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CONSTITUTIVE MATERIAL LAWS IN THE MULTIFRACTAL THEORY OF MOTION (PART I)

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Using the Fractal Theory of Motion in the form of Madelung Scenario, the presence of a permanent interaction between structural units of any complex system and a multifractal medium is highlighted. In such a context, the characterization of the multifractal medium through a multifractal tensor allows the obtainment of material constitutive laws. Moreover, particular types of material constitutive laws are presented: deformations exist even when no tensions are applied to the material, deformations which can be interpreted as intrinsic or pure material properties (in particular, Bell's constitutive laws). Furthermore, it is shown that not only radiation cosmic background, but the electromagnetic field in general, in its Maxwellian form, is in truth the expression of the existence of the multifractal medium.

Keywords: multifractal Theory of Motion, Schrödinger scenario, Madelung scenario, multifractal tensor, multifractal constitutive material laws

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1. Introduction

Recent papers referring to the description of complex system dynamics through the multifractal Theory of Motion (either in the fractal dimension $D_F = 2$ as in the Scale Relativity Theory [1] or in a constant, but arbitrary fractal dimension [2-7]) specify the fact that the description of dynamics, regardless of the scale resolution, are self-similar and are induced by the property of the motion curves (fractal/multifractal curves). As such, the holographic mode of description for complex system dynamics is implemented, both in the form of the Schrödinger multifractal scenario and in the Madelung multifractal scenario. The two description scenarios are not mutually exclusive, rather they are complementary: in any complex system dynamics, local non-linear behaviors (of digital type) and global non-linear behaviors are reciprocally conditioned, regardless of the scale resolution.

In the present paper, the identification of the previously mentioned conjecture is proven to be reducible to material constitutive laws.

2. A short reminder on the multifractal hydrodynamic model

Let it be considered the multifractal Schrödinger equation [3]:

$$2\lambda^2 (dt)^{\left[\frac{4}{f(\alpha)}\right]-2} \partial^l \partial_l \Psi + i\lambda (dt)^{\left[\frac{2}{f(\alpha)}\right]-1} \partial_t \Psi = 0 \tag{1}$$

where

$$\partial^l \partial_l = \frac{\partial^2}{\partial {x_l}^2}, \partial_t = \frac{\partial}{\partial t}, l = 1,2,3$$

In the previous relation, x_l are the multifractal space coordinates, t is a non-multifractal time coordinate, Ψ is the state function, λ is a constant associated to the multifractal-non-multifractal scale transition, dt is the scale resolution, $f(\alpha)$ is the singularity spectrum of order α and $\alpha = \alpha(D_F)$ is the fractal dimension of the motion curves. For other details referring to the meanings of the previously mentioned variables and parameters, please see [2-3].

In such a context, if Ψ is chosen in the form (the Madelung substitution):

$$\Psi = \sqrt{\rho} e^{is}, \tag{2}$$

where $\sqrt{\rho}$ is the amplitude and s is the phase, then the complex velocity field [3]

$$\hat{V}^i = -2i\lambda (dt)^{\left[\frac{2}{f(\alpha)}\right]-1} \partial^i \ln \Psi \tag{3}$$

take the explicit form:

$$\hat{V}^i = 2\lambda(dt)^{\left[\frac{2}{f(\alpha)}\right]-1}\partial^i s - i\lambda(dt)^{\left[\frac{2}{f(\alpha)}\right]-1}\partial^i \ln\rho \tag{4}$$

which implies the real velocity fields:

$$V_D^i = 2\lambda(dt)^{\left[\frac{2}{f(\alpha)}\right]-1}\partial^i s \tag{5}$$

$$V_F^i = \lambda(dt)^{\left[\frac{2}{f(\alpha)}\right]-1}\partial^i \ln\rho. \tag{6}$$

In (5), V_D^i is the differentiable velocity field, while V_F^i is the non-differentiable velocity field.

Now, by (2), 5) and (6) and using the mathematical procedures from [2-3], the equation (1) is reduced to the multifractal hydrodynamic equations:

$$\partial_t V_D^i + V_D^l \partial_l V_D^i = -\partial^i Q \tag{7}$$

$$\partial_t \rho + \partial_l\left(\rho V_D^l\right) = 0 \tag{8}$$

with Q the specific multifractal potential:

$$Q = -2\lambda^2(dt)^{\left[\frac{4}{f(\alpha)}\right]-2}\frac{\partial^l\partial_l\sqrt{\rho}}{\sqrt{\rho}} = -V_F^i V_F^i - \frac{1}{2}\lambda(dt)^{\left[\frac{2}{f(\alpha)}\right]-1}\partial_l V_F^l. \tag{9}$$

Equation (7) corresponds to the specific multifractal momentum conservation law, while equation (8) corresponds to the multifractal states density conservation law. The specific multifractal potential (9) implies the specific multifractal force:

$$F^i = -\partial^i Q = -2\lambda^2(dt)^{\left[\frac{4}{f(\alpha)}\right]-2}\partial^i\frac{\partial^l\partial_l\sqrt{\rho}}{\sqrt{\rho}} \tag{10}$$

which is a measure of the multifractality of the motion curves.

Therefore, for the complex velocity fields (4), the dynamics of any complex system are described through hydrodynamic equations at various scale resolutions.

The following consequences result:

i) Any complex system's structural units are in a permanent interaction with a multifractal medium through the specific multifractal force (10);
ii) Any complex system can be identified with a multifractal fluid, the dynamics of which is described by the multifractal hydrodynamic model (see Eqs. (7) – (9));

iii) The velocity field V_F^i does not represent the contemporary dynamics, but contributes to the transfer of the specific multifractal momentum and to the multifractal energy focus. This can be clearly seen from the absence of V_F^i from the multifractal-type states density conservation law and also from its role in the multifractal variational principles (for details see [8]);

iv) If a multifractal tensor is chosen:

$$\hat{\tau}^{il} = 2\lambda^2 (dt)^{\left[\frac{4}{f(\alpha)}\right]-2} \rho \partial^i \partial^l \ln \rho \tag{11}$$

the equation defining the multifractal "forces" that derive from a multifractal "potential" Q can be written in the form of a multifractal equilibrium equation. This equation can be written in a tensorial form:

$$\rho \partial^i Q = \partial_l \hat{\tau}^{il}. \tag{12}$$

The multifractal tensor $\hat{\tau}^{il}$ can be written in the form:

$$\hat{\tau}^{il} = \eta\left(\partial_l V_F^i + \partial_i V_F^l\right) \tag{13}$$

with

$$\eta = \lambda (dt)^{\left[\frac{2}{f(\alpha)}\right]-1} \rho. \tag{14}$$

This is a multifractal constitutive law for a multifractal "viscous fluid". Moreover, an original interpretation of the η coefficient as a multifractal dynamic viscosity of a multifractal fluid is given [9-11].

3. Multifractal tensions and deformations. Multifractal constitutive material laws

Because the multifractal tensor (11) plays a fundamental role in the definition of a material constitutive equation, in what follows, let it be presented some of its properties and their implications.

In continuum mechanics, one works with tensors of the second order or, more generally, with matrices, to adequately represent stresses and strains, and these are representations essentially non-polar, especially if they are not specified in terms of displacement fields and forces, as is usually done in engineering problems. More than that, when it comes to their reality, this is guaranteed by the so-called constitutive law [12-13].

In general terms, the constitutive law is a relationship between stresses and strains. Since regular representations for these concepts are by matrices, a constitutive law is a mathematical relationship - algebraic or analytical - between two 3×3 matrices. If it is noted with $\boldsymbol{\sigma}$ the stress matrix and $\boldsymbol{\varepsilon}$ the deformation matrix, then a constitutive law is a relation of the form $\boldsymbol{\sigma} = \sum \boldsymbol{\varepsilon}$ where the matrix function is accessible to experimental evaluation or, in any case, evaluation of a different nature than that through pure algebraic calculation.

Here, is of importance the meaning according to which $\boldsymbol{\sigma}$ is the applied stress while $\boldsymbol{\varepsilon}$ is the strain resultant. The reality referred to above then refers to the identity of the material characterized by the constitutive law. In materials science it is claimed that stress and strain matrices are universal mathematical tools, while the function $\sum$ is specific to the material to which the stresses $\boldsymbol{\sigma}$ are applied to induce deformations $\boldsymbol{\varepsilon}$. It can be seen in the concept of tension, extended beyond applied force, a means of eliminating the force in general from the conceptual arsenal of mechanics. Really, only the tension applied from the outside is closely related to the idea of force. Otherwise, moving away from the idea of force, tensions can also be thought of as energy densities that characterize matter, occasionally even independent of any force. The central problem here is to find a function $\sum$ which implicitly contains the physical nature of the continuum to which the stresses and strains refer [14-19].

Now, a specific feature appears, here the problem revolves around uncontrollable manifestations. This is the main reason for maintaining the way of thinking from classical mechanics that considers force as a vector. It is true that any (human) action is executed by force. In other words, it is possible to only control forces and, if it can be anything else, only through forces.

The most general idea of uncontrollability comes in very handy by means of a constitutive law that could be called natural. Indeed, a constitutive law that relates the stresses to the strains must be of the form

$$\boldsymbol{\sigma} = p_0\boldsymbol{e} + p_1\boldsymbol{\varepsilon} + p_2\boldsymbol{\varepsilon}^2 \tag{15}$$

where $\boldsymbol{e}$ is the 3×3 unit matrix. This equation can be called a natural constitutive law, because it naturally derives from the representations for stresses and strains. Indeed, the models for stresses and strains are 3×3 matrices, and if the constitutive law is analytic, equation (15) must automatically be true, because then the relationship between two matrices can be represented in the form of a whole series in the deformation matrix, always reducible to one polynomial of the second degree by the Hamilton-Cayley theorem. For the same reason, the relationship can be written with the interchanged locations of stresses and strains, so that deformations are still quadratic functions in stresses, only with other coefficients. Therefore, for a material there are not only three characteristic numbers, but six: three for the expression of stresses in relation to the deformations and three for the expression of the deformations in relation to the stresses.

In this scheme, the material therefore has, at least apparently, a precise identity, for that it is possible to identify it by the coefficients p_0, p_1, p_2 which are accessible to the experiment [20-21]. This one is, in fact, what is usually meant by "characterizing the material". Too often, however, in experimental practice, these coefficients are considered properties of material pure, of the order of density, but this restriction can lead to confusion in concepts, especially in engineering problems. Let such a situation be better explained. Regardless of what these material properties are, equation (15) shows that each of them can be extracted from experiments of loading a piece of material, either in extension, or in compression. Furthermore, regardless of the nature of this loading, the main directions of the stresses coincide with the main directions of the deformations. On the other hand, if $\sigma_{1,2,3}$ are the main values of the stress matrix and $\varepsilon_{1,2,3}$ those of the deformation matrix, then the constitutive law (15) is equivalent to the system

$$\begin{aligned}\sigma_1 &= p_0 + p_1\varepsilon_1 + p_2\varepsilon_1^2,\\ \sigma_2 &= p_0 + p_1\varepsilon_2 + p_2\varepsilon_2^2,\\ \sigma_3 &= p_0 + p_1\varepsilon_3 + p_2\varepsilon_3^2\end{aligned} \qquad (16)$$

Suppose that it is possible to perform experiments that allow the simultaneous measurement of all three main values of deformations and stresses. Such an experiment cannot be practically carried out, but the theoretical argument implies it always. The result of these experiments will allow the calculation of the properties of material embodied in the coefficients $p_{0,1,2}$ from the system (16). As the material is unique, a unique solution of the system must be sought, which is obtained only if its determinant

$$\begin{vmatrix}1 & \varepsilon_1 & \varepsilon_1^2\\ 1 & \varepsilon_2 & \varepsilon_2^2\\ 1 & \varepsilon_3 & \varepsilon_3^2\end{vmatrix} = (\varepsilon_2-\varepsilon_3)(\varepsilon_3-\varepsilon_1)(\varepsilon_1-\varepsilon_2) \qquad (17)$$

is non-zero. Thus, the parameters p_0, p_1, p_2 are indeed uniquely determined, regardless of the nature of the stresses imposed on the material if, and only if, the main deformations induced are different from each other. Regardless of the fact that they are unique, and therefore very suitable for the characterization of material, the coefficients thus obtained are not pure material properties, because they depend on the stress state impressed on the material. Therefore, the meaning of pure material properties must be further specified.

As such, the formalism shows that deformations exist even when there are no stresses applied to the material. Because their origins are unknown, these deformations can justifiably be taken as intrinsic properties, i.e., pure, of the material, considering that they could be generated by forces whose presence cannot be currently acquiesced [22-25]. These can be then described by the system of equations:

$$\begin{aligned} 0 &= p_0 + p_1\varepsilon_1 + p_2\varepsilon_1^2, \\ 0 &= p_0 + p_1\varepsilon_2 + p_2\varepsilon_2^2, \\ 0 &= p_0 + p_1\varepsilon_3 + p_2\varepsilon_3^2. \end{aligned} \tag{18}$$

Consequently, the intrinsic characterization of the material by experiment is now delegated to finding solutions of this linear and homogeneous system, if they exist. In fact, they always exist, only remaining to decide how many, and this fact depends on what can be measured in reality. If three different deformations in three orthogonal directions from space are always measured, then the material does not respond to the printed stresses. However, because the simultaneous measurement of three main values for a matrix is only conceptually possible, that quality of the material must be equally a conceptual one. In reality, it is only possible to simultaneously measure at most two eigenvalues, a fact which, when taken into consideration, reveals that the material could still respond to stresses, in other words, its deformation is really accompanied by tensions. Thus, if one and the same value of the deformation in any direction in space is measured, there exists a double infinity of stress states of the material, depending on two material parameters. If two values of the deformation are measured, in one direction and in its perpendicular plane for example, then material stress states depending on a single material parameter exist. Assuming it is possible to include one of the material parameters in a measured value, the most general constitutive law satisfied by the material presenting tensions accompanying the deformations will be of the form

$$\boldsymbol{\sigma} = K(\boldsymbol{\varepsilon} - \varepsilon_1\boldsymbol{e})(\boldsymbol{\varepsilon} - \varepsilon_2\boldsymbol{e}) \tag{19}$$

where K is an arbitrary constant. Such a material has three uncontrollable quantities, two of which are measurable.

In conclusion, it is noted that that as long as measurable quantities are of interest, a convenient way to express the deformation matrix characteristic of the material which presents uncontrollable deformations, is in the form of the tensor

$$\varepsilon_{ij} = \varepsilon_2\delta_{ij} + (\varepsilon_1 - \varepsilon_2)\cdot \mathbf{n}_i bf n_j \; i,j = 1,2,3 \tag{20}$$

where $\hat{n}$ is a unit eigenvector corresponding to the principal value ε_1. Such a material has distinctive directional properties in relation to the $\hat{n}$ direction, and these properties are given by the eigenvalues ε_1 and ε_2. In fact, equation (20) includes all the cases in above view of the material, if it is agreed to characterize its intrinsic properties as deformations. Note that this convention is independent of the constitutive description and must be guaranteed by available measurement capability. As such, whenever the material deforms freely, i.e., under no perceptible force, its deformation matrix must be of the form (20) with all special cases included. The deformations, as well as the accompanying stresses, will then manifestly be orthogonal tensors [26-28].

In the same way, it is possible to discuss that category of materials capable of sustaining tensions without responding with deformations. To express it, the opposite law must be considered, namely

$$\boldsymbol{\varepsilon} = q_0\boldsymbol{e} + q_1\boldsymbol{\sigma} + q_2\boldsymbol{\sigma}^2 \tag{21}$$

This time, $\boldsymbol{\sigma}$ could only hardly be called tension; rather, it represents an internal energy density of matter. Then, the defining state of this multifractal medium will be characterized by the system of equations

$$\begin{aligned} 0 &= q_0 + q_1\sigma_1 + q_2\sigma_1^2 \\ 0 &= q_0 + q_1\sigma_2 + q_2\sigma_2^2 \\ 0 &= q_0 + q_1\sigma_3 + q_2\sigma_3^2 \end{aligned} \tag{22}$$

which corresponds to the situation in which no deformation of the multifractal medium is observed. Again, the characterization of this multifractal medium depends on the number of non-trivial solutions of the system, and the most general form of the deformation matrix is here

$$\boldsymbol{\varepsilon} = K_1^{-1}(\boldsymbol{\sigma} - \sigma_1\boldsymbol{e})(\boldsymbol{\sigma} - \sigma_2\boldsymbol{e}) \tag{23}$$

where K_1 is a constant. The relation (23) with $\sigma_1 + \sigma_2 = 0$, in the absence of the multifractality, was found by Bell [29] as a characteristic of metals.

Again, if the interest is in measurable quantities for characterization of this material, its intrinsic stresses assume the following convenient tensor representation, analogous to equation (20):

$$\sigma_{ij} = \sigma_2\delta_{ij} + (\sigma_1 - \sigma_2)\cdot m_i m_j,\ i.j = 1,2,3 \tag{24}$$

where $\hat{m}$ is a unit vector corresponding to the main value σ_1. It can be stated that the general property of the material that does not show deformations under tensions is embedded in the form (23), all particular cases being included.

The case of equations (20) and (24) is specific to the tensors that should further be called equivalent to a vector field. This equivalence can be understood in the following way: let $\vec{v}$ be a vector field, with the help of which the following matrix is built

$$v_{ij} = \alpha\delta_{ij} + \beta v_i v_j. \tag{25}$$

It is obvious that if v_k are the components of a vector, and admitting α and β scalars, v_{ij} are automatically the components of a tensor. One of its main values, namely α, is double. The other main value, different from α, is given by

$$\alpha' = \alpha + \beta v^2 \tag{26}$$

There are some interesting properties of this tensor. First, if either β or v_k is zero, v_{ij} is a purely spherical tensor. Second, if the eigenvector of $\mathbf{v}$ is calculated corresponding to the eigenvalue (26), it is found that it is $\vec{v}$, up to a normalization factor. This one property is independent of the parameter α, and in fact it is what allows defining the afore-mentioned equivalence: given $\vec{v}$, it is possible to directly construct $\mathbf{v}$ as a family of tensors depending on two arbitrary parameters that has this vector as an eigenvector. It could be stated that $\mathbf{v}$ represents some kind of action directed in the general direction of $\vec{v}$, but not exactly in that direction.

4. An exemplification of the model

A tensor which would then describe the multifractal medium, could be of the form:

$$w_{ij} = \alpha\delta_{ij} + \beta u_i u_j + \gamma v_i v_j \tag{27}$$

It can be noted that the calculations are much more symmetrical if (27) is written in a more convenient form, namely

$$w_{ij} = \lambda u_{ij} + \mu v_{ij} \tag{28}$$

where λ and μ are real parameters, which describe the degree of "spatial" and "material" of the multifractal medium, with the matrices $\boldsymbol{u}$ and $\boldsymbol{v}$ defined by

$$\begin{gathered} u_{ij} = u_i u_j - \frac{1}{2}u^2\delta_{ij}; \quad v_{ij} = v_i v_j - \frac{1}{2}v^2\delta_{ij} \\ u^2 = u_1^2 + u_2^2 + u_3^2; \quad v^2 = v_1^2 + v_2^2 + v_3^2. \end{gathered} \tag{29}$$

This tensor contains eight measurable quantities and two intrinsic vectors. Extensively written, the matrix (28) will be

$$w_{ij} = \lambda u_i u_j + \mu v_i v_j - \frac{1}{2}(\lambda u^2 + \mu v^2)\delta_{ij}. \tag{30}$$

It can be observed that this tensor has three real and distinct principal values. Its orthogonal invariants are

$$I_1 = -e;\ I_2 = -e^2 + g^2;\ I_3 = -e(e^2 - g^2) \tag{31}$$

where

$$e \equiv \frac{1}{2}(\lambda u^2 + \mu v^2);\ \vec{g} \equiv \sqrt{\lambda\mu}(\vec{u} \times \vec{v}). \tag{32}$$

The main values of w_{ij} can then be calculated as the roots of the characteristic equation of matrix, and they are

$$w_1 = e,\ w_{2,3} = \pm\sqrt{e^2 - g^2}. \tag{33}$$

It happens that the pair in equation (32) is one of the eigenvectors of w together with its own value. The other two eigenvectors of w are perpendicular, located in the plane of the vectors $\vec{u}$ and $\vec{v}$.

5. Conclusions

The main conclusions of the present paper are the following:

i) A short reminder of the multifractal hydrodynamic models is given. In such a context, the existence of a multifractal medium was highlighted and moreover, a characterization of this medium was made, by means of a multifractal tensor;

ii) The existence of the multifractal tensor allowed the construction of several multifractal material constitutive laws. Since deformations exist even when no tensions are applied to the material, they can be viewed as intrinsic or pure material properties - in particular, see Bell's constitutive laws;

iii) In the same context of the proposed model, it is shown that not only radiation cosmic background, but the electromagnetic field in general, in its Maxwellian form, is in truth the expression of the existence of the multifractal medium.

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