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INJECTIVITY AND PROJECTIVITY OF SOME CLASSES OF FRÉCHET ALGEBRAS

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In this paper for a locally compact group G and a decreasing sequence of weight functions $\{\omega_n\}$ on it with $\omega_n > 1$ ($n \in \mathbb{N}$), we show that Fréchet algebra $\cap_{n\in\mathbb{N}} L^{\infty}(G, \omega_n^{-1})$ is projective if and only if G is finite and Fréchet algebra $\cap_{n\in\mathbb{N}} C_0(G, \omega_n^{-1})$ is projective (injective) if and only if G is compact (finite). Similar result will be shown for Fréchet algebra $\cap_{n\in\mathbb{N}} L_0^{\infty}(G, \omega_n^{-1})$.

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1. Introduction and Preliminaries

Injectivity and projectivity properties in the context of Fréchet algebras out of Banach algebra's category first was considered by Taylor[13] as an interesting field which we can find some examples that there is no symmetric relationship between these two type of properties on non-normable Fréchet algebras at all. Pirkovskii found these examples [10], in fact he introduced some examples of non-zero Fréchet algebras for which there is no any injective module over them, while we know in general there are enough projective modules over Fréchet algebras (see for example [6] and [13]). The main problem in this way is that the study of injectivity on Fréchet modules depends on $\mathcal{B}(\mathcal{A}, X)$, the space of all bounded operator from $\mathcal{A}$ to X, where both of them are non-normable Fréchet algebra and Fréchet module respectively, there is no any reasonable topology on $\mathcal{B}(\mathcal{A}, X)$ making it Fréchet space; however it can be seen in the proposition 1.1 that in the special case when $\mathcal{A}$ is Banach algebra then $\mathcal{B}(\mathcal{A}, X)$ is in the category of Fréchet spaces; in this situation we take a look at these properties on some examples of Fréchet modules which are constructed by a class of Banach modules through projective limit. Foundation of this paper based on the work of Dales and Polayakov[4].

A Fréchet space is a topological vector space whose topology can be given by an increasing sequence of semi-norms. A Fréchet space $\mathcal{A}$ is called Fréchet algebra, when these semi-norms are sub-multiplicative. For a Fréchet algebra $\mathcal{A}$, a Fréchet space X is called Fréchet left $\mathcal{A}$-module (in abbreviation $\mathcal{A}$-mod) if it is an algebraic left module over $\mathcal{A}$ and

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in addition the multiplication $m : \mathcal{A} \times X \to X$ is jointly continuous. Similarly right $\mathcal{A}$-module will be defined. Let $\mathcal{A}$ be a Banach algebra and X be a Fréchet space then we denote $\mathcal{B}(\mathcal{A}, X)$ as the space of all continuous morphisms, that is:

$$\mathcal{B}(\mathcal{A}, X) = \{T : \mathcal{A} \to X : P_n(Tx) \leq C_n\|x\|, n \in \mathbb{N}, x \in \mathcal{A} \text{ and some } C_n \in \mathbb{R}\}$$

where $\{P_n\}$ is a family of semi-norms on X that generates its topology.

Proposition 1.1. *The space $\mathcal{B}(\mathcal{A}, X)$ is a Fréchet space with respect to semi-norms*

$$Q_n(T) = \sup_{\|x\|\leq 1} P_n(Tx), \quad T \in \mathcal{B}(\mathcal{A}, X) \text{ and } n \in \mathbb{N}.$$

Proof. Since the strong topology on $\mathcal{B}(\mathcal{A}, X)$ actually will be generated by the semi-norms $R_{B,n}(T) = \sup_{x\in B} P_n(Tx)$, for $T \in \mathcal{B}(\mathcal{A}, X)$ and $n \in \mathbb{N}$, where B runs over all bounded subsets of $\mathcal{A}$, (see for example [12, P. 81]) and $\mathcal{B}(\mathcal{A}, X)$ with this topology is complete [14, Corollary 2 P. 344] and since $\{Q_n\}_{n\in\mathbb{N}}$ is a subset of $\{R_{B,n}\}$, so the topology on $\mathcal{B}(\mathcal{A}, X)$ that is generated by $\{Q_n\}_{n\in\mathbb{N}}$ is coarsest than the strong topology, so it is enough to prove that it is also finer. Consider fix bounded subset B_0 of $\mathcal{A}$ and $n_0 \in \mathbb{N}$, since B_0 is bounded there exists a constant $c > 0$ such that $\|x\| \leq c$, for all $x \in B_0$ hence for $T \in \mathcal{B}(\mathcal{A}, X)$:

$$R_{B_0,n_0}(T) = \sup_{x\in B_0} P_{n_0}(Tx) \leq \sup_{\|x\|\leq c} P_{n_0}(Tx) = c \sup_{\|x\|\leq 1} P_{n_0}(Tx) = cQ_{n_0}(T),$$

and therefore the open neighbourhood that is generated by Q_{n_0} is a subset of that is generated by R_{B_0,n_0} and the result follows. □

When E is Banach algebra and E is Fréchet right $\mathcal{A}$-module $\mathcal{B}(E, F)$ is also a Fréchet $\mathcal{A}$-mod by the action $a \cdot T(x) = T(x \cdot a), a \in \mathcal{A}$ and $x \in E$. For left Fréchet $\mathcal{A}$-modules E and F we denote all continuous module morphisms from E to F by ${}_{\mathcal{A}}\mathcal{B}(E, F)$.

Let G be a locally compact group and ω a weight on it, that is a positive continuous function with $\omega(xy) \leq \omega(x)\omega(y)$ for all $x, y \in G$ and $\omega(e_G) = 1$ where e_G is the identity of group G, then as in [3] and [8] the spaces $L^\infty(G, \omega^{-1})$ and $L^1(G, \omega)$ will be defined by:

$$L^\infty(G, \omega^{-1}) = \{f \text{ Borel measurable} : ess\sup_{x\in G} \frac{|f(x)|}{\omega(x)} < \infty\}$$

and

$$L^1(G, \omega) = \{f \text{ Borel measurable} : \int_G |f(x)|\omega(x)dm(x) < \infty\}$$

by additional hypothesis that f and g in $L^\infty(G, \omega^{-1})$ are equal if they are equal locally almost every where with respect to the left Haar measure m on G and they are equal in $L^1(G, \omega)$ if they are equal almost every where. $L^\infty(G, \omega^{-1})$ and $L^1(G, \omega)$ respectively with the norms:

$$\|f\|_{\infty,\omega} = ess\sup_{x\in G} \frac{|f(x)|}{\omega(x)} \quad \text{and} \quad \|f\|_\omega = \int_G |f(x)|\omega(x)dm(x)$$

are Banach spaces. $L^1(G, \omega)$ with convolution product,

$$f \star g(x) = \int_G f(y)g(y^{-1}x)dm(y)$$

is a Banach algebra [8, P. 20].

Lemma 1.1. *$L^1(G,\omega)$ is right $L^1(G)$-module with the following product,*

$$f \cdot g = f \star \frac{g}{\omega}, \quad f \in L^1(G,\omega) \text{ and } g \in L^1(G).$$

Proof. Since for all $x, y \in G$ we have $\omega(x) \leq \omega(y)\omega(y^{-1}x)$, so for $f \in L^1(G,\omega)$ and $g \in L^1(G)$:

$$\begin{aligned} \textstyle\int_G |f \star \frac{g}{\omega}(x)|\omega(x)dm(x) &= \textstyle\int_G |\int_G f(y)\frac{g(y^{-1}x)}{\omega(y^{-1}x)}\omega(x)dm(y)|dm(x) \\ &\leq \textstyle\int_G |\int_G f(y)\frac{g(y^{-1}x)}{\omega(y^{-1}x)}\omega(y)\omega(y^{-1}x)dm(y)|dm(x) \\ &\leq \textstyle\int_G \int_G |f(y)g(y^{-1}x)|\omega(y)dm(y)dm(x), \end{aligned}$$

by Fubini's theorem and substitution yx instead of x and this fact that the Haar measure is left invariant we have:

$$\begin{aligned} \textstyle\int_G \int_G |f(y)g(y^{-1}x)|\omega(y)dm(x)dm(y) &= \textstyle\int_G \int_G |f(y)g(x)|\omega(y)dm(x)dm(y) \\ &= \textstyle\int_G |f(y)|\omega(y)dm(y) \int_G |g(x)|dm(x) \\ &= \|f\|_\omega \|g\|_1 < \infty, \end{aligned}$$

so $f \cdot g \in L^1(G,\omega)$. □

Likewise $L^\infty(G,\omega^{-1})$ with product $f \cdot g = g * \tilde{f}$ is a Banach $L^1(G,\omega)$-mod where $\tilde{f}(x) = f(x^{-1})$ for all $x \in G$. $L^\infty(G, \frac{1}{\omega})$ is dual space of $L^1(G,\omega)$ that is defined by:

$$\langle f, g\rangle = \int_G f(x)g(x)dm(x) \quad f \in L^1(G,\omega), g \in L^\infty(G, \frac{1}{\omega})$$

In the other hand similarly to [3, Proposition 7.17] when $\omega(x) \geq 1$ for all $x \in G$ we have:

$$L^1(G,\omega) \cdot L^\infty(G,\omega^{-1}) = LUC(G,\omega^{-1}),$$

where:

$$LUC(G,\omega^{-1}) = \{f \in L^\infty(G,\omega^{-1}) : \frac{f}{\omega} \in LUC(G)\},$$

and $LUC(G)$ is the set of left uniformly continuous functions. Similarly $C_0(G,\omega^{-1})$ will be defined by $C_0(G,\omega^{-1}) = \{f \in L^\infty(G,\omega^{-1}) : \frac{f}{\omega} \in C_0(G)\}$ where $C_0(G)$ is the space of all continuous functions f that vanish at infinity that is for $\epsilon > 0$ there is a compact subset K of G for which $|f(x)| < \epsilon$ for all x in the complement of K. $C_0(G,\omega^{-1})$ is a closed subspace of $L^\infty(G,\omega^{-1})$ that is a Banach $L^1(G,\omega)$-mod.

Similarly to the [1] we define $L_0^\infty(G,\omega^{-1})$ as closed subspace of $L^\infty(G,\omega^{-1})$ consisting of all functions that vanish at infinity.

Lemma 1.2. *Let $f \in L^1(G,\omega)$ and $g \in L_0^\infty(G,\omega^{-1})$ then $f \cdot g \in L_0^\infty(G,\omega^{-1})$ furthermore if $\omega(x) \geq 1$ for all $x \in G$ then $f \cdot g \in C_0(G,\omega^{-1})$*

Proof. Since $|f\omega| \in L^1(G)$ and $|\frac{g}{\omega}| \in L_0^\infty(G)$ clearly $|\frac{g}{\omega}| * |\widetilde{f\omega}| \in L_0^\infty(G)$. Now because $\frac{1}{\omega(x)} \leq \frac{\omega(x^{-1}y)}{\omega(y)}$ so:

$$\begin{aligned} |\frac{g * \widetilde{f}(x)}{\omega(x)}| &= |\int_G \frac{g(y)\tilde{f}(y^{-1}x)}{\omega(x)} dm(y)| = |\int_G \frac{g(y)f(x^{-1}y)}{\omega(x)} dm(y)| \\ &\leq \int_G |\frac{g(y)}{\omega(y)} f(x^{-1}y)\omega(x^{-1}y)| dm(y) = |\frac{g}{\omega}| * |\widetilde{f\omega}|(x) \end{aligned}$$

and therefore $g * \tilde{f} \in L_0^\infty(G, \omega^{-1})$. Furthermore when $f \in L^1(G, \omega)$, $g \in L_0^\infty(G, \omega^{-1})$ and $\omega(x) \geq 1$ for all $x \in G$ then $g * \tilde{f} \in LUC(G, \omega^{-1})$ so in this situation $f \cdot g \in C_0(G, \omega^{-1})$. □

Now consider $\{E_\alpha\}_{\alpha\in\Lambda}$ as a family of Fréchet algebras over a directed set Λ and $\{f_{\alpha\beta}\}_{\alpha,\beta\in\Lambda}$ a family of morphism from E_β into E_α with $f_{\alpha\alpha} = id_{E_\alpha}$ and $f_{\alpha\gamma} = f_{\alpha\beta} \circ f_{\beta\gamma}$ for any α, β, γ in Λ, with $\alpha \leq \beta \leq \gamma$, where id_{E_α} is the identity map on E_α, then $\{(E_\alpha, f_{\alpha\beta})\}$ is called a projective system of Fréchet algebras. With respect to this system, the closed subalgebra E of $F = \Pi_{\alpha\in\Lambda} E_\alpha$ will be defined by $E = \{x = (x_\alpha) \in F : x_\alpha = f_{\alpha\beta}(x_\beta), if \alpha \leq \beta\}$. This algebra is called projective limit of projective system of $\{(E_\alpha, f_{\alpha\beta})\}$ that we denote it by $E = \varprojlim(E_\alpha, f_{\alpha\beta})$. If f_α is the restriction map of projection map π_α on E then $f_\alpha = f_{\alpha\beta} \circ f_\beta$ for all $\alpha \leq \beta$ that induce projective topology on E (For further information see [9]).

In the special case when $\{E_n\}_{n\in\mathbb{N}}$ is a sequence of Banach algebras with $E_n \supseteq E_{n+1}$ for all $n \in \mathbb{N}$, then by $f_{nm} : E_m \to E_n$, $(n \leq m)$ as inclusion maps, $\{(E_n, f_{nm})\}$ is projective system and $E = \varprojlim(E_n, f_{nm})$ is a Fréchet algebra [9, P. 84]. It can be easily seen that E is isomorphic to $\cap_{n\in\mathbb{N}} E_n$.

Proposition 1.2. *Let $\mathcal{A}$ be a Fréchet algebra and let $\{(E_n, f_{nm})\}$ be projective system of Fréchet algebra such that E_n is Fréchet $\mathcal{A}$-mod for all $n \in \mathbb{N}$, then by pointwise product $E = \varprojlim(E_n, f_{nm})$ is Fréchet $\mathcal{A}$-mod.*

Proof. Since for all $m \in \mathbb{N}, a \in \mathcal{A}$ and $x = (x_n) \in E$ we have $f_m(a \cdot x) = f_m((a \cdot x_n)) = a \cdot f_m(x) = a \cdot x_m$ by [12, Theorem 5.2] the module product is continuous. □

Corollary 1.1. *Let $\mathcal{A}$ be a Banach algebra and let $\{E_n\}_{n\in\mathbb{N}}$ be a family of decreasing Banach $\mathcal{A}$-mod then $\cap_{n\in\mathbb{N}} E_n$ is Fréchet $\mathcal{A}$-mod.*

Suppose that $\{\omega_n\}_{n\in\mathbb{N}}$ is a decreasing sequence of weight functions with $\omega_n(x) > 1$ for all $x \in G$ and $n \in \mathbb{N}$, similarly to the Lemma 1.2 we can show that $L^\infty(G, \omega_n^{-1}), L_0^\infty(G, \omega_n^{-1})$ and $C_0(G, \omega_n^{-1})$ are $L^1(G, \omega_1)$-mod for all $n \in \mathbb{N}$, so by Corollary 1.1 Fréchte spaces $\cap_{n\in\mathbb{N}} L^\infty(G, \omega_n^{-1}), \cap_{n\in\mathbb{N}} L_0^\infty(G, \omega_n^{-1})$ and $\cap_{n\in\mathbb{N}} C_0(G, \omega_n^{-1})$ as constructed above are $L^1(G, \omega)$-mod.

2. **Projectivity**

Let $\mathcal{A}$ be a Banach algebra and X a Fréchet $\mathcal{A}$-mod then for $\mathcal{A}^\sharp$, the unit linked of Banach algebra $\mathcal{A}$ we can consider X as Fréchet $\mathcal{A}^\sharp$-mod by the action $(a, \lambda) \cdot x := a \cdot x + \lambda x$, $a \in \mathcal{A}, x \in X$ and $\lambda \in \mathbb{C}$.

Definition 2.1. *Let $\mathcal{A}$ be a Banach algebra then a Fréchet $\mathcal{A}$-mod X is called projective if the product map $\pi_X : \mathcal{A}^\sharp\hat{\otimes}X \to X$ which is defined by $\pi_X(a\otimes x) = a\cdot x, (a \in \mathcal{A}^\sharp, x \in X)$ has right inverse $\mathcal{A}$-module morphism.*

Next proposition and its proof is similar to [6, Proposition IV. 4.4].

Proposition 2.1. *Let $\mathcal{A}$ be a Banach algebra and let X be a projective Fréchet $\mathcal{A}$-mod and at least one of the spaces $\mathcal{A}, X$ have the approximation property then for any $0 \neq x \in X$ there exists a left $\mathcal{A}$-module morphism $\psi : X \to \mathcal{A}^\sharp$ such that $\psi(x) \neq 0$.*

Proof. Let $\rho : X \to \mathcal{A}^\sharp\hat{\otimes}X$ be as in the above definition, so $\pi_X \circ \rho = id_X$ and consider the map $id_{\mathcal{A}^\sharp} \otimes f : \mathcal{A}^\sharp\hat{\otimes}X \to \mathcal{A}^\sharp\hat{\otimes}\mathbb{C}$, for a functional f on X. Since $x \neq 0$ then $\pi_X \circ \rho(x) \neq 0$, so $\rho(x) \neq 0$. Because of approximation property $\mathcal{A}$ or X by [10, Lemma 1.11] there are f and g in the dual space of X and $\mathcal{A}^\sharp$ respectively with respect to the strong topology such that $(g \otimes f)(\rho(x)) \neq 0$, so $(id_{\mathcal{A}^\sharp} \otimes f)\rho(x) \neq 0$. Let $\psi = (id_{\mathcal{A}^\sharp} \otimes f)\rho : X \to \mathcal{A}^\sharp\hat{\otimes}\mathbb{C}$, therefore $\psi(x) \neq 0$. it is easy to see that ψ is left $\mathcal{A}$-module morphism and by applying $\mathcal{A}^\sharp\hat{\otimes}\mathbb{C} \cong \mathcal{A}^\sharp$, we get the result. □

Theorem 2.1. *Let G be locally compact group and E be projective Fréchet $L^1(G,\omega)$-mod which satisfy $C_c(G) \subseteq E$ then G is compact.*

Proof. Toward to the second part of the proof [4, Theorem 3.1] we can find a $f \in C_c(G)$ such that $0 \neq f \cdot f$, so by the above proposition there exists an $\mathcal{A}$-module morphism from E into $\mathcal{A}^\sharp$ such that $0 \neq T(f \cdot f)$, hence by following the exactly method that has been used in the proof of [4, Theorem 3.1] the result follows. □

Corollary 2.1. *Let G be a locally compact group, then $\cap_{n\in\mathbb{N}}C_0(G,\omega_n^{-1})$ is projective as $L^1(G,\omega_1)$-mod if and only if G is compact.*

Proof. Suppose that $\cap_{n\in\mathbb{N}}C_0(G,\omega_n^{-1})$ is projective then by the above theorem G is compact. Now let G be compact since in this situation $\cap_{n\in\mathbb{N}}C_0(G,\omega_n^{-1}) = C_0(G)$ so by [4, Theorem 3.1] $\cap_{n\in\mathbb{N}}C_0(G,\omega_n^{-1})$ is projective. □

Corollary 2.2. *Let G be a locally compact group, then $\cap_{n\in\mathbb{N}}L^\infty(G,\omega_n^{-1})$ is projective as $L^1(G,\omega_1)$-mod if and only if G is finite.*

Proof. Let G be finite then for $n \in \mathbb{N}$, $L^\infty(G,\omega_n^{-1}) = L^\infty(G)$ so $\cap_{n\in\mathbb{N}}L^\infty(G,\omega_n^{-1})$ is projective by [4, Theorem 3.3].

Suppose that $\cap_{n\in\mathbb{N}}L^\infty(G,\omega_n^{-1})$ is projective then by Theorem 2.1 G is compact, so for $n \in \mathbb{N}$, $L^\infty(G,\omega_n^{-1}) = L^\infty(G)$ and therefore $\cap_{n\in\mathbb{N}}L^\infty(G,\omega_n^{-1}) = L^\infty(G)$ so $L^\infty(G)$ is projective and by [4, Theorem 3.3] G is finite. □

Since when G is compact then $\cap_{n\in\mathbb{N}}L_0^\infty(G,\omega_n^{-1}) = \cap_{n\in\mathbb{N}}L^\infty(G,\omega_n^{-1})$, so the above theorem by the same argument is also true for $\cap_{n\in\mathbb{N}}L_0^\infty(G,\omega_n^{-1})$.

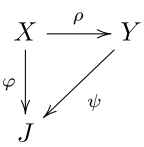
Proposition 2.2. *Let G be a locally compact group and $\omega(x) \geq 1$ for all $x \in G$ then $L^1(G,\omega)$ is projective right $L^1(G,\omega)$-module.*

Proof. By [7, Proposition 1.2] $L^1(G,\omega)\hat{\otimes}L^1(G,\omega)$ is isometric isomorphic to $L^1(G\times G,\omega\times\omega)$ and the rest of proof is similar to the proof of [2, Theorem 3.3.32]. □

3. Injectivity

Definition 3.1. *For Fréchet spaces X and Y, a morphism $\varphi : X \to Y$ is said to be admissible if its kernel complemented in X and its image is closed and complemented in Y.*

Definition 3.2. *Suppose that $\mathcal{A}$ is a Banach algebra, a Fréchet $\mathcal{A}$-mod J is said to be injective if for any admissible monomorphism $\rho : X \to Y$ and any morphism $\varphi : X \to J$ there exists a morphism $\psi : Y \to J$ such that the diagram*



is commutative. Where in the above all spaces are Fréchet $\mathcal{A}$-mod and morphisms are module morphisms.

Theorem 3.1. *Let $\mathcal{A}$ be a Banach algebra and let E be projective limit of a family of injective Banach $\mathcal{A}$-mod $E_n (n \in \mathbb{N})$, then E is injective Fréchet $\mathcal{A}$-mod.*

Proof. Consider Fréchet $\mathcal{A}$-mod X, Y, admissible module monomorphism $\rho : X \to Y$, module morphism $\varphi : X \to E$ and suppose that $f_n : E \to E_n$ is the n'th restriction of the projection map of $\Pi_{n\in\mathbb{N}} E_n$ on projective limit. Since E_n is injective module, for $f_n \circ \varphi : X \to E_n$, there exists a morphism $\psi_n : Y \to E_n$ such that $\psi_n \circ \rho = f_n \circ \varphi$. As ρ is monomorphism so there exists a module morphism ρ' such that $\rho' \circ \rho(x) = x$, for all $x \in X$ and since it is admissible so $Y = Im\rho \oplus Ker\rho'$, where $Im\rho$ is the image of ρ and $Ker\rho'$ is the kernel of ρ', hence we can define $\psi : Y \to E$ by $\psi(y) = (\psi_n(\gamma))$, where $y = \gamma + z$ for some $\gamma \in Im\rho$ and $z \in Ker\rho'$, obviously ψ is a module morphism and

$$\psi \circ \rho(x) = (\psi_n(\rho(x))) = (f_n(\varphi(x))) = \varphi(x),$$

for $x \in E$, so E is injective for all $n \in \mathbb{N}$. □

Corollary 3.1. *Let G be a locally compact group then $\cap_{n\in\mathbb{N}} L^\infty(G, \omega_n^{-1})$ is injective as $L^1(G, \omega_1)$-mod*

Proof. Since for all $n \in \mathbb{N}$ dual of $L^1(G, \omega_n)$ is $L^\infty(G, \omega_n^{-1})$ and by Proposition 2.2 $L^1(G, \omega_n)$ is projective so by [11, Example 5.3.7(b)] $L^\infty(G, \omega_n^{-1})$ is injective and hence by above theorem the result follows. □

For Banach algebra $\mathcal{A}$ and Fréchet $\mathcal{A}$-mod X, consider the embedding map $\Pi : X \to {}_{\mathcal{A}}\mathcal{B}(\mathcal{A}^\sharp, X)$, that is defined by $\Pi(x)(a) = a \cdot x$ for all $a \in \mathcal{A}^\sharp$ and $x \in X$. We know that this map has no left inverse module morphism in the category of non-normable Fréchet modules, but by [5, Lemma 2.1] in the special case when $\mathcal{A}$ is Banach algebra and X is Fréchet space it has left inverse, so by regarding to this fact we can prove the following result.

Theorem 3.2. *Let G be a locally compact group then $\cap_{n\in\mathbb{N}} C_0(G, \omega_n^{-1})$ as an $L^1(G, \omega_1)$-mod is injective if and only if G is finite.*

Proof. Since when G is finite then $\cap_{n\in\mathbb{N}}C_0(G,\omega_n^{-1}) = C_0(G)$ so $\cap_{n\in\mathbb{N}}C_0(G,\omega_n^{-1})$ is clearly injective by [4, Theorem 3.8].

Conversely let $\mathcal{A} = L^1(G,\omega_1)$ and suppose that $\cap_{n\in\mathbb{N}}C_0(G,\omega_n^{-1})$ is injective so the canonical embedding

$$\Pi_1 : \cap_{n\in\mathbb{N}}C_0(G,\omega_n^{-1}) \to {}_{\mathcal{A}}\mathcal{B}(\mathcal{A}^\sharp, \cap_{n\in\mathbb{N}}C_0(G,\omega_n^{-1}))$$

has a left inverse morphism $\rho_1 : {}_{\mathcal{A}}\mathcal{B}(\mathcal{A}^\sharp, \cap_{n\in\mathbb{N}}C_0(G,\omega_n^{-1})) \to \cap_{n\in\mathbb{N}}C_0(G,\omega_n^{-1})$ by [5, Lemma 2.1]. Now Since $\mathcal{A}^\sharp \subseteq L^1(G)^\sharp$ and $C_0(G) \subseteq \cap_{n\in\mathbb{N}}C_0(G,\omega_n^{-1})$, we can define restriction map:

$$\begin{cases} \alpha : {}_{L^1(G)}\mathcal{B}(L^1(G)^\sharp, C_0(G)) & \longrightarrow \quad {}_{\mathcal{A}}\mathcal{B}(\mathcal{A}^\sharp, \cap_{n\in\mathbb{N}}C_0(G,\omega_n^{-1})) \\ \qquad T & \longrightarrow \quad T\big|_{\mathcal{A}^\sharp}. \end{cases}$$

This map by Lemma 1.1 is a module morphism, so the following diagram commutes:

$$\begin{array}{ccc} & & \cap_{n\in\mathbb{N}}C_0(G,\omega_n^{-1}) \\ & \overset{id}{\swarrow} & \uparrow \rho_1 \\ \cap_{n\in\mathbb{N}}C_0(G,\omega_n^{-1}) & \xrightarrow{\Pi_1} & {}_{\mathcal{A}}\mathcal{B}(\mathcal{A}^\sharp, \cap_{n\in\mathbb{N}}C_0(G,\omega_n^{-1})) \\ id \uparrow & & \uparrow \alpha \\ C_0(G) & \xrightarrow[\Pi_2]{} & {}_{L^1(G)}\mathcal{B}(L^1(G)^\sharp, C_0(G)) \end{array}$$

where Π_2 is embedding map. If we define $\rho_2 = \rho_1 \circ \alpha$ then we have:

$$\rho_2 \circ \Pi_2(x) = \rho_1 \circ \alpha \circ \Pi_2(x) = \rho_1 \circ \Pi_1(x) = x,$$

for all $x \in C_0(G)$, so $C_0(G)$ is injective and consequently by [4, Theorem 3.8] G is finite. □

Similarly to the last statement in the above proof if we consider $L_0(G)$ and $\cap_{n\in\mathbb{N}}L_0^\infty(G,\omega_n^{-1})$ instead of $C_0(G)$ and $\cap_{n\in\mathbb{N}}C_0(G,\omega_n^{-1})$ respectively and this fact that when G is compact then $\cap_{n\in\mathbb{N}}L_0^\infty(G,\omega_n^{-1}) = L^\infty(G)$, by [1, Theorem 3.4] we can conclude the following result.

Theorem 3.3. *Let G be a locally compact group then $\cap_{n\in\mathbb{N}}L_0^\infty(G,\omega_n^{-1})$ is an injective $L^1(G,\omega_1)$-mod if and only if G is compact.*

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