WOLFE DUALITY FOR MULTIOBJECTIVE PROGRAMMING PROBLEMS INVOLVING HIGHER-ORDER (Φ,ρ)-V- TYPE I INVEX FUNCTIONS

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A class of functions called higher-order (Φ,ρ)-V-type I invex functions is introduced. Using the assumptions on the functions involved, weak, strong and strict converse duality theorems are established for Wolfe higher-order type multiobjective dual programs in order to relate the efficient solutions of primal and dual problems. Special cases are also discussed to show that the results obtained generalize some existent known papers in the literature.

Keywords: Multiobjective programming, eficiency, higher-order (Φ,ρ)-V-type I invexity, Wolfe type higher-order duality.

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1. Introduction

The study of higher order duality in mathematical programming is significant due to the computational advantage over the first order duality as it provides higher bounds for the value of the objective function of the primal problem when approximations are used, because there are more parameters involved.

Higher-order duality in nonlinear programming has been studied by some researchers, see [1,2,9,11,12,16-21,23-28].

Recently, the concept of first order (Φ,ρ)-invexity for differentiable scalar optimization problems has been introduced by Caristi *et al.* [10] to extend fundamental theoretical results in mathematical programming, while Antczak [6,7] introduced the notion of (Φ,ρ)-V-invexity by combining the concepts of (Φ,ρ)-invexity and of V-invexity.

In [25], Sharma and Gupta proved duality theorems for higher-order Wolfe and Mond-Weir type duals of the vector optimization using the concept of higher-order (Φ,ρ)-V-invex function. In [27], Stancu-Minasian *et al.* introduced the concept of higher-order (Φ,ρ)-V-invexity and presented two types of higher-order dual models for a semi-infinite minimax fractional programming problem. Weak, strong and strict converse duality theorems were discussed under the assumptions of higher-order (Φ,ρ)-V-invexity to establish a relation between the primal and dual problems. In [18], Jayswal *et al.* gave one more generalized dual model for semi-infinite minimax fractionl programming problem involving higher-order (Φ,ρ)-V-invexity and proved duality results.

Singh *et al.* [26] defined a new class of higher order (Φ,ρ)-V-invex functions over cones, named K-higher order (Φ,ρ)-V-invex functions, and formulated two types of higher order dual models for a vector optimization problem over cones containing support functions

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in objectives as well as in constraints. They established several duality results, i.e.,weak and strong duality results.

Mishra *et al.* [23, 24] studied a number of higher-order duals to a nondifferentiable programming problem and established duality results under the higher-order generalized type-I α-invex functions. Kumar *et al.* [22] used the notion of generalized (Φ, ρ)-invexity to establish sufficient optimality conditions and duality results for Wolfe- and Mond-Weir-type duals for a class of interval-valued programming problem where constraints are interval-valued and infinite.

Tripathy [28] introduced K-(Φ, ρ)-convex functions and presented a model of higher order Wolfe type nondifferentiable multiobjective symmetric dual programs and established the weak, strong and converse duality theorems.

Recently, nonconvex optimization problems with so-called (generalized) type I functions have been the object of increasing interest. This class of generalized functions has been introduced in nonlinear scalar optimization problems by Hanson and Mond [15].

Dubey and Mishra [12] presented a higher-order unified dual and established duality results under higher order pseudo quasi/strictly pseudo quasi/weak strictly pseudo quasi-(V, ρ, d)-type-I assumptions for a nondifferentiable multiobjective fractional programming problem in which each component of objective functions contains a term including the support function of a compact convex set.

In [4], Antczak introduced the notions of nondifferentiable (generalized) (Φ, ρ)-type I objective and constraint functions and established sufficient optimality conditions and Mond-Weir duality results for a class of nonconvex nonsmooth multiobjective programming.

But, up to now, there is not literatures dealing with higher-order (Φ, ρ)-V-type I invex functions.

In the present paper, following Stancu-Minasian *et al.* [27] and Sharma and Gupta [25], we introduce the concept of higher-order (Φ, ρ)-V-type I invex functions. Our approach of considering Wolfe higher-order duality for multiobjective programming problem involving higher-order (Φ, ρ)-V-type I invex functions is motivated by Jayswal *et al.*[16] and Sharma and Gupta [25].

The plan of the paper is as follows. In Section 2, we give some notation and definitions used throughout the paper and also we give some new ones. In Section 3, we discuss duality between the primal problem and a higher-order Wolfe type multiobjective program. Special cases are also discussed in Section 4 to show that the results obtained generalize some existent known papers in the literature. Conclusions are given in Section 5.

2. **Notation and Preliminaries**

Let R^n be the n-dimensional Euclidean space and R^n_+ be its nonnegative orthant. The following convention for inequalities will be used throughout the paper. If $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n) \in R^n$, we denote

$$x \leqq y \Leftrightarrow x_i \leqq y_i \quad \forall i = 1, ..., n;$$
$$x \leq y \Leftrightarrow x_i \leqq y_i \ \forall i = 1, ...n \text{ and } x \neq y;$$
$$x < y \Leftrightarrow x_i < y_i \quad \forall i = 1, ..., n;$$

Consider the index sets $K = \{1, \cdots, k\}$ and $M = \{1, \cdots, m\}$. For each $r \in K$ the set $K - \{r\}$ is denoted by K_r.

We consider the following multiobjective programming problem:

$$\text{(P)} \qquad \begin{array}{c} \text{Minimize } f(x) = (f_1(x), \cdots, f_k(x)) \\ \text{subject to } x \in D = \{x \in X : g(x) \leqq 0\} \end{array}$$

where X is an open subset of $R^n, f_i : X \to R, i = 1, \cdots, k$, and $g : X \to R^m$, are differentiable on X.

Definition 2.1 A point $\bar{x} \in D$ is said to be an eficient solution to problem (P) if and only if there exists no other $x \in D$ such that $f(x) \leq f(\bar{x})$.

Definition 2.2 A function $\Phi : X \times X \times R^{n+1} \to R$ is said to be convex in the third argument iff for any fixed $(x,u) \in X \times X$ the inequality

$$\Phi(x,u,\lambda(\xi_1,\rho_1) + (1-\lambda)(\xi_2,\rho_2)) \leqq \lambda\Phi(x,u,(\xi_1,\rho_1)) + \\ +(1-\lambda)\Phi(x,u,(\xi_2,\rho_2)),$$

holds for all $\xi_1, \xi_2 \in R^n,\ \rho_1,\rho_2 \in R$ and for any $\lambda \in [0,1]$.

Now, we give the definition of (strictly) higher-order (Φ,ρ)-V- type I invex function.

Let $f : X \to R^k, g : X \to R^m, h : X \times R^n \to R^k$ and $k : X \times R^n \to R^m$ be differentiable functions. Also, consider the function $\Phi : X \times X \times R^{n+1} \to R$, where $\Phi(x,\bar{x},\cdot)$ is convex on R^{n+1}, $\Phi(x,\bar{x},(0,a)) \geqq 0$ for all $x \in X$ and every $a \in R_+$, the vectors $\alpha = (\alpha_1^1, \cdots, \alpha_k^1, \alpha_1^2, \cdots, \alpha_m^2)$ and $\rho = (\rho_1^1, \cdots, \rho_k^1, \rho_1^2, \cdots, \rho_m^2)$, where $\alpha_i^1, \alpha_j^2 : X \times X \to R_+ \setminus \{0\}$ and $\rho_i^1, \rho_j^2 \in R$ for $i \in K$ and $j \in M$.

Definition 2.3 The pair of functions (f,g) is said to be higher-order (Φ,ρ)-V-type I invex at $\bar{x} \in X$ on X, with respect to (h,k) if

$$f_i(x) - f_i(\bar{x}) - h_i(\bar{x},p) + p'\nabla_p h_i(\bar{x},p) \geqq \\ \geqq \Phi\left(x,\bar{x},\alpha_i^1(x,\bar{x})\left(\nabla f_i(\bar{x}) + \nabla_p h_i(\bar{x},p), \rho_i^1\right)\right), i \in K \tag{2.1}$$

$$-g_j(x) - k_j(\bar{x},p) + p'\nabla_p k_j(\bar{x},p) \\ \geqq \Phi\left(x,\bar{x},\alpha_j^2(x,\bar{x})\left(\nabla g_j(\bar{x}) + \nabla_p k_j(\bar{x},p), \rho_j^2\right)\right), j \in M \tag{2.2}$$

hold for all $(x,p) \in X \times R^n$.

If the pair of functions (f,g) is higher-order (Φ,ρ)-V-type I invex at each $x \in X$, then (f,g) is said to be higher-order (Φ,ρ)-V-type I invex on X.

If in the above definition, the first inequality is satisfied as

$$f_i(x) - f_i(\bar{x}) - h_i(\bar{x},p) + p'\nabla_p h_i(\bar{x},p) > \\ > \Phi\left(x,\bar{x},\alpha_i^1(x,\bar{x})\left(\nabla f_i(\bar{x}) + \nabla_p h_i(\bar{x},p), \rho_i^1\right)\right), i \in K$$

then we say that the pair of functions is strictly higher-order (Φ,ρ)-V-type I invex at $\bar{x}$, with respect to (h,k).

In order to simplify the terminology we will use the shorthands "higher-order (Φ,ρ)-V-type I invex at $\bar{x} \in X$" to mean "higher-order (Φ,ρ)-V-type I invex at $\bar{x} \in X$, with respect to (h,k)".

Remark 2.1 In order to define an analogous class of higher-order (Φ,ρ)-V-type I incave functions, the direction of the inequalities (2.1) and (2.2) should be changed to the opposite one.

In the sequel we need the following results.

Theorem 2.1 *Let x^* be an efficient solution to problem* (P) *at which the Kuhn-Tucker constraint qualification is satisfied. Then there exist $\lambda^* \in R^k$ and $\mu^* \in R^m$ such that*

$$\sum_{i=1}^{k} \lambda_i^* \nabla f_i(x^*) + \sum_{j=1}^{m} \mu_j^* \nabla g_j(x^*) = 0$$

$$\sum_{j=1}^{m} \mu_j^* g_j(x^*) = 0$$

$$\lambda^* \geqq 0, \sum_{i=1}^{k} \lambda_i^* = 1, \mu^* \geqq 0.$$

Lemma 2.1 (Jensen's inequality) Let $X \subseteq R^n$ be a convex set, $f : X \to R$ a convex function on X and let $x^1, \cdots, x^m \in X$.

Then

$$f\left(\sum_{i=1}^{m} \lambda_i x^i\right) \leqq \sum_{i=1}^{m} \lambda_i f\left(x^i\right)$$

for any $\lambda_1, \cdots, \lambda_m \in [0,1]$ satisfying $\sum_{i=1}^{m} \lambda_i = 1$.

3. **Wolfe Type Duality**

In this section, we consider the following Wolfe type higher-order dual to (P) and establish weak, strong and strict converse duality theorems.

$$\text{(WD)}\quad \begin{aligned} &\text{Maximize } \Big(f_1(y) + h_1(y,p) - p'\nabla h_1(y,p) + \\ &+ \sum_{j=1}^{m} \mu_j \left\{g_j(y) + k_j(y,p) - p'\nabla_p k_j(y,p)\right\}, \cdots, \\ &, \cdots, f_k(y) + h_k(y,p) - p'\nabla_p h_k(y,p) + \\ &+ \sum_{j=1}^{m} \mu_j \left\{g_j(y) + k_j(y,p) - p'\nabla_p k_j(y,p)\right\}\Big) \end{aligned}$$

subject to

$$\sum_{i=1}^{k} \lambda_i \left\{\nabla f_i(y) + \nabla_p h_i(y,p)\right\} + \sum_{j=1}^{m} \mu_j \left\{\nabla g_j(y) + \nabla_p k_j(y,p)\right\} = 0 \tag{3.1}$$

$$\lambda \geq 0, \mu \geqq 0 \tag{3.2}$$

where $y, p \in R^n, \lambda \in R^k, \sum_{i=1}^{k} \lambda_i = 1$ and $\mu \in R^m$.

Let W be the set of all feasible solutions of dual problem (WD). Moreover, let $\text{pr}_X W = \{y \in X : (y, \lambda, \mu, p) \in W\}$ be the projection of W on X.

The following theorems generalize Theorems 3.1, 3.3, and 3.4 given by Jayswal *et al.* [16] and Theorems 1-3 given by Sharma and Gupta [25] to higher-order (Φ, ρ)-V-type I invex functions.

In what follows, λf denotes the vector $(\lambda_1 f_1, \cdots, \lambda_k f_k)$. Similarly, μg denotes the vector $(\mu_1 g_1, \cdots, \mu_m g_m)$.

Theorem 3.1 (Weak Duality) *Let x and (y, λ, μ, p) be arbitrary feasible solutions of problems* (P) *and* (WD), *respectively, and assume that*

(i) $(\lambda f, \mu g)$ *is strictly higher-order* (Φ, ρ)-V-*type* I *invex at* y, *on* $D \cup \text{pr}_X W$

(ii) $\sum_{i=1}^{k} \rho_i^1 + \sum_{j=1}^{m} \rho_j^2 \geqq 0$.

Then one cannot have

$$\begin{aligned} f_i(x) \leqq f_i(y) + h_i(y,p) - p'\nabla_p h_i(y,p) \\ + \sum_{j=1}^{m} y_j \left\{g_j(y) + k_j(y,p) - p'\nabla_p k_j(y,p)\right\} \text{ for all } i \in K \end{aligned} \tag{3.3}$$

$$\begin{aligned} f_r(x) < f_r(y) + h_r(y,p) - p'\nabla_p h_r(y,p) \\ + \sum_{j=1}^{m} y_j \left\{g_j(y) + k_j(y,p) - p'\nabla_p k_j(y,p)\right\} \text{ for at least } r \in K_i. \end{aligned} \tag{3.4}$$

Proof. Suppose to the contrary that there exist feasible solutions such that inequalities (3.3) and (3.4) hold. Multiplying by λ_i the relations (3.3) and (3.4) and summing, since $\lambda_i \geqq 0$ and $\sum_{i=1}^{k} \lambda_i = 1$, we obtain

$$\begin{aligned}\sum_{i=1}^{k} \lambda_i f_i(x) \leqq \sum_{i=1}^{k} \lambda_i f_i(y) + \sum_{i=1}^{k} \lambda_i h_i(y,p) - \sum_{i=1}^{k} p' \nabla_p \lambda_i h_i(y,p) + \\ + \sum_{j=1}^{m} \mu_j \left\{ g_j(y) + k_j(y,p) - p' \nabla_p k_j(y,p) \right\}.\end{aligned} \tag{3.5}$$

By assumption (i), we have

$$\begin{aligned}\lambda_i f_i(x) - \lambda_i f_i(y) - \lambda_i h_i(y,p) + p' \nabla_p \lambda_i h_i(y,p) \\ > \Phi\left(x, y, \alpha_i^1(x,y)\left(\nabla \lambda_i f_i(y) + \nabla_p \lambda_i h_i(y,p), \rho_i^1\right)\right),\ i \in K,\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}-\mu_j g_j(y) - \mu_j k_j(y,p) + p' \nabla_p \mu_j k_j(y,p) \\ \geqq \Phi\left(x, y, \alpha_j^2(x,y)\left(\nabla \mu_j g_j(y) + \nabla_p \mu_j k_j(y,p), \rho_j^2\right)\right),\ j \in M.\end{aligned} \tag{3.7}$$

Denote

$$A = \sum_{i=1}^{k} \frac{1}{\alpha_i^1(x,y)} + \sum_{j=1}^{m} \frac{1}{\alpha_j^2(x,y)},$$

$$\tilde{\alpha}_i^1 = \frac{1}{\alpha_i^1(x,y) A}, i \in K \text{ and } \tilde{\alpha}_j^2 = \frac{1}{\alpha_j^2(x,y) A},\ j \in M. \tag{3.8}$$

From (3.8), we see that $0 < \tilde{\alpha}_i^1(x,y) < 1,\ i \in K,\ 0 < \tilde{\alpha}_j^2(x,y) < 1,\ j \in M$ and

$$\sum_{i=1}^{k} \tilde{\alpha}_i^1(x,y) + \sum_{j=1}^{m} \tilde{\alpha}_j^2(x,y) = 1.$$

Adding the two inequalities (3.6) and (3.7) after multiplying (3.6) by $\tilde{\alpha}_i^1(x,y)$ and (3.7) by $\tilde{\alpha}_j^2(x,y)$ and using convexity of $\Phi(x, y, (\cdot,\cdot))$, we get

$$\begin{aligned}&\sum_{i=1}^{k} \tilde{\alpha}_i^1(x,y)\left[\lambda_i f_i(x) - \lambda_i f_i(y) - \lambda_i h_i(y,p) + p' \nabla_p \lambda_i h_i(y,p)\right] + \\ &+ \sum_{j=1}^{m} \tilde{\alpha}_j^2(x,y)\left[-\mu_j g_j(y) - \mu_k k_j(y,p) + p' \nabla_p \mu_j k_j(y,p)\right] \\ &> \Phi\left(x, y, \frac{1}{\sum_{i=1}^{k} \frac{1}{\alpha_i^1(x,y)} + \sum_{j=1}^{m} \frac{1}{\alpha_j^2(x,y)}}\left(\sum_{i=1}^{k}\left(\nabla \lambda_i f_i(y) + \nabla_p \lambda_i h_i(y,p)\right) + \right.\right. \\ &\left.\left.\sum_{j=1}^{m}\left(\nabla \mu_j g_j(y) + \nabla_p \mu_j k_j(y,p)\right), \sum_{i=1}^{k} \rho_i^1 + \sum_{j=1}^{m} \rho_j^2\right)\right) \quad \text{by (3.1)} \\ &= \Phi\left(x, y, \frac{1}{A}\left(0, \sum_{i=1}^{k} \rho_i^1 + \sum_{j=1}^{m} \rho_j^2\right)\right) > 0\end{aligned}$$

because $\Phi(x, y, (0,a)) \geqq 0$ for each $a \in R_+$ and from hypothesis (ii).

Hence

$$\sum_{i=1}^{k} \tilde{\alpha}_i^1(x,y)\left[\lambda_i f_i(x) - \lambda_i f_i(y) - \lambda_i h_i(y,p) + p' \nabla_p \lambda_i h_i(y,p)\right] +$$

$$+\sum_{j=1}^{m}\tilde{\alpha}_j^2(x,y)\left[-\mu_j g_j(y)-\mu_k k_j(y,p)+p'\nabla_p\mu_j k_j(y,p)\right]>0.$$

Finally, in the above inequality we take $\tilde{\alpha}_i^1(x,y)=\tilde{\alpha}_j^2(x,y)=\alpha(x,y)>0$ and obtain

$$\sum_{i=1}^{k}\lambda_i f_i(x)>\sum_{i=1}^{k}\left[\lambda_i f_i(y)+\lambda_i h_i(y,p)-p'\nabla_p\lambda_i h_i(y,p)\right]+$$
$$+\sum_{j=1}^{m}\left[\mu_j g_j(y)+\mu_j k_i(y,p)-p'\nabla_p\mu_j k_j(y,p)\right]$$

which contradicts (3.5). Hence (3.3) and (3.4) cannot hold.

Theorem 3.2 (Strong Duality). *Let x^* be an efficient solution to* (P) *at which the Kuhn-Tucker constraint qualification are satisfied on X. Also, if*

$$h(x^*,0)=0,\ k(x^*,0)=0,\ \nabla_p h(x^*,0)=0,\ \nabla_p k(x^*,0)=0, \tag{3.9}$$

then there exist $\lambda^*\in R^k,\mu^*\in R^m$ and $p^*\in R^n$ such that $(x^*,\lambda^*,\mu^*,p^*=0)$ is feasible for (WD) *and the corresponding objective values of* (P) *and* (WD) *are equal. Further, if the conditions of the weak duality Theorem* 3.1 *also hold for all feasible solution of* (WD), *then $(x^*,\lambda^*,\mu^*,p^*=0)$ is an efficient solution to* (WD).

Proof. Since x^* is an efficient solution to (P) and the Kuhn-Tucker constraint qualification is satisfied, from Theorem 2.1 there exist $\lambda^*\in R^k$ and $\mu^*\in R^m$ such that

$$\sum_{i=1}^{k}\lambda_i^*\nabla f_i(x^*)+\sum_{j=1}^{m}\mu_j^*\nabla g_j(x^*)=0,$$

$$\sum_{j=1}^{m}\mu_j^* g_j(x^*)=0,$$

$$\lambda^*\geqq 0,\sum_{i=1}^{k}\lambda_i^*=1,\mu^*\geqq 0,$$

which by equations 3.9 gives that $(x^*,\lambda^*,\mu^*,p^*=0)$ is feasible for (WD) and the two objectives are equal. Efficiency of $(x^*,\lambda^*,\mu^*,p^*=0)$ for (WD) can be developed by establishing a contradiction with the help of the equation $\sum_{j=1}^{m}\mu_j^* g_j(x^*)=0$, and the weak duality theorem.

Theorem 3.3 (Strict Converse Duality) *Let $\bar{x}$ and $\left(\bar{y},\bar{\lambda},\bar{\mu},\bar{p}\right)$ be eficient solutions to* (P) *and* (WD), *respectively. Assume that*

(i) $\sum_{i=1}^{k}\bar{\lambda}_i f_i(\bar{x})\leqq\sum_{i=1}^{k}\bar{\lambda}_i\left\{f_i(\bar{y})+h_i(\bar{y},\bar{p})-\bar{p}'\nabla_p h_i(\bar{y},\bar{p})\right\}+$
$+\sum_{j=1}^{m}\bar{\mu}_j\left\{g_j(\bar{y})+k_j(\bar{y},\bar{p})-\bar{p}'\nabla_p h_i(\bar{y},\bar{p})\right\},$

(ii) $(\lambda f,\mu g)$ *is strictly higher-order (Φ,ρ)-V-type I at $\bar{y}$, with $\alpha_i^1(\bar{x},\bar{y})=$ $=\alpha_j^2(\bar{x},\bar{y})=\bar{\alpha}(\bar{x},\bar{y})$ for all $i\in K$ and $j\in M$,*

(iii) $\sum_{i=1}^{k}\rho_i^1+\sum_{j=1}^{m}\rho_j^2\geqq 0.$

Then $\bar{x}=\bar{y}$.

Proof. Suppose, contrary to the result that $\bar{x}\neq\bar{y}$. The proof runs on the same lines as the proof of Theorem 3.1 and is hence omitted.

4. **Special cases**

(i) If $h_i(\bar{x}, p) = 0$ and $k_j(\bar{x}, p) = 0$ for $i \in K,\ j \in M$,

$$\Phi\left(x, \bar{x}, \alpha_i^1(x, \bar{x})\left(\nabla f_i(\bar{x}), \rho_i^1\right)\right) = F(x, \bar{x}, \alpha_i^1(x, \bar{x})\nabla f_i(\bar{x})) + \rho_i^1 d^2(x, \bar{x})$$

and

$$\Phi(x, \bar{x}, \alpha_j^2(x, \bar{x})\left(\nabla g_j(\bar{x}), \rho_j^2\right)) = F(x, \bar{x}, \alpha_j^2(x, \bar{x})\nabla g_j(\bar{x})) + \rho_j^2 d^2(x, \bar{x})$$

where $F : X \times X \times R^n \to R$ is a sublinear function in the third argument and $d(\cdot, \cdot) : X \times X \to R$, then the Definition 2.3 becomes that of (F, α, ρ, d)-type I function as defined by Hachimi and Aghezzaf [15].

(ii) If $h_i(x, \cdot) = 0$ and $k_j(x, \cdot) = 0, \alpha_i^1(x, \bar{x}) = 1, i \in K, \alpha_j^2(x, \bar{x}) = 1, j \in M$ and the functions f and g are nondifferentiable, then the Definition 2.3 becomes that of nonsmooth (Φ, ρ)-V-type I objective and constraint functions as defined by Antczak [4].

(iii) If

$$\Phi\left(x, \bar{x}, \alpha_i^1(x, \bar{x})\left(\nabla f_i(\bar{x}), \rho_i^1\right)\right) = F(x, \bar{x}, \alpha_i^1(x, \bar{x})(\nabla f_i(\bar{x}) + \nabla_p h_i(\bar{x}, p))) + \rho_i^1 d^2(x, \bar{x})$$

and

$$\Phi(x, \bar{x}, \alpha_j^2(x, \bar{x})\left(\nabla g_j(\bar{x}), \rho_j^2\right)) = F(x, \bar{x}, \alpha_j^2(x, \bar{x})(\nabla g_j(\bar{x}) + \nabla_p k_j(\bar{x}, p))) + \rho_j^2 d^2(x, \bar{x}),$$

then the Definition 2.3 becomes that of higher-order (F, α, ρ, d)-V-type I function as defined by Jayswal *et al.* [16].

5. **Conclusions**

In this paper, we have introduced the concept of (strictly) higher-order (Φ, ρ)-V-type I invexity. Using the assumptions on the functions involved, weak, strong and strict converse duality theorems are established for Wolfe higher-order type multiobjective dual programs in order to relate the efficient solutions of primal and dual problems. Special cases are also discussed to show that the results obtained generalize some existent known papers in the literature. The methods used here can be extended to the study of nonsmooth variational and nonsmooth control problems which will orient the future research of the authors.

REFERENCES

[1] *I. Ahmad*, Higher-order duality in nondifferentiable minimax fractional programming involving generalized convexity, J. Inequal. Appl., 2012:306, 8 pages.

[2] *I. Ahmad, Z. Husain and S. Sharma*, Higher-order duality in nondifferentiable minimax programming with generalized type I functions, J. Optim. Theory Appl., 141(2009), 1-12.

[3] *T. Antczak*. Nonsmooth minimax programming under locally Lipschitz (Φ, ρ)-invexity, Appl. Math. Comput., 217(2011), 9607-9624.

[4] *T. Antczak,* Nondifferentiable (Φ, ρ)-type I and generalized (Φ, ρ)-type I finctions in nonsmooth vector optimization, J. Appl. Anal., 19(2013), 247-270.

[5] *T. Antczak,* Saddle point criteria and Wolfe duality in nonsmooth (Φ, ρ)-invex vector optimization problems with inequality and equality constraints, Int. J. Comput. Math., 92(2015), 1-26.

[6] *T. Antczak,* Sufficient optimality conditions for semi-infinite multiobjective fractional programming under (Φ, ρ)-V-invexity and generalized (Φ, ρ)-V-invexity, Filomat, 30(2016), 3649-3665.

[7] *T. Antczak,* Semi-infinite minimax fractional programming under (Φ, ρ)-V-invexity and generalized (Φ, ρ)-V-invexity. Optimality, Int. J. Oper. Res., 31(2018), 164-197.

[8] *T. Antczak and G. J. Zalmai*, Second order (Φ, ρ)-V-invexity and duality for semi-infinite minimax fractional programming, Appl. Math. Comput., 227(2014), 831-856.

[9] *M. Bhatia,* Higher-order duality in vector optimization over cones, Optim. Lett., 6(2012), 17-30.

[10] *G. Caristi, M. Ferrara and A. Ştefănescu*, Mathematical programming with (Φ, ρ)-invexity, in: V. I. Konnov, D. T. Luc, A. M. Rubinov (Eds.), Generalized Convexity and Related Topics, Lecture Notes in Economics and Mathematical Systems, vol. 583, Springer, Berlin, Heidelberg, New York, 2007, pp. 167-176.

[11] *X. Chen,* Higher-order symmetric duality in nondifferentiable multiobjective programming problems, J. Math. Anal. Appl., 290(2004), 423-435.

[12] *R. Dubey and V.N. Mishra*, Nondifferentiable higher-order duality theorems for new type of dual model under generalized functions, Proyecciones (Antofagasta), 39(2020), 15-29.

[13] *M. Ferrara and M.V. Ştefănescu,* Optimality conditions and duality in multiobjective programming with (Φ,ρ)-invexity, Yugoslav J. Oper. Res., 18(2008), 153-165.

[14] *M. Hachimi and B. Aghezzaf,* Sufficiency and duality in differentiable multiobjective programming involving generalized type I functions, J. Math. Anal. Appl., 296(2004), 382-392.

[15] *M.A. Hanson and B. Mond*, Necessary and sufficient conditions in constrained optimization, Math. Program., 37(1987),51-58.

[16] *A. Jayswal, I.M. Stancu-Minasian and D. Kumar,* Higher-order duality for multiobjective programming problems involving (F,α,ρ,d)-V-type I functions, J. Math. Model. Algorithms Oper. Res., 13(2014), 125-141.

[17] *A. Jayswal and Ioan Stancu-Minasian*, Higher-order duality for nondifferentiable minimax programming problem with generalized convexity, Nonlinear Anal. Theory Methods Appl., Ser.A, Theory Methods, 74(2011), 616-625.

[18] *A. Jayswal, I. M. Stancu-Minasian and A. M. Stancu*, Generalized duality for semi-infinite minimax fractional programming problem involving higher-order (Φ,ρ)-V-invexity, U.P.B. Sci. Bull., Series A, 80(2018), 29-38.

[19] *C. Y. Jin and C.-Z. Cheng*, Higher-order duality for minimax fractional type programming involving symmetric matrices, Appl. Math., Irvine, 2(2011), 1387-1392.

[20] *M. Kapoor, S.K. Suneja and M.B. Grover,* Higher order optimality and duality in fractional vector optimization aver cones, Tamkang J. Math., 48(2017), 273-287.

[21] *D.S. Kim, H.S. Kang, Y.J. Lee and Y.Y. Seo,* Higher order duality in multiobjective programming with cone constraints, Optimization 59(2010), 29-43.

[22] *P. Kumar, B. Sharma and J. Dagar*, Interval-valued programming problem with infinite constraints, J. Oper. Res. Soc. China, 6(2018), 611-626.

[23] *S.K Mishra, S.Y. Wang and K.K. Lai*, Higher-order duality for a class of nondifferentiable multiobjective programming problems, Int. J. Pure Appl. Math., 11(2004), 217-227.

[24] *S.K. Mishra, J. S. Rautela and R.P. Pant*, Higher-order type-I α-invexity and duality in nondifferentiable mathematical programming, J. Interdiscip. Math., 11(2008), 787-801.

[25] *S. Sharma and S.K. Gupta,* Higher-order (Φ,ρ)-V-invexity and duality for vector optimization problems, Rend. Circ. Mat. Palermo, II. Ser.2, 65(2016), 351-364.

[26] *V. Singh, A. Jayswal, S. Al-Homidan and I. Ahmad,* Higher order duality for cone vector optimization problems, Asian- Eur. J. Math., 13(2020), 15 pages.

[27] *I.M. Stancu-Minasian, K. Kummari and A. Jayswal*, Duality for semi-infinite minimax fractional programming problem involving higher-order (Φ,ρ)-V-invexity, Numer. Func. Anal. Optim., 38(2017), 926-950.

[28] *A.K. Tripathy*, Higher order non-differentiable multi-objective symmetric duality involving generalized K-(Φ,ρ)-convex functions, Electron. J. Math. Analysis. Appl., 7(2019), 314-330.

[29] *R.U. Verma,* Semi-infinite fractional programming, Infosys Science Foundation Series in Mathematical Sciences, Springer-Verlag, Singapore XI+291 pages, 2017.

[30] *G. J. Zalmai and Q. Zhang*, Global parametric sufficient optimality conditions for semi-infinite discrete minmax fractional programming problems involving generalized (η,ρ)-invex functions, Acta Math. Appl. Sinica, 23 (2007), 217-234.

[31] *G. J. Zalmai and Q. Zhang,* Parametric duality models for semi-infinite multiobjective fractional programming problems containing generalized (α,η,ρ)-V-invex functions, Acta Math. Appl. Sinica, 29 (2013), 225-240.

[32] *G. J. Zalmai and Q. Zhang*, Parametric duality models for semi-infinite discrete minimax fractional programming problems containing generalized (α,η,ρ)-V-invex functions, Global J. Math. Math. Sci., 1 (2005), 39-58.

[33] *J. Zhang*, Generalized convexity and higher order duality for mathematical programming problem. Ph.D. Thesis, La Trobe University, Australia, 1998.