UNCONDITIONAL STABILITY OF A FULLY DISCRETE SCHEME FOR THE KELVIN-VOIGT MODEL

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The purpose of the current paper is to show unconditional stability of a second-order and two-step full discretization scheme for the nonstationary Kelvin-Voigt model. The proposed scheme deals with spatial discretization by mixed finite element method and temporal discretization by a Crank-Nicolson-type scheme. Further, we prove the unconditional stability of the considered scheme, i.e., it has no time step restrictions. Finally, we verify the unconditional stability numerically.

Keywords: unconditional stability, mixed finite element method, Crank-Nicolson-type scheme, Kelvin-Voigt model.

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1. Introduction

In this article, we discuss stability of a fully discrete scheme for the following system of equations of motion arising in the Kelvin-Voigt fluids:

$$\begin{aligned}
&\mathbf{u}_t - \kappa\Delta\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f}(\mathbf{x},t),\ \mathbf{x}\in\Omega,\ t>0,\\
&\nabla\cdot\mathbf{u} = 0,\ \mathbf{x}\in\Omega,\ t>0,\\
&\mathbf{u} = 0,\ \text{on } \partial\Omega,\ t\geq 0,\\
&\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}),\ \text{in } \mathbf{x}\in\Omega,
\end{aligned} \tag{1}$$

where Ω is a convex bounded domain in $\mathbb{R}^d$ (d=2, 3) with boundary $\partial\Omega$. Here $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ represents the velocity vector, $p = p(\mathbf{x}, t)$ is the pressure, ν is the kinematic coefficient of viscosity and κ is the retardation time. The Kelvin-Voigt model was first introduced and studied by Oskolkov [13] as a model for certain viscoelastic fluids known as Kelvin-Voigt fluids. Apart from several applications of this model in the study of organic polymers and food industry, it also appears in the mechanism of diffuse axonal injury that is unexplained by traumatic brain injury models proposed earlier but now it can base on the Kelvin-Voigt model for more detailed description [2, 3].

Recently, a scheme called Crank-Nicolson Leap-Frog (CNLF), which is a classic two-step method within a class of so-called implicit-explicit linear multistep methods and is frequently used in atmosphere, ocean, climate codes and computational fluid dynamics and is based on the classic Crank-Nicolson scheme [4, 5], is proposed [1, 16, 19]. In [17], Verwer discussed convergence of CNLF governed by a special condition allowing a wider class of splittings than commonly used in computational fluid dynamics. Although the firstly analyzed in [9], stability of CNLF for systems is only recently proven in [12], where Layton and Trenchea have proved stability for the coupled system under time step

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condition suggested by linear stability theory for the Leap-Frog scheme. The time step condition can not be eliminated, because the unstable mode of Leap-Frog is not damped by Crank-Nicolson unless the time step condition is met. In fact, Hurl et al. have analysed the stability of the unstable mode in [7] and proved that under time step condition the CNLF unstable mode is asymptotically stable for the linear system. Besides, Kubacki [10] has promoted the CNLF scheme to solve the uncoupling groundwater-surface water flows and derived a time step condition stability.

In order to remove time step restriction, one popular way to counteract this effect of the unstable mode in CNLF is to use time filters, such as Robert-Asselin filter [1, 14] or Robert-Asselin-Williams filter [6, 19]. On other hand, unlike time filters, stabilized method of the CNLF scheme can also remove all time step conditions for stability and control the unstable mode. In [8], Jiang et al. have proposed and analyzed a linear stabilization of the CNLF scheme which is unconditional stable for linear evolution equations. Besides, Kubacki and Moraiti have proved that the CNLF stabilization scheme is unconditionally stable and second-order convergent for the evolutionary Stokes-Darcy equations in [11]. Hence, the stabilized method is an effective technique to counteract the effect of unstable mode for the linear equations. Moreover, for the nonlinear equations, Tang and Huang [15] have proved the almost unconditional stability of the CNLF scheme for the unsteady incompressible Navier-Stokes equations.

In this paper, inspired by [12], we will present a Crank-Nicolson-type scheme based on a mixed finite element approximation for numerically solving the Kelvin-Voigt equations. The main work is to obtain unconditional stability of the proposed scheme and verify the numerical theory result by numerical experiments.

2. **Preliminaries**

In this paper, we employ the standard notation of vector Sobolev spaces and denote the usual $L^2(\Omega)$ norm and its inner product by $\|\cdot\|$ and $(\cdot,\cdot)$ respectively. The natural function spaces for our problem are

$$\mathbf{X} := \mathbf{H}_0^1(\Omega)^d = \{\mathbf{v} \in (L^2(\Omega))^d : \nabla\mathbf{v} \in (L^2(\Omega))^{d\times d},\ \mathbf{v} = 0 \ on\ \partial\Omega\},$$
$$Q := L_0^2(\Omega) = \{q \in L^2(\Omega) : \int_\Omega q dx = 0\}.$$

For $\mathbf{f}$ an element in the dual space of $\mathbf{X}$, its norm is defined by

$$\|\mathbf{f}\|_{-1} = \sup_{\mathbf{v}\in\mathbf{X}} \frac{|(\mathbf{f},\mathbf{v})|}{\|\nabla\mathbf{v}\|}.$$

Furthermore, in the rest of the paper, we adopt a bilinear form:

$$a(\mathbf{u},\mathbf{v}) := (\nabla\mathbf{u},\nabla\mathbf{v}),$$

and a skew-symmetrized trilinear form [18]:

$$b(\mathbf{u},\mathbf{v},\mathbf{w}) := \frac{1}{2}(\mathbf{u}\cdot\nabla\mathbf{v},\mathbf{w}) - \frac{1}{2}(\mathbf{u}\cdot\nabla\mathbf{w},\mathbf{v}) = (\mathbf{u}\cdot\nabla\mathbf{v},\mathbf{w}) + \frac{1}{2}((\nabla\cdot\mathbf{u})\mathbf{v},\mathbf{w}). \tag{2}$$

Then, by using the Green formulas, the variational formulation of problem (1) is to find $(\mathbf{u}(t), p(t)) \in \mathbf{X}\times Q$ such that $\mathbf{u}(0) = \mathbf{u}_0$ and

$$\begin{aligned} (\mathbf{u}_t,\mathbf{v}) + \kappa a(\mathbf{u}_t,\mathbf{v}) + \nu a(\mathbf{u},\mathbf{v}) + b(\mathbf{u},\mathbf{u},\mathbf{v}) - (p,\nabla\cdot\mathbf{v}) &= (\mathbf{f},\mathbf{v}),\\ (\nabla\cdot\mathbf{u}, q) &= 0, \end{aligned} \tag{3}$$

for all $(\mathbf{v}, q) \in \mathbf{X}\times Q$ and $t > 0$.

From now on, let T_h be a regular and quasi-uniform triangulation partition of Ω with element diameters bounded by a real positive parameter h ($h \to 0$). The conforming subspace pair $(\mathbf{X}^h, Q^h)$ of $(\mathbf{X}, Q)$ is constructed based on T_h. In this paper, we employ the well-known Taylor-Hood element to approximate the velocity and pressure.

$$\mathbf{X}^h = (P_2(K))^d \cap \mathbf{X},\ Q^h = \{q_h \in C^0(\Omega) : v_h|_K \in P_1(K), \forall K \in T_h\}.$$

Apparently, the mixed finite element space pair $(\mathbf{X}^h, Q^h)$ satisfies the so-called inf-sup condition :

$$\inf_{q_h \in Q^h} \sup_{\mathbf{v}_h \in \mathbf{X}^h} \frac{(q_h, \nabla \cdot \mathbf{v}_h)}{\|\nabla \mathbf{v}_h\| \|q_h\|} \geq \beta > 0,$$

where β is independent of h. Besides, the discrete divergence free subspace of $\mathbf{X}^h$ is

$$\mathbf{V}^h := \{\mathbf{v}_h \in \mathbf{X}^h : (\nabla \cdot \mathbf{v}_h, q_h) = 0,\ \forall q_h \in Q^h\}.$$

With above statements, Galerkin mixed finite element approximation of problem (3) is to find $(\mathbf{u}_h(t), p_h(t)) \in (\mathbf{X}^h, Q^h)$ such that $\mathbf{u}_h(0) = \mathbf{u}_{0h}$ and

$$\begin{aligned} (\mathbf{u}_{ht}, \mathbf{v}_h) + \kappa a(\mathbf{u}_{ht}, \mathbf{v}_h) + \nu a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) - (p_h, \nabla \cdot \mathbf{v}_h) &= (\mathbf{f}, \mathbf{v}_h), \\ (\nabla \cdot \mathbf{u}_h, q_h) &= 0, \end{aligned} \tag{4}$$

for all $t > 0$ and $(\mathbf{v}_h, q_h) \in (\mathbf{X}^h, Q^h)$.

3. **Stability of a Crank-Nicolson-type scheme for the Kelvin-Voigt model**

In this section, we consider stability of a Crank-Nicolson-type scheme for time discretization and a mixed finite element method in spatial direction of the problem (1). Let $\{t_n\}_{n=0}^N$ be a uniform partition of $[0, T]$ and $t_n = n\tau$, where $\tau > 0$ is time step and T is final time.

Algorithm 3.1.

Step I: Find $(\mathbf{u}_h^1, p_h^1) \in (\mathbf{X}^h, Q^h)$ such that

$$\left(\frac{\mathbf{u}_h^1 - \mathbf{u}_h^0}{\tau}, \mathbf{v}_h\right) + \kappa a\left(\frac{\mathbf{u}_h^1 - \mathbf{u}_h^0}{\tau}, \mathbf{v}_h\right) + \nu a\left(\frac{\mathbf{u}_h^1 + \mathbf{u}_h^0}{2}, \mathbf{v}_h\right) + b\left(\frac{\mathbf{u}_h^1 + \mathbf{u}_h^0}{2}, \frac{\mathbf{u}_h^1 + \mathbf{u}_h^0}{2}, \mathbf{v}_h\right)$$

$$- \left(p_h^1, \nabla \cdot \mathbf{v}_h\right) = (\mathbf{f}^{\frac{1}{2}}, \mathbf{v}_h), \tag{5}$$

$$(\nabla \cdot \mathbf{u}_h^1, q_h) = 0, \tag{6}$$

where $\mathbf{f}^{\frac{1}{2}} = \frac{\mathbf{f}(t_0)+\mathbf{f}(t_1)}{2}$ and the initial level $\mathbf{u}_h^0 = \mathbf{u}_{0h}$.

Step II: For $n \geq 1$, given $(\mathbf{u}_h^{n-1}, p_h^{n-1}), (\mathbf{u}_h^n, p_h^n) \in (\mathbf{X}^h, Q^h)$, find $(\mathbf{u}_h^{n+1}, p_h^{n+1}) \in (\mathbf{X}^h, Q^h)$ such that for all $(\mathbf{v}_h, q_h) \in (\mathbf{X}^h, Q^h)$:

$$\left(\frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^{n-1}}{2\tau}, \mathbf{v}_h\right) + \kappa a\left(\frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^{n-1}}{2\tau}, \mathbf{v}_h\right) + \nu a\left(\frac{\mathbf{u}_h^{n+1} + \mathbf{u}_h^{n-1}}{2}, \mathbf{v}_h\right)$$

$$+ b\left(\frac{\mathbf{u}_h^{n+1} + \mathbf{u}_h^{n-1}}{2}, \frac{\mathbf{u}_h^{n+1} + \mathbf{u}_h^{n-1}}{2}, \mathbf{v}_h\right) - \left(\frac{p_h^{n+1} + p_h^{n-1}}{2}, \nabla \cdot \mathbf{v}_h\right) = (\mathbf{f}(t_n), \mathbf{v}_h), \tag{7}$$

$$(\nabla \cdot \mathbf{u}_h^{n+1}, q_h) = 0, \tag{8}$$

where $\mathbf{f}^n = \frac{\mathbf{f}(t_{n+1})+\mathbf{f}(t_{n-1})}{2}$.

Next, we will prove unconditional stability of the fully discrete Crank-Nicolson-type scheme by tracking a discrete energy, which is denoted by

$$\mathbf{E}^{n+\frac{1}{2}} := \|\mathbf{u}_h^{n+1}\|^2 + \|\mathbf{u}_h^n\|^2 + \kappa(\|\nabla \mathbf{u}_h^{n+1}\|^2 + \|\nabla \mathbf{u}_h^n\|^2).$$

Theorem 3.1. *Let $\mathbf{f} \in L^2(0, T; H^{-1}(\Omega)^d)$. Then, the Algorithm 3.1 is unconditionally stable: for any $\tau > 0$, solutions to the Algorithm 3.1 satisfy*

$$\frac{\mathbf{E}^{N+\frac{1}{2}}}{2} + \frac{\nu\tau}{4}\sum_{n=1}^{N} \|\nabla(\mathbf{u}_h^{n+1} + \mathbf{u}_h^{n-1})\|^2 \leq \|\mathbf{u}_h^0\|^2 + \kappa\|\nabla\mathbf{u}_h^0\|^2 + \frac{\tau}{4\nu}\|\mathbf{f}^{\frac{1}{2}}\|_{-1}^2 + \frac{\tau}{\nu}\sum_{n=1}^{N}\|\mathbf{f}(t_n)\|_{-1}^2.$$

Proof. Setting $\mathbf{v}_h = 2\tau(\mathbf{u}_h^{n+1} + \mathbf{u}_h^{n-1}) \in \mathbf{V}^h$ in (7) gives

$$\begin{aligned}&\|\mathbf{u}_h^{n+1}\|^2 - \|\mathbf{u}_h^{n-1}\|^2 + \kappa\left(\|\nabla\mathbf{u}_h^{n+1}\|^2 - \|\nabla\mathbf{u}_h^{n-1}\|^2\right) + \nu\tau\left\|\nabla(\mathbf{u}_h^{n+1} + \mathbf{u}_h^{n-1})\right\|^2 \\ &= 2\tau(\mathbf{f}(t_n), \mathbf{u}_h^{n+1} + \mathbf{u}_h^{n-1}),\end{aligned}$$

since $b(\frac{1}{2}(\mathbf{u}_h^{n+1} + \mathbf{u}_h^{n-1}), \mathbf{u}_h^{n+1} + \mathbf{u}_h^{n-1}, \mathbf{u}_h^{n+1} + \mathbf{u}_h^{n-1}) = 0$. Next, add and subtract $\|\mathbf{u}_h^n\|^2 + \kappa\|\nabla\mathbf{u}_h^n\|^2$ to get

$$\mathbf{E}^{n+\frac{1}{2}} - \mathbf{E}^{n-\frac{1}{2}} + \nu\tau\|\nabla(\mathbf{u}_h^{n+1} + \mathbf{u}_h^{n-1})\|^2 = 2\tau(\mathbf{f}(t_n), \mathbf{u}_h^{n+1} + \mathbf{u}_h^{n-1}). \tag{9}$$

Applying the Young's inequality, we obtain $\mathbf{E}^{n+\frac{1}{2}} - \mathbf{E}^{n-\frac{1}{2}} + \frac{\nu\tau}{2}\|\nabla(\mathbf{u}_h^{n+1} + \mathbf{u}_h^{n-1})\|^2 \leq \frac{2\tau}{\nu}\|\mathbf{f}(t_n)\|_{-1}^2$. Sum up the above inequality from $n = 1, ..., N$ to find

$$\mathbf{E}^{N+\frac{1}{2}} + \frac{\nu\tau}{2}\sum_{n=1}^{N}\|\nabla(\mathbf{u}_h^{n+1} + \mathbf{u}_h^{n-1})\|^2 \leq \mathbf{E}^{\frac{1}{2}} + \frac{2\tau}{\nu}\sum_{n=1}^{N}\|\mathbf{f}(t_n)\|_{-1}^2. \tag{10}$$

For the first time level, taking $\mathbf{v}_h = \mathbf{u}_h^1 + \mathbf{u}_h^0 \in \mathbf{V}^h$ in (5). Analogously, applying the Young's inequality yields

$$\frac{\|\mathbf{u}_h^1\|^2 - \|\mathbf{u}_h^0\|^2}{\tau} + \kappa\left(\frac{\|\nabla\mathbf{u}_h^1\|^2 - \|\nabla\mathbf{u}_h^0\|^2}{\tau}\right) + \frac{\nu}{2}\|\nabla(\mathbf{u}_h^1 + \mathbf{u}_h^0)\|^2 \leq \frac{1}{2\nu}\|\mathbf{f}^{\frac{1}{2}}\|_{-1}^2.$$

Multiplying the above inequality by τ, we obtain that

$$\|\mathbf{u}_h^1\|^2 + \kappa\|\nabla\mathbf{u}_h^1\|^2 \leq \|\mathbf{u}_h^0\|^2 + \kappa\|\nabla\mathbf{u}_h^0\|^2 + \frac{\tau}{2\nu}\|\mathbf{f}^{\frac{1}{2}}\|_{-1}^2. \tag{11}$$

Finally, taking (11) into (10), we achieve the desired result. □

4. Numerical experiments

In this section, we present some numerical experiments to verify the unconditional stability of the proposed algorithm for the 2D/3D unsteady Kelvin-Voigt model (1).

The prescribed solutions are in $\Omega = [0, 1]^d$ and d=2, 3. Choose the source term $\mathbf{f}$ with equation parameters $\nu = 1$ and $\kappa = 0.01$ such that the exact solutions are

$$\begin{aligned}&u_1 = x^2\varphi^2(x)y\varphi(y)(2\varphi(y) + 1)e^{-t}, \qquad u_2 = -x\varphi(x)(\varphi(x) + 1)y^2\varphi^2(y)e^{-t}, \\ &p = \Big(3\big(\varphi^2(x) + \varphi^2(y)\big) + 6\big(\varphi(x) + \varphi(y)\big) - 8\Big)e^{-t},\end{aligned}$$

for $d = 2$ and

$$\begin{aligned}u_1 &= 10x^2\varphi^2(x)\Big(\big(2y\varphi^2(y) + 2y^2\varphi(y)\big)z^2\varphi^2(z) - y^2\varphi^2(y)\big(2z\varphi^2(z) + 2z^2\varphi(z)\big)\Big)\cos(t), \\ u_2 &= 10y^2\varphi^2(y)\Big(\big(2z\varphi^2(z) + 2z^2\varphi(z)\big)x^2\varphi^2(x) - z^2\varphi^2(z)\big(2x\varphi^2(x) + 2x^2\varphi(x)\big)\Big)\cos(t), \\ u_3 &= 10z^2\varphi^2(z)\Big(\big(2x\varphi^2(x) + 2x^2\varphi(x)\big)y^2\varphi^2(y) - x^2\varphi^2(x)\big(2y\varphi^2(y) + 2y^2\varphi(y)\big)\Big)\cos(t), \\ p &= 10(2\varphi(x) + 1)(2\varphi(y) + 1)(2\varphi(z) + 1)\cos(t),\end{aligned}$$

for $d = 3$. Here, we denote $\varphi(\xi) = (\xi - 1)$, $\xi = x$, y or z, and compute the final time $T = 1$.

In order to validate Theorem 3.1, we compute the values of $||\mathbf{u}_h^n||_0$ and $||\nabla \mathbf{u}_h^n||_0$ for 2D and 3D problems with different time steps listed in Table 1-4 and compare the values with different space meshes under the same time step. We can observe from these tables that the value of $||\mathbf{u}_h^n||_0$ and $||\nabla \mathbf{u}_h^n||_0$ tends to be a constant, which shows that no time-step restriction is need.

TABLE 1. $||\mathbf{u}_h^n||_0$ of the considered scheme for the 2D problem.

$\frac{1}{h}$	$\frac{1}{\tau}$				
	2^6	2^5	2^4	2^3	2^2
2^4	1.4526E-3	1.4765E-3	1.6241E-3	4.0477E-3	6.8058E-3
2^5	1.4527E-3	1.4767E-3	1.6240E-3	4.0477E-3	6.8064E-3
2^6	1.4527E-3	1.4767E-3	1.6240E-3	4.0477E-3	6.8065E-3
2^7	1.4527E-3	1.4767E-3	1.6240E-3	4.0477E-3	6.8065E-3

TABLE 2. $||\nabla \mathbf{u}_h^n||_0$ of the considered scheme for the 2D problem.

$\frac{1}{h}$	$\frac{1}{\tau}$				
	2^6	2^5	2^4	2^3	2^2
2^4	1.0690E-2	1.1019E-2	1.2998E-2	3.0652E-2	5.0252E-2
2^5	1.0698E-2	1.1029E-2	1.2987E-2	3.0643E-2	5.0252E-2
2^6	1.0699E-2	1.1031E-2	1.2988E-2	3.0643E-2	5.0257E-2
2^7	1.0699E-2	1.1031E-2	1.2988E-2	3.0643E-2	5.0257E-2

TABLE 3. $||\mathbf{u}_h^n||_0$ of the considered scheme for the 3D problem.

$\frac{1}{h}$	$\frac{1}{\tau}$				
	2^6	2^5	2^4	2^3	2^2
5	4.3858E-3	4.4058E-3	4.4678E-3	5.6704E-3	8.6080E-3
10	4.3754E-3	4.3952E-3	4.4519E-3	5.6354E-3	8.6026E-3

TABLE 4. $||\nabla \mathbf{u}_h^n||_0$ of the considered scheme for the 3D problem.

$\frac{1}{h}$	$\frac{1}{\tau}$				
	2^6	2^5	2^4	2^3	2^2
5	2.0746E-2	2.0641E-2	2.1725E-2	3.4693E-2	5.8571E-2
10	1.6348E-2	1.5974E-2	1.5141E-2	2.0479E-2	4.5093E-2

5. **Conclusions**

In this work, we present the fully discrete Crank-Nicolson-type scheme in solving the Kelvin-Voigt problem. We find out that the Crank-Nicolson-type scheme is unconditionally stable without using any time filters or stabilized methods for this nonlinear system. Numerical tests verify our result.

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