \big|r(s,t)-r(s',t')\big|\\ &+\lambda\int_c^d\int_a^b | H(s,t,x,y)-H(s^{'},t^{'},x,y)) \,||\, \psi\big(x,y,X_{k-1}(x,y)\big) \,|\, dxdy\\ &\leq \theta(\big|s-s'\big|+\big|t-t'\big|)+\lambda(b-a)(d-c)M\zeta(\big|s-s'\big|+\big|t-t'\big|)\\ &= L_0(\big|s-s'\big|+\big|t-t'\big|)\end{aligned}$$

with $L_0=\theta+\lambda(b-a)(d-c)M\zeta$ and

$$\begin{aligned}\big|\Psi_0(s,t)-\Psi_0(s',t')\big| &\leq \alpha(\big|s-s'\big|+\big|t-t'\big|)+\beta\big|X_0(s,t)-X_0(s',t')\big|\\ &\leq (\alpha+\beta\theta)(\big|s-s'\big|+\big|t-t'\big|)\end{aligned}$$

$$\begin{aligned}\big|\Psi_m(s,t)-\Psi_k(s',t')\big| &\leq \alpha(\big|s-s'\big|+\big|t-t'\big|)+\beta\big|X_k(s,t)-X_k(s',t')\big|\\ &\leq \alpha(\big|s-s'\big|+\big|t-t'\big|)+\beta L_0(\big|s-s'\big|+\big|t-t'\big|)\\ &= L'(\big|s-s'\big|+\big|t-t'\big|).\end{aligned}$$

where $L'=\alpha+\beta L_0=\big(\alpha+\beta(\theta+\lambda(b-a)(d-c)M\zeta)\big)$ □

Corollary 3.1. *The functions $H(s_p,t_q,x,y)\psi(x,y,X_k(x,y)), p=\overline{0,n}\ q=\overline{0,m}, k\in N$ are uniformly Lipschitz with constant $L=\eta M+M_H\big(\alpha+\beta(\theta+\lambda(b-a)(d-c)M\zeta)\big)$*

Proof. Let arbitrary $(s,t),(s',t') \in [a,b]\times[c,d]$. We define the function $\Psi_{k,p,q}: [a,b]\times[c,d]\to\mathbb{R}$, $\Psi_{k,p,q}(x,y) = H(s_p,t_q,x,y)\psi(x,y,X_k(x,y))$, $\quad p=\overline{0,n}\ q=\overline{0,m}$. Then

$$
\begin{aligned}
&\left|\Psi_{k,s,t}(x,y)-\Psi_{k,s,t}(x',y')\right|\\
&=\left|H(s_p,t_q,x,y)\psi(x,y,X_k(x,y))-H(s_p,t_q,x',y')\psi(x',y',X_k(x',y'))\right|\\
&\le\left|H(s_p,t_q,x,y)\psi(x,y,X_k(x,y))-H(s_p,t_q,x',y')\psi(x,y,X_k(x,y))\right|\\
&+\left|H(s_p,t_q,x',y')\psi(x,y,X_k(x,y))-H(s_p,t_q,x',y')\psi(x',y',X_k(x',y'))\right|\\
&\le M\left|H(s_p,t_q,x,y)-H(s_p,t_q,x',y')\right|\\
&+M_H\left|\psi(x,y,X_k(x,y))-\psi(x',y',X_k(x',y'))\right|\\
&\le M\eta(\left|s-s'\right|+\left|t-t'\right|)+M_HL'(\left|s-s'\right|+\left|t-t'\right|)\\
&\le L(\left|s-s'\right|+\left|t-t'\right|).\quad k\in\mathbf{N}
\end{aligned}
\tag{12}
$$

where $L = M\eta + M_HL' = \eta M + M_H\big(\alpha+\beta(\theta+\lambda(b-a)(d-c)M\zeta)\big)$ □

Since any finite interval $[a,b]$ can be transformed to $[0,1]$ by linear maps, it is supposed that $[a,b]=[c,d]=[0,1]$ without any loss of generality.
Now, we present a sequence of successive approximations for numerical solution of (5) using two-dimensional Bernstein operators. To this end, first, we assume the uniform partition $D=(D_x,D_y)$ of the square $S=[0,1]\times[0,1]$ with

$$D_x: 0=s_0<s_1<s_2<\ldots<s_n=1, D_y: 0=t_0<t_1<t_2<\ldots<t_2=1, \tag{13}$$

and $s_i = ih_x$, $t_j = jh_y$, where $h_x=\frac{1}{n}, h_y=\frac{1}{m}$, $i=\overline{0,n}, j=\overline{0,m}$. Then the following iterative procedure, gives the approximate solution of Eq. (1) in point (s,t):

$$
\begin{aligned}
&\overline{X}_0(s,t)=r(s,t),\\
&\overline{X}_k(s,t)=r(s,t)\\
&+\lambda\int_0^1\int_0^1\sum_{i=0}^{n}\sum_{j=0}^{m}H(s,t,s_i,t_j)\psi(s_i,t_j,\overline{X}_{k-1}(s_i,t_j))P_{n,i}(x)P_{m,j}(y)dxdy,\ k\ge1.
\end{aligned}
\tag{14}
$$

Remark 3.1. Since $\int_0^1 t^{k_1}(1-t)^{k_2}dt = \frac{k_1!k_2!}{(k_1+k_2+1)!}, k_1,k_2\in\mathbb{N}$, we get $\int_0^1\int_0^1 P_{n,i}(x)P_{m,j}(y)dxdy = \frac{1}{(n+1)(m+1)}$. Also, (see [45]).

3.2. Algorithm of the approach

Consider the uniform partitions (13) with $s_p=\frac{p}{n}, p=\overline{0,n}$ and $t_q=\frac{q}{m}, q=\overline{0,m}$. On these knots the terms of the sequence of successive approximations are:

$$X_0(s_p,t_q)=r(s_p,t_q), \tag{15}$$

$$X_k(s_p,t_q)=r(s_p,t_q)+\lambda\int_c^d\int_a^b H(s_p,t_q,x,y)\psi(x,y,X_{k-1}(x,y))dxdy,\ k\ge1,$$

and applying two-dimensional Bernstein operators (3) we obtain following iterative algorithm:

- Step 1: Input the values $a, b, c, d, \lambda, n, m$ and the functions r, H, ψ.
- Step 2: Set $h_x = \frac{b-a}{n}$ and $h_y = \frac{d-c}{m}$.
- Step 3: Choose $\varepsilon' > 0$ and for $p = \overline{0,n}$, $q = \overline{0,m}$, set $\overline{X}_0(s_p, t_q) = r(s_p, t_q)$.
- Step 4: For all $p = \overline{0,n}$, $q = \overline{0,m}$, Compute

$$\overline{X}_1(s_p, t_q) = r(s_p, t_q) + \frac{\lambda}{(n+1)(m+1)} \sum_{i=0}^{n} \sum_{j=0}^{m} H(s_p, t_q, s_i, t_j)\psi(s_i, t_j, r(s_i, t_j))$$

- Step 5: For $k \geq 2$, and for all $p = \overline{0,n}$, $q = \overline{0,m}$, Compute

$$\overline{X}_k(s_p, t_q) = r(s_p, t_q) + \frac{\lambda}{(n+1)(m+1)} \sum_{i=0}^{n} \sum_{j=0}^{m} H(s_p, t_q, s_i, t_j)\psi(s_i, t_j, \overline{X}_{k-1}(s_i, t_j)).$$

- Step 6: We use the values computed at the previous step and obtain for $p = \overline{0,n}$, $q = \overline{0,m}$, the values:

$$\mid \overline{X}_k(s_p, t_q) - \overline{X}_{k-1}(s_p, t_q) \mid$$

- Step 7: If $\mid \overline{X}_k(s_p, t_q) - \overline{X}_{k-1}(s_p, t_q) \mid < \varepsilon'$, print k and print $\overline{X}_k(s_p, t_q)$, for all $p = \overline{0,n}$, $q = \overline{0,m}$, stop.; otherwise, set $k = k + 1$ and go to Step 5.

This algorithm has a practical criterion presented below in Remark (3.3).

3.3. The convergence analysis

The convergence property of the proposed method can be obtained from the following theorem.

Theorem 3.2. *Suppose that the conditions (i)-(iv) are satisfied with* $a = c = 0, b = d = 1$. *If* $\beta\lambda M_H < 1$, *then, the sequence* $\{\overline{X_k}(s_p, t_q)\}_{k \in N}$ *converges to the unique solution of Eq. (1), and the error estimate is:*

$$d(X^*, X_k) \leq \frac{(\lambda\beta M_H)^{k+1}}{\beta(1 - \lambda\beta M_H)} M_0 + \frac{L\lambda}{2}\left(\frac{1}{1 - \lambda\beta M_H}\right)\left(\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{m}}\right),$$

where M_0 *is given in (10).*

Proof. Choosing $X_0 \in \Omega, X_0 = r$ we have

$$\big\|X_0 - X_1\big\| = \sup_{\substack{0\leq s\leq 1\\ 0\leq t\leq 1}} \left| r(s,t) - r(s,t) + \lambda \int_0^1 \int_0^1 H(s,t,x,y)\psi(x,y,X_0(x,y))dxdy \right|$$
$$\leq \sup_{\substack{0\leq s\leq 1\\ 0\leq t\leq 1}} \lambda \int_0^1 \int_0^1 \big|H(s,t,x,y)\psi(x,y,X_0(x,y))\big| dxdy$$
$$\leq M_H \lambda \int_0^1 \int_0^1 \sup_{\substack{0\leq s\leq 1\\ 0\leq t\leq 1}} \big|\psi(x,y,r(x,y))\big| dxdy = \lambda M_H M_0, \tag{16}$$

so, by (6) and (16) we obtain

$$\big\|X^* - X_k\big\| \leq \frac{(\beta\lambda M_H)^{k+1}}{\beta(1-\beta\lambda M_H)} M_0, \tag{17}$$

Using (17) we have

$$d(X^*, \overline{X}_k) \leq d(X^*, X_k) + d(X_k, \overline{X}_k) \leq \frac{(\beta\lambda M_H)^{k+1}}{\beta(1-\beta\lambda M_H)} M_0 + d(X_k, \overline{X}_k) \tag{18}$$

therefore, we shall obtain the estimates for $d(X_k, \overline{X}_k)$. Form (14) and (5) forall $(s,t) \in [0,1] \times [0,1]$ we obtain

$$\big|X_k(s,t) - \overline{X}_k(s,t)\big| \leq \lambda \int_0^1 \int_0^1 \Big(|H(s,t,x,y)\psi(x,y,X_{k-1}(x,y))dxdy$$
$$- \sum_{i=0}^{n} \sum_{j=0}^{m} H(s,t,s_i,t_j)\psi(s_i,t_j,\overline{X}_{k-1}(s_i,t_j))P_{n,i}(x)P_{m,j}(y)dxdy| \Big)$$
$$\leq \lambda \int_0^1 \int_0^1 |H(s,t,x,y)\psi(x,y,X_{k-1}(x,y))dxdy$$
$$- \lambda \int_0^1 \int_0^1 \sum_{i=0}^{n} \sum_{j=0}^{m} H(s,t,s_i,t_j)\psi(s_i,t_j,X_{k-1}(s_i,t_j))P_{n,i}(x)P_{m,j}(y)|dxdy$$
$$+ \lambda \int_0^1 \int_0^1 \sum_{i=0}^{n} \sum_{j=0}^{m} |H(s,t,s_i,t_j)\psi(s_i,t_j,X_{k-1}(s_i,t_j))P_{n,i}(x)P_{m,j}(y)dxdy$$
$$- \lambda \int_0^1 \int_0^1 \sum_{i=0}^{n} \sum_{j=0}^{m} H(s,t,s_i,t_j)\psi(s_i,t_j,\overline{X}_{k-1}(s_i,t_j))P_{n,i}(x)P_{m,j}(y)|dxdy$$

therefore,

$$\begin{aligned}\big|X_k(s,t)-\overline{X}_k(s,t)\big| &\leq \lambda\int_0^1\int_0^1 \big|H(s,t,x,y)\psi(x,y,X_{k-1}(x,y))\\ &\quad -\sum_{i=0}^{n}\sum_{j=0}^{m}H(s,t,s_i,t_j)\psi(s_i,t_j,X_{k-1}(s_i,t_j))P_{n,i}(x)P_{m,j}(y)\big|dxdy\\ &\quad +\lambda\beta M_H\int_0^1\int_0^1\sum_{i=0}^{n}\sum_{j=0}^{m}\big|X_{k-1}(s_i,t_j)-\overline{X}_{k-1}(s_i,t_j)\big||P_{n,i}(x)||P_{m,j}(y)|dxdy\end{aligned}$$

According to Theorem 2.1, Corollary 3.1 and taking into account that $\int_0^1\int_0^1 P_n(x)P_n(y)dxdy=\frac{1}{(n+1)(m+1)}$, we have

$$\begin{aligned}\big|X_k(s,t)-\overline{X}_k(s,t)\big| &\leq \frac{L\lambda}{2}(\frac{1}{\sqrt{n}}+\frac{1}{\sqrt{m}})+\lambda\beta M_H\frac{1}{(n+1)(m+1)}\sum_{i=0}^{n}\sum_{j=0}^{m}\big\|X_{k-1}-\overline{X}_{k-1}\big\|\\ &\leq \frac{L\lambda}{2}(\frac{1}{\sqrt{n}}+\frac{1}{\sqrt{m}})+\lambda\beta M_H\big\|X_{k-1}-\overline{X}_{k-1}\big\|,\end{aligned}$$

taking supremum for $(s,t)\in[0,1]\times[0,1]$, we have

$$\begin{aligned}\big\|X_k-\overline{X}_k\big\| &\leq \frac{L\lambda}{2}(\frac{1}{\sqrt{n}}+\frac{1}{\sqrt{m}})+\lambda\beta M_H\big\|X_{k-1}-\overline{X}_{k-1}\big\|,\\ \big\|X_{k-1}-\overline{X}_{k-1}\big\| &\leq \frac{L\lambda}{2}(\frac{1}{\sqrt{n}}+\frac{1}{\sqrt{m}})+\lambda\beta M_H\big\|X_{k-2}-\overline{X}_{k-2}\big\|,\\ &\vdots\\ \big\|X_2-\overline{X}_2\big\| &\leq \frac{L\lambda}{2}(\frac{1}{\sqrt{n}}+\frac{1}{\sqrt{m}})+\lambda\beta M_H\big\|X_1-\overline{X}_1\big\|,\\ \big\|X_1-\overline{X}_1\big\| &\leq \frac{L\lambda}{2}(\frac{1}{\sqrt{n}}+\frac{1}{\sqrt{m}})+\lambda\beta M_H\big\|X_0-\overline{X}_0\big\|,\end{aligned}$$

multiplying the above inequalities by $1, B, B^2, ..., B^{k-1}$, respectively we obtain

$$\begin{aligned}\big\|X_k-\overline{X}_k\big\| &\leq (1+\lambda\beta M_H+(\lambda\beta M_H)^2+...+(\lambda\beta M_H)^{k-1})\frac{L\lambda}{2}(\frac{1}{\sqrt{n}}+\frac{1}{\sqrt{m}})\\ &\leq \frac{1-(\lambda\beta M_H)^{k-1})}{1-\lambda\beta M_H}\frac{L\lambda}{2}(\frac{1}{\sqrt{n}}+\frac{1}{\sqrt{m}})\end{aligned}$$

therefore

$$d(X_k,\overline{X}_k)\leq\frac{L\lambda}{2}(\frac{1}{1-\lambda\beta M_H})(\frac{1}{\sqrt{n}}+\frac{1}{\sqrt{m}}) \tag{19}$$

Hence, from (17) and (19) we conclude that

$$d(X^*,\overline{X}_k) \leq \frac{(\lambda\beta M_H)^{k+1}}{\beta(1-\lambda\beta M_H)}M_0 + \frac{L\lambda}{2}(\frac{1}{1-\lambda\beta M_H})(\frac{1}{\sqrt{n}}+\frac{1}{\sqrt{m}})$$

Since $\lambda\beta M_H < 1$, it is easy to see that $\lim_{\substack{k\to\infty \\ h_x,h_y\to 0}} d(X^*,\overline{X}_k) = 0$, which is the convergence of the proposed method.

3.4. The stability analysis

In order to investigate the numerical stability of the proposed method, we consider another first iteration term $X_0(s,t) = f(s,t) \in C([a,b]\times[c,d],\mathbb{R})$ such that $\exists\varepsilon > 0$ for which $\left|r(s,t)-f(s,t)\right| < \varepsilon, \forall (s,t) \in [a,b]\times[c,d]$. Applying the iterative method presented above to the Hammerstein integral equation:

$$X(s,t) = f(s,t) + \lambda\int_c^d\int_a^b H(s,t,x,y)\psi(x,y,X(x,y))dxdy,\ (s,t)\in I, \quad (20)$$

we obtained the sequence of successive approximations on the knots $s_p = a + p\frac{b-a}{n}, p = \overline{0,n}$ and $t_q = c + q\frac{d-c}{m}, q = \overline{0,m}$:

$$Y_0(s_p,t_p) = f(s_p,t_p), \quad (21)$$
$$Y_k(s_p,t_p) = f(s_p,t_p) + \lambda\int_0^1\int_0^1 H(s_p,t_p,x,y)\psi(x,y,Y_{k-1}(x,y))dxdy,\ k\geq 1$$

and appling the same iterative procedure (14), the computed values are:

$$\overline{Y}_0(s,t) = f(s,t),$$
$$\overline{Y}_k(s,t) = f(s,t)$$
$$+\lambda\int_0^1\int_0^1\sum_{i=0}^{n}\sum_{j=0}^{m} H(s,t,s_i,t_j)\psi(s_i,t_j,\overline{Y}_{k-1}(s_i,t_j))P_{n,i}(x)P_{m,j}(y)dxdy \quad (22)$$

Theorem 3.3. *Under conditions of Theorem 3.2, the iterative procedure (14) is numerically stable with respect to the choice of the first iteration.*

Proof. To obtain the numerical stability, for $\overline{X}_k$ and $\overline{Y}_k$ and for $k \geq 1$, $\forall (s,t) \in [0,1] \times [0,1]$, we have

$$
\begin{aligned}
| \overline{X}_k(s,t) - \overline{Y}_k(s,t) | &\leq |r(s,t) - f(s,t)| \\
&+ |\lambda \int_0^1 \int_0^1 \sum_{i=0}^{n} \sum_{j=0}^{m} |H(s,t,s_i,t_j)\psi(s_i,t_j,\overline{X}_{k-1}(s_i,t_j))P_{n,i}(x)P_{m,j}(y)dxdy \\
&- \lambda \int_0^1 \int_0^1 \sum_{i=0}^{n} \sum_{j=0}^{m} H(s,t,s_i,t_j)\psi(s_i,t_j,\overline{Y}_{k-1}(s_i,t_j))P_{n,i}(x)P_{m,j}(y)dxdy| \\
&\leq \varepsilon + \lambda\beta M_H \int_0^1 \int_0^1 \sum_{i=0}^{n} \sum_{j=0}^{m} \left|\overline{X}_{k-1}(s_i,t_j) - \overline{Y}_{k-1}(s_i,t_j)\right| |P_{n,i}(x)||P_{m,j}(y)|dxdy \\
&\leq \varepsilon + \lambda\beta M_H \frac{1}{(n+1)(m+1)} \sum_{i=0}^{n} \sum_{j=0}^{m} \left|\overline{X}_{k-1}(s_i,t_j) - \overline{Y}_{k-1}(s_i,t_j)\right|
\end{aligned}
$$

Taking supremum for $(s,t) \in I$ from the above inequality, we observe that

$$\|\overline{X}_k(s,t) - \overline{Y}_k(s,t)\| \leq \varepsilon + \lambda\beta M_H \left\|\overline{X}_{k-1}(s_i,t_j) - \overline{Y}_{k-1}(s_i,t_j)\right\|$$

Now, by successive substitutions in the above obtained inequality, and according to the condition $\beta\lambda M_H < 1$, we obtain

$$
\begin{aligned}
\|\overline{X}_k(s,t) - \overline{Y}_k(s,t)\| &\leq \varepsilon + \lambda\beta M_H\varepsilon + (\lambda\beta M_H)^2\varepsilon + ... + (\lambda\beta M_H)^{k-1}\varepsilon \\
&\leq \frac{1}{1-\beta\lambda M_H}\varepsilon
\end{aligned}
$$

□

Remark 3.2. Since $\lambda\beta M_H < 1$, we conclude that the stability of the numerical method is proved. Indeed, we have

$$\lim_{\substack{k\to\infty \\ \varepsilon\to 0}} \left\|\overline{X}_k - \overline{Y}_k\right\| = 0.$$

Remark 3.3. The "a-posteriori error" estimate is useful to get the stopping criterion. Such estimate can be obtained as follows:
For given $\varepsilon' > 0$ (previously chosen), there is determined the first natural number k for which $| \overline{X}_k(s_p,t_q) - \overline{X}_{k-1}(s_p,t_q) | < \varepsilon'$ and we stop to this k retaining the approximations $X_k(s,t)$ of solution. We observe

$$
\begin{aligned}
\|X^* - \overline{X}_k\| &\leq \|X^* - X_k\| + \|X_k - \overline{X}_k\| \\
&\leq \frac{\beta\lambda M_H}{1-\beta\lambda M_H}\|X_{k-1} - X_k\| + \frac{L\lambda}{2}\left(\frac{1}{1-\lambda\beta M_H}\right)\left(\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{m}}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\|X_k - X_{k-1}\| &\leq \|X_k - \overline{X}_k\| + \|\overline{X}_k - \overline{X}_{k-1}\| + \|\overline{X}_{k-1} - X_{k-1}\| \\
&\leq L\lambda\left(\frac{1}{1-\lambda\beta M_H}\right)\left(\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{m}}\right) + \|\overline{X}_k - \overline{X}_{k-1}\|
\end{aligned}
$$

So,

$$\|X^{*}-\overline{X}_{k}\| \leq \frac{\beta\lambda M_{H}}{1-\beta\lambda M_{H}}\|\overline{X}_{k}-\overline{X}_{k-1}\|, + \frac{3\beta\lambda M_{H}}{2(1-\beta\lambda M_{H})}L\lambda(\frac{1}{1-\lambda\beta M_{H}})(\frac{1}{\sqrt{n}}+\frac{1}{\sqrt{m}})$$

and therefore, in order to obtain $\mid X^{*}(s,t)-\overline{X}_{k}(s,t)\mid<\varepsilon$, we require

$$\frac{3\beta\lambda M_{H}}{2(1-\beta\lambda M_{H})}L\lambda(\frac{1}{1-\lambda\beta M_{H}})(\frac{1}{\sqrt{n}}+\frac{1}{\sqrt{m}})<\frac{\varepsilon}{2} \tag{23}$$

and

$$\frac{\beta\lambda M_{H}}{1-\beta\lambda M_{H}}\|\overline{X}_{k}-\overline{X}_{k-1}\|<\frac{\varepsilon}{2}.$$

We can choose the least natural numbers n, m, for which inequality (23) holds. Finally, we find the smallest natural number $k\in\mathbb{N}$ (this is the last iterative step to be made) for which, $\|\overline{X}_{k}-\overline{X}_{k-1}\|<\frac{\varepsilon}{2}.\frac{1-\beta\lambda M_{H}}{\beta\lambda M_{H}}=\varepsilon'$. With these, the inequality $\mid\overline{X}_{k}(s_{p},t_{q})-\overline{X}_{k-1}(s_{p},t_{q})\mid<\varepsilon'$ leads to $\mid X^{*}(s_{p},t_{q})-\overline{X}_{k}(s_{p},t_{q})\mid<\varepsilon$, and the desired accuracy ε is obtained.

4. Numerical experiments

The proposed iterative method of successive approximations in Section 3 was tested on three numerical examples to providing the accuracy of the method and illustrating the correctness of the theoretical results. In these examples, we assumed that $[a,b]\times[c,d]=[0,1]\times[0,1]$, $\lambda=1$. The absolute values of the errors at the selected grid points which are proposed as $(s_{r},t_{r})=(\frac{r}{10},\frac{r}{10}), r=\overline{0,10}$, are reported. In order to analyze the error of the method, we introduce the notations: $\|E_{n}\|_{\infty}:=\max\{E_{p,q}|p,q=0,1,...,n\}$ where $E_{p,q}:=|X^{*}(s_{p},t_{q})-\overline{X}_{k}(s_{p},t_{q})|$, X^{*} is the exact solution and $\overline{X}_{k}$ is the approximate solution obtained by the proposed method. The absolute error in the solution are compared with the similar method in [47].

Example 4.1. [46] *Consider two-dimensional nonlinear Fredholm integral equation*

$$X(s,t)=s^{2}+r^{2}+1+\frac{0.09565}{(s+1)(t+3)}+\int_{0}^{1}\int_{0}^{1}\frac{xy}{(s+1)(t+3)}\cos(X(x,y))dxdy$$

The exact solution is given by $X(s,t)=s^{2}+r^{2}+1$. Applying the algorithm for $n=m=10, \varepsilon'=10^{-15}$, we obtain the number of iterations $k=10$ iterations. For more details, please see Table 1. In order to test the numerical stability regarding the choice of the first iteration, we take $\varepsilon=0.1$ $(r(s,t):=r(s,t)+0.1)$, and the differences between the effective computed values $D_{p,q}=|\overline{X}_{10}(s_{p},t_{q})-\overline{Y}_{10}(s_{p},t_{q})|$,$p,q=\overline{0,10}$, are in Table 1, that confirm the numerical stability of the algorithm.

In order to more detailed testing of convergence, we consider $n=m=100$ and *for* $\varepsilon^{'} = 10^{-25}$ *the number of iterations is* $k = 12$. *It is seen that* $E_{p,q}, p, q = \overline{0, n}$, *tend to zero as* h_x, h_y *decrease. The numerical results are shown in Table 2. For* $n = m = 1000, \varepsilon^{'} = 10^{-25}$, *we have* $k = 14$ *iterations and the results are in Table 3. The results* $\|E_n\|_\infty$ *for* $\varepsilon^{'} = 10^{-15}$ *and* $n = m \in \{10, 100, 1000\}$, *respectively, are* $3.428 \times 10^{-5}, 8.965 \times 10^{-7}$ *and* 2.602×10^{-9}. *The results in Table 1-3 confirm the convergence of the numerical method, that is* $E_{p,q} \to 0$ *as* $h_x, h_y \to 0$.

Table 1. Numerical results for $n = m = 10$***, in Example 4.1.***

(s_p, t_q)	$X^*(s_p, t_q)$	$\overline{X}_{10}(s_p, t_q)$	$E_{p,q}$	$D_{p,q}$
(0.0,0.0)	1.00	1.00003428589077	3.42859 $\times 10^{-5}$	0.106
(0.1,0.1)	1.02	1.02003016354027	3.01635 $\times 10^{-5}$	0.104
(0.2,0.2)	1.08	1.08002678585216	2.67858 $\times 10^{-5}$	0.102
(0.3,0.3)	1.18	1.18002397614739	2.39761 $\times 10^{-5}$	0.101
(0.4,0.4)	1.32	1.32002160875469	2.16087 $\times 10^{-5}$	0.100
(0.5,0.5)	1.50	1.50001959193758	1.95919 $\times 10^{-5}$	0.100
(0.6,0.6)	1.72	1.72001785723478	1.78572 $\times 10^{-5}$	0.100
(0.7,0.7)	1.98	1.98001635257111	1.63525 $\times 10^{-5}$	0.100
(0.8,0.8)	2.28	2.28001503767139	1.50376 $\times 10^{-5}$	0.100
(0.9,0.9)	2.62	2.62001388092744	1.38809 $\times 10^{-5}$	0.100
(1.0,1.0)	3.00	3.00001285720904	1.28572 $\times 10^{-5}$	0.100

Table 2. Numerical results for $n = m = 100$***, in Example 4.1.***

(s_p, t_q)	$X^*(s_p, t_q)$	$\overline{X}_{12}(s_p, t_q)$	$E_{p,q}$
(0.0,0.0)	1.00	1.00000089648041	8.96480 $\times 10^{-7}$
(0.1,0.1)	1.02	1.02000078869244	7.88692 $\times 10^{-7}$
(0.2,0.2)	1.08	1.08000070037532	7.00375 $\times 10^{-7}$
(0.3,0.3)	1.18	1.18000062690938	6.26909 $\times 10^{-7}$
(0.4,0.4)	1.32	1.32000056500866	5.65009 $\times 10^{-7}$
(0.5,0.5)	1.50	1.50000051227452	5.12275 $\times 10^{-7}$
(0.6,0.6)	1.72	1.72000046691688	4.66917 $\times 10^{-7}$
(0.7,0.7)	1.98	1.98000042757412	4.27574 $\times 10^{-7}$
(0.8,0.8)	2.28	2.28000039319316	3.93193 $\times 10^{-7}$
(0.9,0.9)	2.62	2.62000036294753	3.62947 $\times 10^{-7}$
(1.0,1.0)	3.00	3.00000033618015	3.36180 $\times 10^{-7}$

Table 3. Numerical results for $n = m = 1000$***, in Example 4.1.***

(s_p,t_q)	$X^*(s_p,t_q)$	$\overline{X}_{14}(s_p,t_q)$	$E_{p,q}$
(0.0,0.0)	*1.00*	*1.00000000260202*	*2.60202* $\times 10^{-9}$
(0.1,0.1)	*1.02*	*1.02000000228917*	*2.28917* $\times 10^{-9}$
(0.2,0.2)	*1.08*	*1.08000000203283*	*2.03283* $\times 10^{-9}$
(0.3,0.3)	*1.18*	*1.18000000181959*	*1.81959* $\times 10^{-9}$
(0.4,0.4)	*1.32*	*1.32000000163993*	*1.63993* $\times 10^{-9}$
(0.5,0.5)	*1.50*	*1.50000000148687*	*1.48687* $\times 10^{-9}$
(0.6,0.6)	*1.72*	*1.72000000135522*	*1.35522* $\times 10^{-9}$
(0.7,0.7)	*1.98*	*1.98000000124103*	*1.24103* $\times 10^{-9}$
(0.8,0.8)	*2.28*	*2.28000000114124*	*1.14124* $\times 10^{-9}$
(0.9,0.9)	*2.62*	*2.62000000105345*	*1.05345* $\times 10^{-9}$
(1.0,1.0)	*3.00*	*3.00000000097576*	*9.75760* $\times 10^{-10}$

Example 4.2. *Consider two-dimensional nonlinear Fredholm integral equation*

$$X(s,t) = r(s,t) + \int_0^1 \int_0^1 s^2 tyx(x^2 + y^2 + X^3(x,y))dxdy, \tag{24}$$

where

$$r(s,t) = t\sin(s) + \frac{1}{15}\big((\sin(1)^2(\cos(1) - \frac{1}{3}\sin(1)^3 + 2cos1 - 2sin1 - \frac{15}{4}\big)s^2 t,$$

and exact solution $X(s,t) = t\sin(s)$*. Here, we apply proposed iterative method for* $n = m = 10$*,* $\varepsilon' = 10^{-20}$ *and we get* $k = 14$ *(iterations to be made). In order to test the convergence we consider* $n = 10, n = 100$ *and* $n = 1000$*, and we obtain the accuracy* $O(10^{-5} - 10^{-8})$*,* $O(10^{-7} - 10^{-10})$*, and* $O(10^{-9} - 10^{-12})$ *respectively. Table 4 illustrates the numerical results for this example.*

Table 4. Numerical results for $n = 10, n = 100, n = 1000$***, in Ex 4.2.***

(s_p,t_q)	$e_{p,q}, n = 10$	$e_{p,q}, n = 100$	$e_{p,q}, n = 1000$
(0.0,0.0)	0	0	0
(0.1,0.1)	*3.905753519* $\times 10^{-8}$	*5.0330356* $\times 10^{-10}$	*4.07376*$\times 10^{-12}$
(0.2,0.2)	*3.124602815* $\times 10^{-7}$	*4.0264285* $\times 10^{-9}$	*3.25901*$\times 10^{-11}$
(0.3,0.3)	*1.054553450* $\times 10^{-6}$	*1.3589196* $\times 10^{-8}$	*1.09991*$\times 10^{-10}$
(0.4,0.4)	*2.499682252* $\times 10^{-6}$	*3.2211428* $\times 10^{-8}$	*2.60721*$\times 10^{-10}$
(0.5,0.5)	*4.882191899* $\times 10^{-6}$	*6.2912945* $\times 10^{-8}$	*5.09220*$\times 10^{-10}$
(0.6,0.6)	*8.436427601* $\times 10^{-6}$	*1.0871357* $\times 10^{-7}$	*8.79932*$\times 10^{-10}$
(0.7,0.7)	*1.339673457* $\times 10^{-5}$	*1.7263312* $\times 10^{-7}$	*1.39730*$\times 10^{-9}$
(0.8,0.8)	*1.999745802* $\times 10^{-5}$	*2.5769142* $\times 10^{-7}$	*2.08576*$\times 10^{-9}$
(0.9,0.9)	*2.847294315* $\times 10^{-5}$	*3.6690829* $\times 10^{-7}$	*2.96977*$\times 10^{-9}$
(1.0,1.0)	*3.905753519* $\times 10^{-5}$	*5.0330356* $\times 10^{-7}$	*4.07376*$\times 10^{-9}$
k	14	16	16
$\|E_n\|_\infty$	3.906×10^{-5}	5.033×10^{-7}	4.074×10^{-9}

Example 4.3. [47] *Consider two-dimensional nonlinear Fredholm integral equation*

$$X(s,t) = r(s,t) + \int_0^1 \int_0^1 H(s,t,x,y) X^2(x,y) dx dy, \ (s,t) \in [0,1]\times[0,1], \quad (25)$$

with
$r(s,t) = \frac{1}{3}s + t - 1 - \frac{1}{90}(s+1)(t^2+s-1)^2, \quad H(s,t,x,y) = (s+1)(t^2+s-1)^2 y x^2,$ *and exact solution* $X(s,t) = \frac{1}{3}s + t - 1$. *For this example, we apply proposed iterative method for* $n = m = 10, n = m = 50, n = m = 100$, $\varepsilon' = 10^{-20}$. *The Table 4 shows that our method in comparison with the method in* [47] *are more accurate.*

Table 4. Numerical results for $n = 10, n = 50, n = 100$***, in Ex 4.3.***

	Method of [47]			*Presented Method*		
(s , t)	*n=10*	*n=50*	*n=100*	*n=10*	*n=50*	*n=100*
(0.1,0.1)	*4.204e-5*	*1.637e-6*	*4.089e-7*	*3.0964e-6*	*4.3562e-7*	*2.0645e-8*
(0.3,0.3)	*2.334e-5*	*9.088e-7*	*2.270e-7*	*1.1437e-7*	*1.6091e-8*	*7.6210e-9*
(0.5,0.5)	*4.523e-6*	*1.761e-7*	*4.399e-8*	*2.9349e-6*	*1.2569e-7*	*1.9570e-8*
(0.7,0.7)	*2.961e-6*	*1.153e-7*	*2.880e-8*	*5.9716e-6*	*1.0409e-7*	*1.9814e-8*
(0.9,0.9)	*4.621e-5*	*1.799e-6*	*4.495e-7*	*2.0416e-6*	*8.8722e-7*	*1.3612e-7*

5. Conclusions

In this paper, we used the two-dimensional Bernstein operators for the numerical solution of two-dimensional nonlinear Hammerstein-Fredholm integral equations. We presented an efficient iterative algorithm based on the method of successive approximations. It is observed that the given method is simple and gives excellent approximate solution. In Theorem 3.1, we obtain the existence and uniqueness of the solution and prove some the uniformly boundedness and uniformly Lipschitz properties for the terms of the sequence of successive approximations. The convergence and the error estimation of this presented successive approximations method is proved in Theorem 3.2. Also, the numerical stability regarding the choice of the first iteration is shown in Theorem 3.3. The convergence and the numerical stability of the proposed technique are approved by taking some numerical problems. The convergence is tested for stepsize $h_x = h_y = 0.1, h_x = h_y = 0.01$, and $h_x = h_y = 0.001$, and the order of effective error is $O(10^{-5} - 10^{-8})$, $O(10^{-7} - 10^{-10})$, and $O(10^{-9} - 10^{-12})$, respectively. The absolute errors in the solutions by our method are accurate in comparison with [47].

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