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ON ROUGH (m, n) BI-Γ-HYPERIDEALS IN Γ-SEMIHYPERGROUPS

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In this paper, we introduced the concept of (m, n) bi-Γ-hyperideals and rough (m, n) bi-Γ-hyperideals in Γ-semihypergroups and some properties of (m, n) bi-Γ-hyperideals in Γ-semihypergroups are presented.

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1. Introduction

The notion of (m, n)-ideals of semigroups was introduced by Lajos [13, 14]. Later (m, n) quasi-ideals and (m, n) bi-ideals and generalized (m, n) bi-ideals were studied in various algebraic structures.

The notion of a rough set was originally proposed by Pawlak [16] as a formal tool for modeling and processing incomplete information in information systems. Some authors have studied the algebraic properties of rough sets. Kuroki, in [12], introduced the notion of a rough ideal in a semigroup. Anvariyeh et al. [3], introduced Pawlak's approximations in Γ-semihypergroups. Abdullah et al. [1], introduced the notion of M-hypersystem and N-hypersystem in Γ-semihypergroups and Aslam et al. [6], studied rough M-hypersystems and fuzzy M-hypersystems in Γ-semihypergroups, also see [4, 5, 19]. Yaqoob et al. [18], Applied rough set theory to Γ-hyperideals in left almost Γ-semihypergroups.

The algebraic hyperstructure notion was introduced in 1934 by a French mathematician Marty [15], at the 8th Congress of Scandinavian Mathematicians. He published some notes on hypergroups, using them in different contexts: algebraic functions, rational fractions, non commutative groups.

In 1986, Sen and Saha [17], defined the notion of a Γ-semigroup as a generalization of a semigroup. One can see that Γ-semigroups are generalizations of semigroups. Many classical notions of semigroups have been extended to Γ-semigroups and a lot of results on Γ-semigroups are published by a lot of mathematicians, for instance, Chattopadhyay [7], Chinram and Jirojkul [8], Chinram and Siammai [9], Hila [11]. Then, in [2, 10], Davvaz et al. introduced the notion of Γ-semihypergroup

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as a generalization of a semigroup, a generalization of a semihypergroup and a generalization of a Γ-semigroup. They presented many interesting examples and obtained a several characterizations of Γ-semihypergroups.

In this paper, we have introduced the notion of (m,n) bi-Γ-hyperideals and we have applied the concept of rough set theory to (m,n) bi-Γ-hyperideals, which is a generalization of (m,n) bi-Γ-hyperideals of Γ-semihypergroups.

2. **Preliminaries**

In this section, we recall certain definitions and results needed for our purpose.

Definition 2.1. *A map $\circ : S\times S \to \mathcal{P}^*(S)$ is called hyperoperation or join operation on the set S, where S is a non-empty set and $\mathcal{P}^*(S)$ denotes the set of all non-empty subsets of S. A hypergroupoid is a set S with together a (binary) hyperoperation. A hypergroupoid $(S,\circ)$, which is associative, that is $x\circ(y\circ z) = (x\circ y)\circ z$, $\forall x,y,z\in S$, is called a semihypergroup.*

Let A and B be two non-empty subsets of S. Then, we define

$$A\Gamma B = \bigcup_{\gamma\in\Gamma} A\gamma B = \bigcup \{a\gamma b \mid a\in A,\ b\in B \text{ and } \gamma\in\Gamma\}.$$

Let $(S,\circ)$ be a semihypergroup and let $\Gamma = \{\circ\}$. Then, S is a Γ-semihypergroup. So, every semihypergroup is Γ-semihypergroup.

Let S be a Γ-semihypergroup and $\gamma\in\Gamma$. A non-empty subset A of S is called a sub Γ-semihypergroup of S if $x\gamma y\subseteq A$ for every $x,y\in A$. A Γ-semihypergroup S is called *commutative* if for all $x,y\in S$ and $\gamma\in\Gamma$, we have $x\gamma y = y\gamma x$.

Example 2.1. [2] *Let $S=[0,1]$ and $\Gamma=\mathbb{N}$. For every $x,y\in S$ and $\gamma\in\Gamma$, we define $\gamma : S\times S \longrightarrow \wp^*(S)$ by $x\gamma y = \left[0,\frac{xy}{\gamma}\right]$. Then, γ is hyperoperation. For every $x,y,z\in S$ and $\alpha,\beta\in\Gamma$, we have $(x\alpha y)\beta z = \left[0,\frac{xyz}{\alpha\beta}\right] = x\alpha(y\beta z)$. This means that S is Γ-semihypergroup.*

Example 2.2. [2] *Let $(S,\circ)$ be a semihypergroup and Γ be a non-empty subset of S. We define $x\gamma y = x\circ y$ for every $x,y\in S$ and $\gamma\in\Gamma$. Then, S is a Γ-semihypergroup.*

Definition 2.2. [2] *A non-empty subset A of a Γ-semihypergroup S is a right (left) Γ-hyperideal of S if $A\Gamma S\subseteq A$ ($S\Gamma A\subseteq A$), and is a Γ-hyperideal of S if it is both a right and a left Γ-hyperideal.*

Definition 2.3. [2] *A sub Γ-semihypergroup B of a Γ-semihypergroup S is called a bi-Γ-hyperideal of S if $B\Gamma S\Gamma B\subseteq B$.*

A bi-Γ-hyperideal B of a Γ-semihypergroup S is proper if $B\neq S$.

Lemma 2.1. *In a Γ-semihypergroup S, $(A\Gamma B)^m = A^m\Gamma B^m$ holds if $A\Gamma B = B\Gamma A$ for all $A,B\in S$ and m is a positive integer.*

Proof. We prove the result $(A\Gamma B)^m = A^m\Gamma B^m$ by induction on m. For $m=1$, $A\Gamma B = A\Gamma B$, which is true. For $m=2$, $(A\Gamma B)^2 = (A\Gamma B)\Gamma(A\Gamma B) = A\Gamma(B\Gamma A)\Gamma B =$

$A^2\Gamma B^2$. Suppose that the result is true for $m=k$. That is, $(A\Gamma B)^k = A^k\Gamma B^k$. Now for $m=k+1$, we have

$$\begin{aligned}(A\Gamma B)^{k+1} &= (A\Gamma B)^k\Gamma(A\Gamma B) = (A^k\Gamma B^k)\Gamma(A\Gamma B) = A^k\Gamma(B^k\Gamma A)\Gamma B\\ &= (A^k\Gamma A)\Gamma(B^k\Gamma B) = A^{k+1}\Gamma B^{k+1}.\end{aligned}$$

Thus, the result is true for $m=k+1$. By induction hypothesis the result $(A\Gamma B)^m = A^m\Gamma B^m$ is true for all positive integers m. □

3. (m,n) Bi-Γ-hyperideals in Γ-semihypergroups

From [14], a subsemigroup A of a semigroup S is called an (m,n)-ideal of S if $A^mSA^n \subseteq A$.

A subset A of a Γ-semihypergroup S is called an $(m,0)$ Γ-hyperideal ($(0,n)$ Γ-hyperideal) if $A^m\Gamma S \subseteq A$ ($S\Gamma A^n \subseteq A$). A sub Γ-semihypergroup A of a Γ-semihypergroup S is called (m,n) bi-Γ-hyperideal of S, if A satisfies the condition

$$A^m\Gamma S\Gamma A^n \subseteq A,$$

where m, n are non-negative integers (A^m is suppressed if $m=0$). Here if $m=n=1$ then A is called bi-Γ-hyperideal of S. By a proper (m,n) bi-Γ-hyperideal we mean an (m,n) bi-Γ-hyperideal, which is a proper subset of S.

Example 3.1. *Let $(S,\circ)$ be a semihypergroup and Γ be a non-empty subset of S. Define a mapping $S\times\Gamma\times S \to \mathcal{P}^*(S)$ by $x\gamma y = x\circ y$ for every $x,y\in S$ and $\gamma\in\Gamma$. By Example 2.2, we know that S is a Γ-semihypergroup. Let B be an (m,n) bi-hyperideal of the semihypergroup S. Then, $B^m\circ S\circ B^n \subseteq B$. So, $B^m\Gamma S\Gamma B^n = B^m\circ S\circ B^n \subseteq B$. Hence, B is an (m,n) bi-Γ-hyperideal of S.*

Example 3.2. *Let $S=[0,1]$ and $\Gamma=\mathbb{N}$. Then, S together with the hyperoperation $x\gamma y = \left[0,\frac{xy}{\gamma}\right]$ is a Γ-semihypergroup. Let $t\in[0,1]$ and set $T=[0,t]$. Then, clearly it can be seen that T is a sub Γ-semihypergroup of S. Since $T^m\Gamma S = [0,t^m] \subseteq [0,t] = T$ ($S\Gamma T^n = [0,t^n] \subseteq [0,t] = T$), so T is an $(m,0)$ Γ-hyperideal ($(0,n)$ Γ-hyperideal) of S. Since $T^m\Gamma S\Gamma T^n = [0,t^{m+n}] \subseteq [0,t] = T$, then T is an (m,n) bi-Γ-hyperideal of Γ-semihypergroup S.*

Example 3.3. *Let $S=[-1,0]$ and $\Gamma=\{-1,-2,-3,\cdots\}$. Define the hyperoperation $x\gamma y = \left[\frac{xy}{\gamma},0\right]$ for all $x,y\in S$ and $\gamma\in\Gamma$. Then, clearly S is a Γ-semihypergroup. Let $\lambda\in[-1,0]$ and the set $B=[\lambda,0]$. Then, clearly B is a sub Γ-semihypergroup of S. Since $B^m\Gamma S = [\lambda^{2m+1},0] \subseteq [\lambda,0] = B$ ($S\Gamma B^n = [\lambda^{2n+1},0] \subseteq [\lambda,0] = B$), so B is an $(m,0)$ Γ-hyperideal ($(0,n)$ Γ-hyperideal) of S. Since $B^m\Gamma S\Gamma B^n = [\lambda^{2(m+n)+1},0] \subseteq [\lambda,0] = B$, then B is an (m,n) bi-Γ-hyperideal of Γ-semihypergroup S.*

Proposition 3.1. *Let S be a Γ-semihypergroup, B be a sub Γ-semihypergroup of S and let A be an (m,n) bi-Γ-hyperideal of S. Then, the intersection $A\cap B$ is an (m,n) bi-Γ-hyperideal of Γ-semihypergroup B.*

Proof. The intersection $A\cap B$ evidently is a sub Γ-semihypergroup of S. We show that $A\cap B$ is an (m,n) bi-Γ-hyperideal of B, for this

$$(A\cap B)^m\Gamma B\Gamma(A\cap B)^n \subseteq A^m\Gamma S\Gamma A^n \subseteq A, \tag{1}$$

because of A is an (m,n) bi-Γ-hyperideal of S. Secondly

$$(A\cap B)^m\Gamma B\Gamma(A\cap B)^n \subseteq B^m\Gamma B\Gamma B^n \subseteq B. \quad (2)$$

Therefore, (1) and (2) imply that $(A\cap B)^m\Gamma B\Gamma(A\cap B)^n \subseteq A\cap B$, that is, the intersection $A\cap B$ is an (m,n) bi-Γ-hyperideal of B. □

Theorem 3.1. *Suppose that $\{A_i : i\in I\}$ be a family of (m,n) bi-Γ-hyperideals of a Γ-semihypergroup S. Then, the intersection $\bigcap_{i\in I} A_i \neq \emptyset$ is an (m,n) bi-Γ-hyperideal of S.*

Proof. Let $\{A_i : i\in I\}$ be a family of (m,n) bi-Γ-hyperideals in a Γ-semihypergroup S. We know that the intersection of sub Γ-semihypergroups is a sub Γ-semihypergroup. Let $B=\bigcap_{i\in I} A_i$. Now we have to show that $B=\bigcap_{i\in I} A_i$ is an (m,n) bi-Γ-hyperideal of S. Here we need only to show that $B^m\Gamma S\Gamma B^n\subseteq B$. Let $x\in B^m\Gamma S\Gamma B^n$. Then, $x=a_1^m\alpha s\beta a_2^n$ for some $a_1^m, a_2^n\subseteq B$, $s\in S$ and $\alpha,\beta\in\Gamma$. Thus, for any arbitrary $i\in I$ as $a_1^m, a_2^n\subseteq B_i$. So, $x\in B_i^m\Gamma S\Gamma B_i^n$. Since B_i is an (m,n) bi-Γ-hyperideal so $B_i^m\Gamma S\Gamma B_i^n\subseteq B_i$ and therefore $x\in B_i$. Since i was chosen arbitrarily so $x\in B_i$ for all $i\in I$ and hence $x\in B$. So, $B^m\Gamma S\Gamma B^n\subseteq B$ and hence $B=\bigcap_{i\in I} A_i$ is an (m,n) bi-Γ-hyperideal of S. □

It is obvious that the intersection of two or more $(m,0)$ Γ-hyperideals ($(0,n)$ Γ-hyperideals) is an $(m,0)$ Γ-hyperideal ($(0,n)$ Γ-hyperideal). Similarly, the union of two or more $(m,0)$ Γ-hyperideals ($(0,n)$ Γ-hyperideals) is an $(m,0)$ Γ-hyperideal ($(0,n)$ Γ-hyperideal).

Theorem 3.2. *Let S be a Γ-semihypergroup. If A is an $(m,0)$ Γ-hyperideal and also $(0,n)$ Γ-hyperideal of S, then A is an (m,n) bi-Γ-hyperideal of S.*

Proof. Suppose that A is an $(m,0)$ Γ-hyperideal and also $(0,n)$ Γ-hyperideal of S. Then,

$$A^m\Gamma S\Gamma A^n \subseteq A\Gamma A^n \subseteq S\Gamma A^n \subseteq A,$$

which implies that A is an (m,n) bi-Γ-hyperideal of S. □

Theorem 3.3. *Let m,n be arbitrary positive integers. Let S be a Γ-semihypergroup, B be an (m,n) bi-Γ-hyperideal of S and A be a sub Γ-semihypergroup of S. Suppose that $A\Gamma B=B\Gamma A$. Then,*

(1) $B\Gamma A$ is an (m,n) bi-Γ-hyperideal of S.

(2) $A\Gamma B$ is an (m,n) bi-Γ-hyperideal of S.

Proof. (1) The suppositions of the theorem imply that

$$(B\Gamma A)\Gamma(B\Gamma A) = (B\Gamma A\Gamma B)\Gamma A = B\Gamma A.$$

This shows that $B\Gamma A$ is a sub Γ-semihypergroup of S. On the other hand, as B is an (m,n) bi-Γ-hyperideal of S, so

$$(B\Gamma A)^m\Gamma S\Gamma(B\Gamma A)^n = (B^m\Gamma A^m\Gamma S\Gamma B^n)\Gamma A^n \subseteq B\Gamma A^n \subseteq B\Gamma A.$$

Hence, the product $B\Gamma A$ is an (m,n) bi-Γ-hyperideal of S.

(2) The proof is similar to (1). □

Theorem 3.4. *Let S be a Γ-semihypergroup and for a positive integer n, $B_1, B_2, \cdots, B_n$ be (m,n) bi-Γ-hyperideals of S. Then, $B_1\Gamma B_2\Gamma\cdots\Gamma B_n$ is an (m,n) bi-Γ-hyperideal of S.*

Proof. We prove the theorem by induction. By Theorem 3.3, $B_1\Gamma B_2$ is an (m,n) bi-Γ-hyperideal of S. Next, for $k \leq n$, suppose that $B_1\Gamma B_2\Gamma...\Gamma B_k$ is an (m,n) bi-Γ-hyperideal of S. Then, $B_1\Gamma B_2\Gamma...\Gamma B_k\Gamma B_{k+1} = (B_1\Gamma B_2\Gamma...\Gamma B_k)\Gamma B_{k+1}$ is an (m,n) bi-Γ-hyperideal of S by Theorem 3.3. □

Theorem 3.5. *Let S be a Γ-semihypergroup, A be an (m,n) bi-Γ-hyperideal of S, and B be an (m,n) bi-Γ-hyperideal of the Γ-semihypergroup A such that $B^2 = B\Gamma B = B$. Then, B is an (m,n) bi-Γ-hyperideal of S.*

Proof. It is trivial that B is a sub Γ-semihypergroup of S. Secondly, since $A^m\Gamma S\Gamma A^n \subseteq A$ and $B^m\Gamma A\Gamma B^n \subseteq B$, we have

$$B^m\Gamma S\Gamma B^n = B^m\Gamma(B^m\Gamma S\Gamma B^n)\Gamma B^n \subseteq B^m\Gamma(A^m\Gamma S\Gamma A^n)\Gamma B^n \subseteq B^m\Gamma A\Gamma B^n \subseteq B.$$

Therefore, B is an (m,n) bi-Γ-hyperideal of S. □

4. Lower and Upper Approximations in Γ-semihypergroups

In what follows, let S denote a Γ-semihypergroup unless otherwise specified.

Definition 4.1. *Let S be a Γ-semihypergroup. An equivalence relation ρ on S is called regular on S if*

$$(a,b) \in \rho \quad \textit{implies} \quad (a\gamma x, b\gamma x) \in \rho \quad \textit{and} \quad (x\gamma a, x\gamma b) \in \rho,$$

for all $x \in S$ and $\gamma \in \Gamma$.

If ρ is a regular relation on S, then, for every $x \in S$, $[x]_\rho$ stands for the class of x with the represent ρ. A regular relation ρ on S is called complete if $[a]_\rho\gamma[b]_\rho = [a\gamma b]_\rho$ for all $a, b \in S$ and $\gamma \in \Gamma$. In addition, ρ on S is called congruence if, for every $(a,b) \in S$ and $\gamma \in \Gamma$, we have $c \in [a]_\rho\gamma[b]_\rho \Longrightarrow [c]_\rho \subseteq [a]_\rho\gamma[b]_\rho$.

Let A be a non-empty subset of a Γ-semihypergroup S and ρ be a regular relation on S. Then, the sets

$$\underline{Apr}_\rho(A) = \left\{x \in S : [x]_\rho \subseteq A\right\} \quad \text{and} \quad \overline{Apr}_\rho(A) = \left\{x \in S : [x]_\rho \cap A \neq \emptyset\right\}$$

are called ρ-lower and ρ-upper approximations of A, respectively. For a non-empty subset A of S, $Apr_\rho(A) = (\underline{Apr}_\rho(A), \overline{Apr}_\rho(A))$ is called a rough set with respect to ρ if $\underline{Apr}_\rho(A) \neq \overline{Apr}_\rho(A)$.

Theorem 4.1. [3] *Let ρ be a regular relation on a Γ-semihypergroup S and let A and B be non-empty subsets of S. Then,*

(1) $\overline{Apr}_\rho(A)\Gamma\overline{Apr}_\rho(B) \subseteq \overline{Apr}_\rho(A\Gamma B)$;

(2) If ρ is complete, then $\underline{Apr}_\rho(A)\Gamma\underline{Apr}_\rho(B) \subseteq \underline{Apr}_\rho(A\Gamma B)$.

Theorem 4.2. [3] *Let ρ be a regular relation on a Γ-semihypergroup S. Then,*

(1) Every sub Γ-semihypergroup of S is a ρ-upper rough sub Γ-semihypergroup of S.

(2) Every right (left) Γ-hyperideal of S is a ρ-upper rough right (left) Γ-hyperideal of S.

Theorem 4.3. [3] *Let $\emptyset \neq A \subseteq S$ and let ρ be a complete regular relation on S such that the ρ-lower approximation of A is non-empty. Then,*

(1) If A is a sub Γ-semihypergroup of S, then A is a ρ-lower rough sub Γ-semihypergroup of S.

(2) If A is a right (left) Γ-hyperideal of S, then A is a ρ-lower rough right (left) Γ-hyperideal of S.

A subset A of a Γ-semihypergroup S is called a ρ-upper [ρ-lower] rough bi-Γ-hyperideal of S if $\overline{Apr}_\rho(A)[\underline{Apr}_\rho(A)]$ is a bi-Γ-hyperideal of S.

Theorem 4.4. [3] *Let ρ be a regular relation on S and A be a bi-Γ-hyperideal of S. Then,*

(1) A is a ρ-upper rough bi-Γ-hyperideal of S.

(2) If ρ is complete such that the ρ-lower approximation of A is non-empty, then A is a ρ-lower rough bi-Γ-hyperideal of S.

Lemma 4.1. *Let ρ be a regular relation on a Γ-semihypergroup S. Then, for a non-empty subset A of S*

(1) $\left(\overline{Apr}_\rho(A)\right)^n \subseteq \overline{Apr}_\rho(A^n)$ for all $n \in \mathbb{N}$.

(2) If ρ is complete, then $\left(\underline{Apr}_\rho(A)\right)^n \subseteq \underline{Apr}_\rho(A^n)$ for all $n \in \mathbb{N}$.

Proof. (1) Let A be a non-empty subset of S, then for $n = 2$, and by Theorem 4.1(1), we get

$$\left(\overline{Apr}_\rho(A)\right)^2 = \overline{Apr}_\rho(A)\Gamma\overline{Apr}_\rho(A) \subseteq \overline{Apr}_\rho(A\Gamma A) = \overline{Apr}_\rho(A^2).$$

Now for $n = 3$, we get

$$\begin{aligned}\left(\overline{Apr}_\rho(A)\right)^3 &= \overline{Apr}_\rho(A)\Gamma\left(\overline{Apr}_\rho(A)\right)^2 \subseteq \overline{Apr}_\rho(A)\Gamma\overline{Apr}_\rho(A^2)\\ &\subseteq \overline{Apr}_\rho(A\Gamma A^2) = \overline{Apr}_\rho(A^3).\end{aligned}$$

Suppose that the result is true for $n = k-1$, such that $(\overline{Apr}_\rho(A))^{k-1} \subseteq \overline{Apr}_\rho(A^{k-1})$, then for $n = k$, we get

$$\begin{aligned}\left(\overline{Apr}_\rho(A)\right)^k &= \overline{Apr}_\rho(A)\Gamma\left(\overline{Apr}_\rho(A)\right)^{k-1} \subseteq \overline{Apr}_\rho(A)\Gamma\overline{Apr}_\rho(A^{k-1})\\ &\subseteq \overline{Apr}_\rho(A\Gamma A^{k-1}) = \overline{Apr}_\rho(A^k).\end{aligned}$$

Hence, this shows that $\left(\overline{Apr}_\rho(A)\right)^k \subseteq \overline{Apr}_\rho(A^k)$. This implies that $\left(\overline{Apr}_\rho(A)\right)^n \subseteq \overline{Apr}_\rho(A^n)$ is true for all $n \in \mathbb{N}$. By using Theorem 4.1(2), the proof of (2) can be seen in a similar way. This completes the proof. □

5. Rough (m,n) Bi-Γ-hyperideals in Γ-semihypergroups

Let ρ be a regular relation on a Γ-semihypergroup S. A subset A of S is called a ρ-upper rough $(m,0)$ Γ-hyperideal ($(0,n)$ Γ-hyperideal) of S if $\overline{Apr}_\rho(A)$ is an $(m,0)$ Γ-hyperideal ($(0,n)$ Γ-hyperideal) of S. Similarly, a subset A of a Γ-semihypergroup S is called a ρ-lower rough $(m,0)$ Γ-hyperideal ($(0,n)$ Γ-hyperideal) of S if $\underline{Apr}_\rho(A)$ is an $(m,0)$ Γ-hyperideal ($(0,n)$ Γ-hyperideal) of S.

Theorem 5.1. *Let ρ be a regular relation on a Γ-semihypergroup S and A be an $(m,0)$ Γ-hyperideal ($(0,n)$ Γ-hyperideal) of S. Then,*

(1) $\overline{Apr}_\rho(A)$ is an $(m,0)$ Γ-hyperideal ($(0,n)$ Γ-hyperideal) of S.

(2) If ρ is complete, then $\underline{Apr}_\rho(A)$ is, if it is non-empty, an $(m,0)$ Γ-hyperideal ($(0,n)$ Γ-hyperideal) of S.

Proof. (1) Let A be an $(m,0)$ Γ-hyperideal of S, that is, $A^m\Gamma S \subseteq A$. Note that $\overline{Apr}_\rho(S) = S$. Then, by Theorem 4.1(1) and Lemma 4.1(1), we have

$$\begin{aligned}\left(\overline{Apr}_\rho(A)\right)^m \Gamma S &= \left(\overline{Apr}_\rho(A)\right)^m \Gamma \overline{Apr}_\rho(S) \subseteq \overline{Apr}_\rho(A^m)\Gamma\overline{Apr}_\rho(S) \\ &\subseteq \overline{Apr}_\rho(A^m\Gamma S) \subseteq \overline{Apr}_\rho(A).\end{aligned}$$

This shows that $\overline{Apr}_\rho(A)$ is an $(m,0)$ Γ-hyperideal of S, that is, A is a ρ-upper rough $(m,0)$ Γ-hyperideal of S. Similarly, we can show that the ρ-upper approximation of a $(0,n)$ Γ-hyperideal is a $(0,n)$ Γ-hyperideal of S.

(2) Let A be an $(m,0)$ Γ-hyperideal of S, that is, $A^m\Gamma S \subseteq A$. Note that $\underline{Apr}_\rho(S) = S$. Then, by Theorem 4.1(2) and Lemma 4.1(2), we have

$$\begin{aligned}\left(\underline{Apr}_\rho(A)\right)^m \Gamma S &= \left(\underline{Apr}_\rho(A)\right)^m \Gamma \underline{Apr}_\rho(S) \subseteq \underline{Apr}_\rho(A^m)\Gamma\underline{Apr}_\rho(S) \\ &\subseteq \underline{Apr}_\rho(A^m\Gamma S) \subseteq \underline{Apr}_\rho(A).\end{aligned}$$

This shows that $\underline{Apr}_\rho(A)$ is an $(m,0)$ Γ-hyperideal of S, that is, A is a ρ-lower rough $(m,0)$ Γ-hyperideal of S. Similarly, we can show that the ρ-lower approximation of a $(0,n)$ Γ-hyperideal is a $(0,n)$ Γ-hyperideal of S. This completes the proof. □

A subset A of a Γ-semihypergroup S is called a ρ-upper [ρ-lower] rough (m,n) bi-Γ-hyperideal of S if $\overline{Apr}_\rho(A)$ [$\underline{Apr}_\rho(A)$] is an (m,n) bi-Γ-hyperideal of S.

Theorem 5.2. *Let ρ be a regular relation on a Γ-semihypergroup S. If A is an (m,n) bi-Γ-hyperideal of S, then it is a ρ-upper rough (m,n) bi-Γ-hyperideal of S.*

Proof. Let A be an (m,n) bi-Γ-hyperideal of S. Then, by Theorem 4.1(1) and Lemma 4.1(1), we have

$$\begin{aligned}(\overline{Apr}_\rho(A))^m\Gamma S\Gamma(\overline{Apr}_\rho(A))^n &= (\overline{Apr}_\rho(A))^m\Gamma\overline{Apr}_\rho(S)\Gamma(\overline{Apr}_\rho(A))^n \\ &\subseteq \overline{Apr}_\rho(A^m)\Gamma\overline{Apr}_\rho(S)\Gamma\overline{Apr}_\rho(A^n) \\ &\subseteq \overline{Apr}_\rho(A^m\Gamma S)\Gamma\overline{Apr}_\rho(A^n) \\ &\subseteq \overline{Apr}_\rho(A^m\Gamma S\Gamma A^n) \subseteq \overline{Apr}_\rho(A).\end{aligned}$$

From this and Theorem 4.2(1), we obtain that $\overline{Apr}_\rho(A)$ is an (m,n) bi-Γ-hyperideal of S, that is, A is a ρ-upper rough (m,n) bi-Γ-hyperideal of S. This completes the proof. □

Theorem 5.3. *Let ρ be a complete regular relation on a Γ-semihypergroup S. If A is an (m,n) bi-Γ-hyperideal of S, then $\underline{Apr}_\rho(A)$ is, if it is non-empty, an (m,n) bi-Γ-hyperideal of S.*

Proof. Let A be an (m,n) bi-Γ-hyperideal of S. Then, by Theorem 4.1(2) and Lemma 4.1(2), we have

$$\begin{aligned}(\underline{Apr}_\rho(A))^m\Gamma S\Gamma(\underline{Apr}_\rho(A))^n &= (\underline{Apr}_\rho(A))^m\Gamma\underline{Apr}_\rho(S)\Gamma(\underline{Apr}_\rho(A))^n\\ &\subseteq \underline{Apr}_\rho(A^m)\Gamma\underline{Apr}_\rho(S)\Gamma\underline{Apr}_\rho(A^n)\\ &\subseteq \underline{Apr}_\rho(A^m\Gamma S)\Gamma\underline{Apr}_\rho(A^n)\\ &\subseteq \underline{Apr}_\rho(A^m\Gamma S\Gamma A^n)\subseteq\underline{Apr}_\rho(A).\end{aligned}$$

From this and Theorem 4.3(1), we obtain that $\underline{Apr}_\rho(A)$ is, if it is non-empty, an (m,n) bi-Γ-hyperideal of S. This completes the proof. □

The following example shows that the converse of Theorem 5.2 and Theorem 5.3 does not hold.

Example 5.1. *Let $S=\{x,y,z\}$ and $\Gamma=\{\beta,\gamma\}$ be the sets of binary hyperoperations defined below:*

β	x	y	z
x	x	$\{x,y\}$	z
y	$\{x,y\}$	$\{x,y\}$	z
z	z	z	z

γ	x	y	z
x	$\{x,y\}$	$\{x,y\}$	z
y	$\{x,y\}$	y	z
z	z	z	z

Clearly S is a Γ-semihypergroup. Let ρ be a complete regular relation on S such that the ρ-regular classes are the subsets $\{x,y\}$, $\{z\}$. Now for $A=\{x,z\}\subseteq S$, $\overline{Apr}_\rho(A)=\{x,y,z\}$ and $\underline{Apr}_\rho(A)=\{z\}$. It is clear that $\overline{Apr}_\rho(A)$ and $\underline{Apr}_\rho(A)$ are (m,n) bi-Γ-hyperideals of S, but A is not an (m,n) bi-Γ-hyperideal of S. Because $A^m\Gamma S\Gamma A^n=S\nsubseteq A$.

6. Rough (m,n) Bi-Γ-hyperideals in the Quotient Γ-semihypergroups

Let ρ be a regular relation on a Γ-semihypergroup S. We put $\widehat{\Gamma}=\{\widehat{\gamma}\ :\ \gamma\in\Gamma\}$. For every $[a]_\rho,[b]_\rho\in S/\rho$, we define $[a]_\rho\widehat{\gamma}[b]_\rho=\{[z]_\rho : z\in a\gamma b\}$.

Theorem 6.1. *(*[3, Theorem 4.1]*) If S is a Γ-semihypergroup, then S/ρ is a $\widehat{\Gamma}$-semihypergroup.*

Definition 6.1. *Let ρ be a regular relation on a Γ-semihypergroup S. The ρ-lower approximation and ρ-upper approximation of a non-empty subset A of S can be presented in an equivalent form as shown below:*

$$\underline{\underline{Apr}}_\rho(A)=\left\{[x]_\rho\in S/\rho:[x]_\rho\subseteq A\right\}\quad\text{and}\quad\overline{\overline{Apr}}_\rho(A)=\left\{[x]_\rho\in S/\rho:[x]_\rho\cap A\neq\emptyset\right\},$$

respectively.

Theorem 6.2. *(*[3, Theorems 4.3, 4.4]*) Let ρ be a regular relation on a Γ-semihypergroup S. If A is a sub Γ-semihypergroup of S. Then,*

(1) $\overline{\overline{Apr}}_\rho(A)$ is a sub $\widehat{\Gamma}$-semihypergroup of S/ρ.

(2) $\underline{\underline{Apr}}_\rho(A)$ is, if it is non-empty, a sub $\widehat{\Gamma}$-semihypergroup of S/ρ.

Theorem 6.3. *Let ρ be a regular relation on a Γ-semihypergroup S. If A is an $(m, 0)$ Γ-hyperideal ($(0, n)$ Γ-hyperideal) of S. Then,*

(1) $\overline{\overline{Apr}}_\rho(A)$ is an $(m, 0)$ $\widehat{\Gamma}$-hyperideal ($(0, n)$ $\widehat{\Gamma}$-hyperideal) of S/ρ.

(2) $\underline{\underline{Apr}}_\rho(A)$ is, if it is non-empty, an $(m, 0)$ $\widehat{\Gamma}$-hyperideal ($(0, n)$ $\widehat{\Gamma}$-hyperideal) of S/ρ.

Proof. (1) Assume that A is a $(0, n)$ Γ-hyperideal of S. Let $[x]_\rho$ and $[s]_\rho$ be any elements of $\overline{\overline{Apr}}_\rho(A)$ and S/ρ, respectively. Then, $[x]_\rho \cap A \neq \emptyset$. Hence, $x \in \overline{Apr}_\rho(A)$. Since A is a $(0, n)$ Γ-hyperideal of S, by Theorem 10(1), $\overline{Apr}_\rho(A)$ is a $(0, n)$ Γ-hyperideal of S. So, for $\gamma \in \Gamma$, we have $s\gamma x^n \subseteq \overline{Apr}_\rho(A)$. Now, for every $t \in s\gamma x^n$, we have $[t]_\rho \cap A \neq \emptyset$. On the other hand, from $t \in s\gamma x^n$, we obtain $[t]_\rho \in [s]_\rho \widehat{\gamma}[x]^n_\rho$. Therefore, $[s]_\rho \widehat{\gamma}[x]^n_\rho \subseteq \overline{\overline{Apr}}_\rho(A)$. This means that $\overline{\overline{Apr}}_\rho(A)$ is a $(0, n)$ $\widehat{\Gamma}$-hyperideal of S/ρ.

(2) Let A be a $(0, n)$ Γ-hyperideal of S. Let $[x]_\rho$ and $[s]_\rho$ be any elements of $\underline{\underline{Apr}}_\rho(A)$ and S/ρ, respectively. Then, $[x]_\rho \subseteq A$, which implies $x \in \underline{Apr}_\rho(A)$. Since A is a $(0, n)$ Γ-hyperideal of S, by Theorem 10(2), $\underline{Apr}_\rho(A)$ is a $(0, n)$ Γ-hyperideal of S. Thus, for every $\gamma \in \Gamma$, we have $s\gamma x^n \subseteq \underline{Apr}_\rho(A)$. Now, for every $t \in s\gamma x^n$, we have $t \in \underline{Apr}_\rho(A)$, which implies that $[t]_\rho \subseteq A$. Hence, $[t]_\rho \in \underline{\underline{Apr}}_\rho(A)$. On the other hand, from $t \in s\gamma x^n$, we have $[t]_\rho \in [s]_\rho \widehat{\gamma}[x]^n_\rho$. Therefore, $[s]_\rho \widehat{\gamma}[x]^n_\rho \subseteq \underline{\underline{Apr}}_\rho(A)$. This means that $\underline{\underline{Apr}}_\rho(A)$ is, if it is non-empty, a $(0, n)$ $\widehat{\Gamma}$-hyperideal of S/ρ.

The other cases can be seen in a similar way. This completes the proof. □

Theorem 6.4. *Let ρ be a regular relation on a Γ-semihypergroup S. If A is an (m, n) bi-Γ-hyperideal of S. Then,*

(1) $\overline{\overline{Apr}}_\rho(A)$ is an (m, n) bi-$\widehat{\Gamma}$-hyperideal of S/ρ.

(2) $\underline{\underline{Apr}}_\rho(A)$ is, if it is non-empty, an (m, n) bi-$\widehat{\Gamma}$-hyperideal of S/ρ.

Proof. (1) Let $[x]_\rho$ and $[y]_\rho$ be any elements of $\overline{\overline{Apr}}_\rho(A)$ and $[s]_\rho$ be any element of S/ρ. Then,

$$[x]_\rho \cap A \neq \emptyset \quad \text{and} \quad [y]_\rho \cap A \neq \emptyset.$$

Hence, $x \in \overline{Apr}_\rho(A)$ and $y \in \overline{Apr}_\rho(A)$. By Theorem 11, $\overline{Apr}_\rho(A)$ is an (m, n) bi-$\widehat{\Gamma}$-hyperideal of S. So, for every $\alpha, \beta \in \Gamma$, we have $x^m \alpha s \beta y^n \subseteq \overline{Apr}_\rho(A)$. Now, for every $t \in x^m \alpha s \beta y^n$, we obtain $[t]_\rho \in [x]^m_\rho \widehat{\alpha} s \widehat{\beta} [y]^n_\rho$. On the other hand, since $t \in \overline{Apr}_\rho(A)$, we have $[t]_\rho \cap A \neq \emptyset$. Thus,

$$[x]^m_\rho \widehat{\alpha} s \widehat{\beta} [y]^n_\rho \subseteq \overline{\overline{Apr}}_\rho(A).$$

Therefore, $\overline{\overline{Apr}}_\rho(A)$ is an (m, n) bi-$\widehat{\Gamma}$-hyperideal of S/ρ.

(2) Let $[x]_\rho$ and $[y]_\rho$ be any elements of $\underline{\underline{Apr}}_\rho(A)$ and $[s]_\rho$ be any element of S/ρ. Then,

$$[x]_\rho \subseteq A \quad \text{and} \quad [y]_\rho \subseteq A.$$

Hence, $x \in \underline{Apr}_\rho(A)$ and $y \in \underline{Apr}_\rho(A)$. By Theorem 12, $\underline{Apr}_\rho(A)$ is an (m, n) bi-$\widehat{\Gamma}$-hyperideal of S. So, for every $\alpha, \beta \in \Gamma$, we have $x^m \alpha s \beta y^n \subseteq \underline{Apr}_\rho(A)$. Then, for every $t \in x^m \alpha s \beta y^n$, we obtain $[t]_\rho \in [x]_\rho^m \widehat{\alpha} a \widehat{\beta} [y]_\rho^n$. On the other hand, since $t \in \underline{Apr}_\rho(A)$, we have $[t]_\rho \subseteq A$. So,

$$[x]_\rho^m \widehat{\alpha} a \widehat{\beta} [y]_\rho^n \subseteq \underline{\underline{Apr}}_\rho(A).$$

Therefore, $\underline{\underline{Apr}}_\rho(A)$ is, if it is non-empty, an (m,n) bi-$\widehat{\Gamma}$-hyperideal of S/ρ. This completes the proof. □

7. Conclusion

The relations between rough sets and algebraic systems have been already considered by many mathematicians. In this paper, the properties of (m,n) bi-Γ-hyperideal in Γ-semihypergroup are investigated and hence the concept of rough set theory is applied to (m,n) bi-Γ-hyperideals.

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