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STRONG CONVERGENCE OF AN EXTRAGRADIENT ALGORITHM FOR VARIATIONAL INEQUALITY AND FIXED POINT PROBLEMS

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In this paper, we consider the fixed point and variational inequality problems in Hilbert spaces. We suggest an extragradient algorithm for finding a common element of the set of fixed points of a pseudocontractive operator and the set of solutions of the variational inequality problem. Strong convergence of the proposed algorithm is proved.

Keywords: Extragradient algorithm, fixed point; variational inequality, pseudocontraction.

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1. Introduction

Let H denote a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\|\cdot\|$. Let C be a nonempty subset of H. Let $A\colon C \to H$ be a nonlinear operator. Let us consider the variational inequality

$$\text{find } \tilde{u} \in C \text{ such that } \langle A\tilde{u}, u - \tilde{u} \rangle \geq 0, \ \ \forall u \in C. \tag{1}$$

Denote the solution set of variational inequality (1) by $VI(C, A)$.

Assume that the following conditions are fulfilled:

(i) a set $C \subset H$ is convex and closed;
(ii) the operator A is monotone and Lipschitz with a constant $\kappa > 0$;
(iii) $VI(C, A) \neq \emptyset$.

Many practical problems in mathematical physics and operations research can be modelled as variational inequalities. A great number of algorithms have been proposed for solving variational inequality (1), see, e.g., [2, 8, 9, 11, 16, 18, 19, 20, 25, 29, 33, 34, 37, 41]. In particular, gradient algorithms are offered to date to solve them. For fixed initial value x_0, compute the sequence $\{x_n\}$ iteratively by

$$x_{n+1} = \text{proj}_C[x_n - \lambda A x_n], \ n \geq 0, \tag{2}$$

where $\text{proj}_C\colon H \to C$ denotes the orthogonal projection and $\lambda > 0$ is a constant.

Note that for the convergence of the gradient algorithm (2), some strengthened monotonicity assumptions (strongly monotone or inverse strongly monotone) should be fulfilled ([45]). In order to overcome this restrictive condition, Korpelevich [14] introduced the so-called extragradient algorithm for solving variational inequality (1). For fixed initial value

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x_0, compute the sequence $\{x_n\}$ iteratively by

$$\begin{cases} y_n = \text{proj}_C[x_n - \lambda A x_n], \\ x_{n+1} = \text{proj}_C[x_n - \lambda A y_n], \ n \geq 0, \end{cases} \tag{3}$$

where $\lambda \in (0, 1/\kappa)$.

In Korpelevich's algorithm (3), one has to compute projections twice. Korpelevich's algorithm (3) has received so much attention by a range of scholars, who improved it in several ways; see, e.g., [5, 6, 10, 31, 38, 39]. Especially, Tseng [28] proposed the following modified extragradient algorithm for solving variational inequality (1): for fixed initial value x_0, compute the sequence $\{x_n\}$ iteratively by

$$\begin{cases} y_n = \text{proj}_C[x_n - \lambda A x_n], \\ x_{n+1} = y_n + \lambda(A x_n - A y_n), \ n \geq 0, \end{cases} \tag{4}$$

where $\lambda \in (0, 1/\kappa)$.

Obviously, in Tseng's algorithm (4), we only need to compute one projection. But, we have to compute the values of A at two different points. Recently, Malitsky [17] suggested the following iteration for solving variational inequality (1): for fixed initial value x_0, compute the sequence $\{x_n\}$ iteratively by

$$x_{n+1} = \text{proj}_C[x_n - \lambda A(2x_n - x_{n-1})], \ n \geq 0, \tag{5}$$

where $\lambda \in (0, \frac{\sqrt{2}-1}{\kappa})$.

Remark 1.1. In Malitsky's algorithm (5), we need to compute one projection and one values of A in each iterative step.

Observe that we need to estimate the Lipschitz constant κ of operator A in algorithms (3), (4) and (5). In order to overcome this drawback, Armijo-like search rule was used in iterative algorithms, see, for instance, [12, 13, 15, 27].

On the other hand, iterative computation of fixed points of nonlinear operators has been growing interest due to its applications in science and engineering. Iterative methods for finding fixed points of nonexpansive operators and pseudocontractive operators have received vast investigation, see, e.g., [3, 4, 7, 21, 22, 23, 24, 26, 30, 35, 40, 42, 43]. Our another main purpose of the present paper is to find the fixed points of pseudocontractive operators.

Very recently, Yao, Postolache and Yao [46] presented an iterative algorithm for finding a common element of the set of fixed points of a pseudocontractive operator and the set of solutions of the variational inequality problem and weak convergence analysis is given.

Motivated and inspired by the works in this field, the purpose of this paper is to find a common element of the set of fixed points of a pseudocontractive operator and the set of solutions of the variational inequality problem (1). We suggest an extragradient algorithm and strong convergence of the proposed algorithm is proved.

2. **Preliminaries**

Let C be a nonempty closed convex subset of a real Hilbert space H. Recall that an operator $S\colon C \to C$ is said to be

(i) ρ-Lipschitzian if $\|S(u) - S(u^\dagger)\| \leq \rho\|u - u^\dagger\|, \forall u, u^\dagger \in C$, where $\rho > 0$ is a constant. If $\rho = 1$, S is called nonexpansive.

(ii) pseudocontractive if $\|Su - Su^\ddagger\|^2 \leq \|u - u^\ddagger\|^2 + \|(I - S)u - (I - S)u^\ddagger\|^2, \forall u, u^\ddagger \in C$.

An operator $B\colon H \to H$ is said to be strongly positive, if there is a constant $\xi > 0$ with the property

$$\langle Bu, u\rangle \geq \xi\|u\|^2, \ \forall u \in H.$$

In a real Hilbert space H, there holds the following equality

$$\|\varsigma z + (1-\varsigma)z^\dagger\|^2 = \varsigma\|z\|^2 + (1-\varsigma)\|z^\dagger\|^2 - \varsigma(1-\varsigma)\|z - z^\dagger\|^2, \tag{6}$$

$\forall z, z^\dagger \in H$ and $\forall \varsigma \in [0,1]$.

For fixed $z \in H$, there exists a unique $z^\dagger \in C$ satisfying

$$\|z - z^\dagger\| = \inf\{\|z - \tilde{z}\| : \tilde{z} \in C\}.$$

Denote $z^\dagger$ by $\mathrm{proj}_C[z]$.

The following inequality is an important property of projection proj_C([36]): for given $x \in H$,

$$\langle x - \mathrm{proj}_C[x], y - \mathrm{proj}_C[x]\rangle \leq 0, \ \forall y \in C. \tag{7}$$

In what follows, we shall use the following expressions:

- $u_n \rightharpoonup z^\dagger$ denotes the weak convergence of u_n to $z^\dagger$.
- $u_n \to z^\dagger$ stands for the strong convergence of u_n to $z^\dagger$.
- $\mathrm{Fix}(S)$ means the set of fixed points of S.
- $w_\omega(u_n) = \{u^\dagger : \exists\{u_{n_i}\} \subset \{u_n\}$ such that $u_{n_i} \rightharpoonup u^\dagger (i \to \infty)\}$.

Lemma 2.1 ([32]). *Let C a nonempty closed convex subset of a real Hilbert space H. Let $S\colon C \to C$ be an L-Lipschitz pseudocontractive operator. Let δ be a constant such that $0 < \delta < \frac{1}{\sqrt{1+L^2}+1}$. Then,*

$$\|S((1-\delta)\tilde{u} + \delta S\tilde{u}) - u^\dagger\|^2 \leq \|\tilde{u} - u^\dagger\|^2 + (1-\delta)\|S((1-\delta)\tilde{u} + \delta S\tilde{u}) - \tilde{u}\|^2,$$

for all $\tilde{u} \in C$ and $u^\dagger \in Fix(S)$.

Lemma 2.2 ([44]). *Let H be a real Hilbert space, C a closed convex subset of H. Let $S\colon C \to C$ be a continuous pseudocontractive operator. Then*

(i) $\mathrm{Fix}(S)$ *is a closed convex subset of C,*

(ii) *S is demi-closed, i.e., $u_n \rightharpoonup \tilde{u}$ and $S(u_n) \to u^\dagger$ imply that $S(\tilde{u}) = u^\dagger$*

Lemma 2.3 ([30]). *Suppose $\{\varpi_n\} \subset [0,\infty)$, $\{\mu_n\} \subset (0,1)$ and $\{\varrho_n\}$ are three real number sequences satisfying*

(i) $\varpi_{n+1} \leq (1-\mu_n)\varpi_n + \varrho_n, \ \forall n \geq 1$;

(ii) $\sum_{n=1}^{\infty} \mu_n = \infty$;

(iii) $\limsup_{n\to\infty} \frac{\varrho_n}{\mu_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\varrho_n| < \infty$.

Then $\lim_{n\to\infty} \varpi_n = 0$.

3. Main results

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A\colon C \to H$ be a κ-Lipschitz continuous and monotone operator. Let $S\colon C \to C$ be an L-Lipschitz pseudocontractive operator with $L > 1$. Let $B\colon C \to H$ be a strongly positive linear bounded operator with coefficient $\xi > 0$. Let $f\colon C \to H$ be a ρ-contractive operator.

Next, we present our procedure for solving monotone variational inequality (1) and fixed point problem of Lipschitz pseudocontractive operator S.

Algorithm 3.1. *Let $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\delta_n\}_{n=0}^{\infty}$ be three sequences in $(0,1)$. Let $\mu > 0$, $\gamma > 0$, $\lambda \in (0,1)$ and $\tau \in (0,1)$ be four constants.*

Step 1. *Set an initial guess $x_0 \in C$ arbitrarily and let $n = 0$.*

Step 2. *For given current sequence x_n, compute the iterates y_n and z_n via the following form*

$$y_n = \mathrm{proj}_C[x_n - \mu\lambda^{\sigma_n} A x_n], \tag{8}$$

and

$$z_n = \mathrm{proj}_C[y_n - \mu\lambda^{\sigma_n}(Ay_n - Ax_n)], \tag{9}$$

where $\sigma_n \in \{0, 1, 2, \ldots\}$ *is the smallest nonnegative integer satisfying*

$$\mu\lambda^{\sigma_n}\|Ax_n - Ay_n\| \le \tau\|x_n - y_n\|. \tag{10}$$

Step 3. *Compute*

$$u_n = \mathrm{proj}_C[\alpha_n\gamma f(x_n) + (I - \alpha_n B)z_n], \tag{11}$$

and

$$x_{n+1} = (1 - \beta_n)u_n + \beta_n S[(1 - \delta_n)u_n + \delta_n S u_n]. \tag{12}$$

Set $n := n + 1$ *and return to step 1.*

Remark 3.1. There is an additional assumption with A being κ-Lipschitz continuous. However, the information of κ is not necessary priority to be known. That is, we need not to estimate the value of κ.

Lemma 3.1 ([1, 13])**.** *The line rule* (10) *is well-defined. Especially,* $\min\{1, \dfrac{\tau\lambda}{\kappa\mu}\} \le \lambda^{\sigma_n} \le 1$.

Now, we prove the iterative procedure generated by the above algorithm converges strongly to a common element in $VI(C, A) \cap \mathrm{Fix}(S)$.

Theorem 3.1. *Assume that* $\Omega = VI(C, A) \cap Fix(S) \neq \emptyset$. *Assume that the involved parameters satisfy the following conditions:*

(C1): $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(C2): $0 < \vartheta < \beta_n < \delta_n < \frac{1}{\sqrt{1+L^2}+1} (\forall n \ge 0)$;

(C3): $0 \le \rho < \min\{\xi/\gamma, 1\}$.

Then the sequence $\{x_n\}$ *generated by Algorithm* 3.1 *converges strongly to* $x^* = \mathrm{proj}_\Omega(I - B + \gamma f)x^*$.

Proof. First, note that the operator $\mathrm{proj}_\Omega(I - B + \gamma f)$ is contractive with unique fixed point denoted by x^*. From (11), we obtain

$$\begin{aligned}\|u_n - x^*\| &= \|\mathrm{proj}_C[\alpha_n\gamma f(x_n) + (I - \alpha_n B)z_n] - x^*\| \\ &\le \|\alpha_n\gamma(f(x_n) - f(x^*)) + (I - \alpha_n B)(z_n - x^*) \\ &\quad + \alpha_n(\gamma f(x^*) - Bx^*)\| \\ &\le \alpha_n\gamma\|f(x_n) - f(x^*)\| + \|I - \alpha_n B\|\|z_n - x^*\| \\ &\quad + \alpha_n\|\gamma f(x^*) - Bx^*\| \\ &\le \alpha_n\gamma\rho\|x_n - x^*\| + (1 - \alpha_n\xi)\|z_n - x^*\| + \alpha_n\|\gamma f(x^*) - Bx^*\|.\end{aligned} \tag{13}$$

According to (9), we have

$$\begin{aligned}\|z_n - x^*\|^2 &= \|\mathrm{proj}_C[y_n - \mu\lambda^{\sigma_n}(Ay_n - Ax_n)] - x^*\|^2 \\ &\le \|y_n - x^* - \mu\lambda^{\sigma_n}(Ay_n - Ax_n)\|^2 \\ &= \|y_n - x^*\|^2 + \mu^2\lambda^{2\sigma_n}\|Ay_n - Ax_n\|^2 \\ &\quad - 2\mu\lambda^{\sigma_n}\langle Ay_n - Ax_n, y_n - x^*\rangle \\ &= \|y_n - x_n\|^2 + 2\langle y_n - x_n, x_n - x^*\rangle + \|x_n - x^*\|^2 \\ &\quad + \mu^2\lambda^{2\sigma_n}\|Ay_n - Ax_n\|^2 - 2\mu\lambda^{\sigma_n}\langle Ay_n - Ax_n, y_n - x^*\rangle \\ &= \|x_n - x^*\|^2 - \|y_n - x_n\|^2 + \mu^2\lambda^{2\sigma_n}\|Ay_n - Ax_n\|^2 \\ &\quad + 2\langle y_n - x^*, y_n - x_n\rangle - 2\mu\lambda^{\sigma_n}\langle Ay_n - Ax_n, y_n - x^*\rangle.\end{aligned} \tag{14}$$

By (7) and (8), we get

$$\langle x_n - \mu\lambda^{\sigma_n} Ax_n - y_n, y_n - x^* \rangle \geq 0.$$

It follows that

$$\langle y_n - x_n, y_n - x^* \rangle \leq -\mu\lambda^{\sigma_n} \langle Ax_n, y_n - x^* \rangle. \tag{15}$$

In the light of (14), (15) and together with (10), we obtain

$$\begin{aligned} \|z_n - x^*\|^2 &\leq \|x_n - x^*\|^2 - \|y_n - x_n\|^2 - 2\mu\lambda^{\sigma_n} \langle y_n - x^*, Ax_n \rangle \\ &\quad + \mu^2\lambda^{2\sigma_n} \|Ay_n - Ax_n\|^2 - 2\mu\lambda^{\sigma_n} \langle Ay_n - Ax_n, y_n - x^* \rangle \\ &\leq \|x_n - x^*\|^2 - (1-\tau^2)\|y_n - x_n\|^2 - 2\mu\lambda^{\sigma_n} \langle y_n - x^*, Ax^* \rangle \\ &\quad - 2\mu\lambda^{\sigma_n} \langle y_n - x^*, Ay_n - Ax^* \rangle \end{aligned} \tag{16}$$

By the monotonicity of A, we have $\langle y_n - x^*, Ay_n - Ax^* \rangle \geq 0$. Owing to $y_n \in C$ and $x^* \in VI(C, A)$, we deduce that $\langle y_n - x^*, Ax^* \rangle \geq 0$. Thus, in view of (16), we obtain

$$\begin{aligned} \|z_n - x^*\|^2 &\leq \|x_n - x^*\|^2 - (1-\tau^2)\|y_n - x_n\|^2 \\ &\leq \|x_n - x^*\|^2. \end{aligned} \tag{17}$$

On the basis of (13) and (17), we have

$$\|u_n - x^*\| \leq [1 - (\xi - \gamma\rho)\alpha_n]\|x_n - x^*\| + \alpha_n\|\gamma f(x^*) - Bx^*\|. \tag{18}$$

Set $v_n = (1-\delta_n)u_n + \delta_n Su_n$ for all $n \geq 0$. Using (6), (12) and Lemma 2.1, we deduce that

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \|(1-\beta_n)(u_n - x^*) + \beta_n(Sv_n - x^*)\|^2 \\ &= (1-\beta_n)\|u_n - x^*\|^2 - \beta_n(1-\beta_n)\|Sv_n - u_n\|^2 \\ &\quad + \beta_n\|Sv_n - x^*\|^2 \\ &\leq (1-\beta_n)\|u_n - x^*\|^2 - \beta_n(1-\beta_n)\|Sv_n - u_n\|^2 \\ &\quad + \beta_n(\|u_n - x^*\|^2 + (1-\delta_n)\|u_n - Sv_n\|^2) \\ &= \|u_n - x^*\|^2 - \beta_n(\delta_n - \beta_n)\|u_n - Sv_n\|^2. \end{aligned} \tag{19}$$

By virtue of (18) and (19), we derive

$$\begin{aligned} \|x_{n+1} - x^*\| &\leq [1 - (\xi - \gamma\rho)\alpha_n]\|x_n - x^*\| + \alpha_n\|\gamma f(x^*) - Bx^*\| \\ &\leq \max\{\|x_n - x^*\|, \|\gamma f(x^*) - Bx^*\|/(\xi - \gamma\rho)\}. \end{aligned} \tag{20}$$

This implies that the sequence $\{x_n\}$ is bounded. Subsequently, the sequences $\{Ax_n\}$, $\{y_n\}$, $\{Ay_n\}$, $\{z_n\}$ and $\{u_n\}$ are all bounded.

Based on (11) and the firmly-nonexpansivity of $proj_C$, we have

$$\begin{aligned} \|u_n - x^*\|^2 &= \|\mathrm{proj}_C[\alpha_n\gamma f(x_n) + (I - \alpha_n B)z_n] - x^*\|^2 \\ &\leq \langle \alpha_n(\gamma f(x_n) - Bx^*) + (I - \alpha_n B)(z_n - x^*), u_n - x^* \rangle \\ &\leq (1 - \alpha_n\xi)\|z_n - x^*\|\|u_n - x^*\| + \alpha_n\gamma\rho\|x_n - x^*\|\|u_n - x^*\| \\ &\quad + \alpha_n\langle \gamma f(x^*) - Bx^*, u_n - x^* \rangle \\ &\leq \frac{1}{2}[(1-\alpha_n\xi)\|z_n - x^*\| + \alpha_n\gamma\rho\|x_n - x^*\|]^2 + \frac{1}{2}\|u_n - x^*\|^2 \\ &\quad + \alpha_n\langle \gamma f(x^*) - Bx^*, u_n - x^* \rangle. \end{aligned}$$

It follows that

$$\begin{aligned}\|u_n - x^*\|^2 &\le [(1-\alpha_n\xi)\|z_n - x^*\| + \alpha_n\gamma\rho\|x_n - x^*\|]^2 \\ &\quad + 2\alpha_n\langle \gamma f(x^*) - Bx^*, u_n - x^*\rangle \\ &= (1-\alpha_n\xi)^2\|z_n - x^*\|^2 + (\alpha_n\gamma\rho)^2\|x_n - x^*\|^2 \\ &\quad + 2(1-\alpha_n\xi)\alpha_n\gamma\rho\|z_n - x^*\|\|x_n - x^*\| \\ &\quad + 2\alpha_n\langle \gamma f(x^*) - Bx^*, u_n - x^*\rangle.\end{aligned}$$

This together with (17) implies that

$$\begin{aligned}\|u_n - x^*\|^2 &\le [1-(\xi-\gamma\rho)\alpha_n]\|x_n - x^*\|^2 - (1-\alpha_n\xi)^2(1-\tau^2)\|y_n - x_n\|^2 \\ &\quad + 2\alpha_n\langle \gamma f(x^*) - Bx^*, u_n - x^*\rangle.\end{aligned} \tag{21}$$

Combining (19) and (21), we get

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\le [1-(\xi-\gamma\rho)\alpha_n]\|x_n - x^*\|^2 + 2\alpha_n\langle \gamma f(x^*) - Bx^*, u_n - x^*\rangle \\ &\quad - \beta_n(\delta_n - \beta_n)\|u_n - Sv_n\|^2 - (1-\alpha_n\xi)^2(1-\tau^2)\|y_n - x_n\|^2 \\ &= [1-(\xi-\gamma\rho)\alpha_n]\|x_n - x^*\|^2 + (\xi-\gamma\rho)\alpha_n[-\frac{\beta_n(\delta_n-\beta_n)}{(\xi-\gamma\rho)\alpha_n} \\ &\quad \times \|u_n - Sv_n\|^2 + \frac{2}{\xi-\gamma\rho}\langle \gamma f(x^*) - Bx^*, u_n - x^*\rangle \\ &\quad - \frac{(1-\alpha_n\xi)^2(1-\tau^2)}{(\xi-\gamma\rho)\alpha_n}\|y_n - x_n\|^2].\end{aligned} \tag{22}$$

Set $\eta_n = \|x_n - z\|^2$ and

$$\begin{aligned}\sigma_n = &-\frac{\beta_n(\delta_n-\beta_n)}{(\xi-\gamma\rho)\alpha_n}\|u_n - Sv_n\|^2 - \frac{(1-\alpha_n\xi)^2(1-\tau^2)}{(\xi-\gamma\rho)\alpha_n}\|y_n - x_n\|^2 \\ &+ \frac{2}{\xi-\gamma\rho}\langle \gamma f(x^*) - Bx^*, u_n - x^*\rangle.\end{aligned} \tag{23}$$

for all $n \ge 0$.

By (22) and (23), we obtain

$$\delta_{n+1} \le [1-(\xi-\gamma\rho)\alpha_n]\delta_n + (\xi-\gamma\rho)\alpha_n\sigma_n, n \ge 0. \tag{24}$$

Next, we show that $\limsup_{n\to\infty}\sigma_n$ is finite. From (23), we get

$$\sigma_n \le \frac{2}{\xi-\gamma\rho}\|f(x^*) - Bx^*\|\|u_n - x^*\|.$$

It follows that

$$\limsup_{n\to\infty}\sigma_n < +\infty.$$

Next we prove $\limsup_{n\to\infty}\sigma_n \ge -1$ by contradiction. If we assume on the contrary $\limsup_{n\to\infty}\sigma_n < -1$, then there exists m_0 such that $\sigma_n \le -1$ for all $n \ge m_0$. It then follows from (24) that

$$\begin{aligned}\delta_{n+1} &\le [1-(\xi-\gamma\rho)\alpha_n]\delta_n - (\xi-\gamma\rho)\alpha_n \\ &\le \delta_n - (\xi-\gamma\rho)\alpha_n\end{aligned}$$

for all $n \ge m_0$.

By induction, we have

$$\delta_{n+1} \le \delta_{m_0} - (\xi-\gamma\rho)\sum_{i=m_0}^{n}\alpha_i. \tag{25}$$

By taking lim sup as $n \to \infty$ in (25), we have

$$\limsup_{n\to\infty} \delta_n \le \delta_{m_0} - (\xi - \gamma\rho) \lim_{n\to\infty} \sum_{i=m_0}^{n} \alpha_i = -\infty,$$

which induces a contradiction. So, $-1 \le \limsup_{n\to\infty} \sigma_n < +\infty$. Hence, $\limsup_{n\to\infty} \sigma_n$ exists. Thus, we can take a subsequence $\{n_k\}$ such that

$$\begin{aligned}\limsup_{n\to\infty} \sigma_n &= \lim_{k\to\infty} \sigma_{n_k} \\ &= \lim_{k\to\infty} \Big[-\frac{\beta_{n_k}(\delta_{n_k} - \beta_{n_k})}{(\xi - \gamma\rho)\alpha_{n_k}} \|u_{n_k} - Sv_{n_k}\|^2 \\ &\quad - \frac{(1-\alpha_{n_k}\xi)^2(1-\tau^2)}{(\xi - \gamma\rho)\alpha_{n_k}} \|y_{n_k} - x_{n_k}\|^2 \\ &\quad + \frac{2}{\xi - \gamma\rho} \langle \gamma f(x^*) - Bx^*, u_{n_k} - x^* \rangle \Big].\end{aligned} \tag{26}$$

Since $\langle \gamma f(x^*) - Bx^*, u_{n_k} - x^* \rangle$ is a bounded real sequence, without loss of generality, we may assume $\lim_{k\to\infty} \langle \gamma f(x^*) - Bx^*, u_{n_k} - x^* \rangle$ exists. Consequently, from (26), we have

$$\lim_{k\to\infty} -\frac{\beta_{n_k}(\delta_{n_k} - \beta_{n_k})}{(\xi - \gamma\rho)\alpha_{n_k}} \|u_{n_k} - Sv_{n_k}\|^2 \text{ exists,} \tag{27}$$

and

$$\lim_{k\to\infty} -\frac{(1-\alpha_{n_k}\xi)^2(1-\tau^2)}{(\xi - \gamma\rho)\alpha_{n_k}} \|y_{n_k} - x_{n_k}\|^2 \text{ exists.} \tag{28}$$

Note that $\lim_{n\to\infty} \alpha_n = 0$, $\liminf_{n\to\infty} \beta_n(\delta_n - \beta_n) > 0$ and $\lim n \to \infty (1-\alpha_n\xi)^2(1-\tau^2) > 0$. From (27) and (28), we get

$$\lim_{k\to\infty} \|u_{n_k} - Sv_{n_k}\| = 0 \tag{29}$$

and

$$\lim_{k\to\infty} \|y_{n_k} - x_{n_k}\| = 0. \tag{30}$$

From (9) and (10), we deduce that

$$\|z_n - y_n\| \le \mu\lambda^{\sigma_n} \|Ay_n - Ax_n\| \le \tau \|y_n - x_n\|.$$

This together with (30) implies that

$$\|z_{n_k} - y_{n_k}\| \to 0. \tag{31}$$

In accordance with (11), we get

$$\|u_n - z_n\| \le \alpha_n(\gamma \|f(x_n)\| + \|Bz_n\|) \to 0. \tag{32}$$

Observe that

$$\begin{aligned}\|u_n - Su_n\| &\le \|u_n - Sv_n\| + \|Sv_n - Su_n\| \\ &\le \|u_n - Sv_n\| + L\|v_n - u_n\| \\ &\le \|u_n - Sv_n\| + \delta_n L \|u_n - Su_n\|.\end{aligned}$$

It follows that

$$\|u_n - Su_n\| \le \frac{1}{1 - \delta_n L} \|u_n - Sv_n\|. \tag{33}$$

Combine (29) and (33) to get

$$\lim_{k\to\infty} \|u_{n_k} - Su_{n_k}\| = 0. \tag{34}$$

It follows that any weak cluster point of $\{u_{n_k}\}$ belongs to $\mathrm{Fix}(\mathcal{S})$ by Lemma 2.2.

At the same time, note that

$$\|x_{n_k+1} - u_{n_k}\| = \beta_{n_k}\|u_{n_k} - Sv_{n_k}\| \to 0.$$

Hence,

$$\|u_{n_k+1} - u_{n_k}\| \to 0. \tag{35}$$

This implies that any weak cluster point of $\{u_{n_k+1}\}$ (and hence $\{x_{n_k+1}\}$) also belongs to $\mathrm{Fix}(S)$. Without loss of generality, we assume that $\{u_{n_k+1}\}$ (and hence $\{x_{n_k+1}\}$) converges weakly to $\bar{x} \in \Omega$. Therefore,

$$\begin{aligned}\limsup_{n\to\infty} \sigma_n &\le \lim_{k\to\infty} \frac{2}{\xi - \gamma\rho}\langle \gamma f(x^*) - Bx^*, u_{n_k} - x^*\rangle \\ &= \frac{2}{\xi - \gamma\rho}\langle \gamma f(x^*) - Bx^*, \bar{x} - x^*\rangle \le 0\end{aligned}$$

due to the fact that $x^* = \mathrm{proj}_\Omega(I - B + \gamma f)x^*$ and (7).

From (22), we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 \le{}& [1 - (\xi - \gamma\rho)\alpha_n]\|x_n - x^*\|^2 \\ &+ (\xi - \gamma\rho)\alpha_n[\frac{2}{\xi - \gamma\rho}\langle \gamma f(x^*) - Bx^*, u_n - x^*\rangle].\end{aligned} \tag{36}$$

Finally, applying Lemma 2.3 to (36), we get $x_n \to x^*$. This completes the proof. □

Algorithm 3.2. *Let $\{\alpha_n\}_{n=0}^\infty$, $\{\beta_n\}_{n=0}^\infty$ and $\{\delta_n\}_{n=0}^\infty$ be three sequences in $(0,1)$. Let $\mu > 0$, $\gamma > 0$, $\lambda \in (0,1)$ and $\tau \in (0,1)$ be four constants.*

Step 1. Set an initial guess $x_0 \in C$ arbitrarily and let $n = 0$.

Step 2. For given current sequence x_n, compute the iterates y_n and z_n via the following form

$$y_n = \mathrm{proj}_C[x_n - \mu\lambda^{\sigma_n} Ax_n],$$

and

$$z_n = \mathrm{proj}_C[y_n - \mu\lambda^{\sigma_n}(Ay_n - Ax_n)],$$

where $\sigma_n \in \{0,1,2,\ldots\}$ is the smallest nonnegative integer satisfying

$$\mu\lambda^{\sigma_n}\|Ax_n - Ay_n\| \le \tau\|x_n - y_n\|.$$

Step 3. Compute

$$x_{n+1} = \mathrm{proj}_C[\alpha_n\gamma f(x_n) + (I - \alpha_n B)z_n].$$

Set $n := n+1$ and return to step 1.

Corollary 3.1. *Assume that $VI(C,A) \neq \emptyset$. Assume that the involved parameters satisfy the following conditions:*

(C1): $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^\infty \alpha_n = \infty$*;*

(C2): $0 < \vartheta < \beta_n < \delta_n < \frac{1}{\sqrt{1+L^2}+1} (\forall n \ge 0)$*;*

(C3): $0 \le \rho < \min\{\xi/\gamma, 1\}$.

Then the sequence $\{x_n\}$ generated by Algorithm 3.2 converges strongly to $\tilde{x} = \mathrm{proj}_{VI(C,A)}(I - B + \gamma f)\tilde{x}$.

4. Conclusion

We considered fixed point and variational inequality problems in Hilbert spaces. An extragradient algorithm for finding a common element of the set of fixed points of a pseudocontractive operator and the set of solutions of the variational inequality problem is presented. Strong convergence of the proposed algorithm is proved.

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