FIXED POINT RESULTS FOR COUPLINGS ON METRIC SPACES

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In this paper we define the concept of coupling between two non-empty subsets in metric space. The definition is motivated by the concept of cyclic mapping of a metric space. We show that these coupling have strong unique coupled fixed point whenever they satisfy Banach type or Chatterjea type contractive inequalities. We illustrate our main results with examples.

Keywords: Coupling, coupled fixed point, strong coupled fixed point.
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1. Introduction

In this paper we introduce two types of couplings defined on metric spaces, namely, Banach type and Chatterjea type couplings. These are actually coupled cyclic mappings with respect to two given subsets of a metric space. We establish the existence and uniqueness of strong coupled fixed points for both types of couplings. The celebrated work of Banach [2], in which he established the Contraction Mapping Principle is widely recognized as the source of metric fixed point theory. Another category of contraction which is separate from Banach contraction, and does not imply continuity, was proposed by Kannan[18, 19] who also established in the same work that such mappings necessarily have unique fixed points in complete metric spaces. Mappings belonging to this category are known as Kannan type mappings. These mappings, their extension and generalizations, have a large literature. References [13, 14, 24] are some instances of these works. In the same vein, Chatterjea[7] established another class of contraction different from the above two categories. Extension of Chatterjea contraction, sometimes also called C-contraction, have been studied in good number of papers. Some recent references are of there works are [1, 8, 17, 27]. Cyclic contractions were introduced by Kirk et al.[21]. These are nonself contractions from one subset to another subset of a metric space. Several types of cyclic contractions have been studied in fixed point theory in works like [12, 15, 20, 24, 25].

Coupled fixed point was introduced in the work of Guo et al.[16]. It was after the appearance of a coupled contration mapping theorem by Bhaskar et al.[3], the coupled fixed point results appeared in a large number of works like [4, 5, 9, 10, 11, 22, 23, 26].

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Against the above background we define couplings of Banach and Chatterjea types which are the cyclic coupled mappings satisfying contractions of the two categories respectively mentioned above. Such mappings satisfying Kannan type contractive condition have already been considered by Choudhury et al.[12]. We establish strong unique fixed point theorems for both type of couplings. In the following we recall some definitions for the purpose of further discussion in this paper.

Definition 1.1 (Coupled fixed point)**.** [3] An element $(x,\ y) \in X \times X$, where X is any non-empty set, is called a coupled fixed point of the mapping $F : X \times X \to X$ if $F(x,\ y) = x$ and $F(y,\ x) = y$.

Definition 1.2 (Strong Coupled fixed point)**.** [12] An element $(x,\ y) \in X \times X$, where X is any non-empty set, is a strong coupled fixed point of the mapping $F : X \times X \to X$ if $(x,\ y)$ is a coupled fixed point and $x = y$; that is, if $F(x,\ x) = x$.

Definition 1.3 (Coupled Banach Contraction mapping)**.** [3] Let (X, d) be a metric space. A mapping $F : X \times X \to X$ is called a coupled Banach contraction if there exists $k \in (0,\ 1)$ such that for all $(x,\ y), (u,\ v) \in X \times X$, the following inequality holds:

$$d(F(x,y), F(u,v)) \leq \frac{k}{2}[d(x,\ u) + d(y,\ v)].$$

Definition 1.4 (Chatterjea Contraction)**.** [7] Let (X, d) be a metric space. A self-mapping $T : X \to X$ is called a Chatterjea contraction if there exists $k \in (0, \frac{1}{2})$ such that for all $x, y \in X$, the following inequality holds:

$$d(Tx,\ Ty) \leq k[d(x, Ty) + d(y, Tx)].$$

Definition 1.5 (Cyclic mapping)**.** [21] Let A and B be two nonempty subsets of a given set X. Any function $f : X \to X$ is said to be cyclic(with respect to A and B) if $f(A) \subset B$ and $f(B) \subset A$.

Next we define a ‘coupling’ by extending the idea behind a cyclic mapping.

Definition 1.6. [12] Let $(X,\ d)$ be a metric space and let A and B be two non-empty subsets of X. A coupling with respect to A and B is a function $F : X \times X \to X$ such that $F(x,\ y) \in B$ and $F(y,\ x) \in A$ whenever $x \in A$ and $y \in B$

Later, in the examples in section 2 and 3, we give the illustrations of the coupling function.

2. **Results for Banach Type Coupling**

In this section we define Banach type coupling by putting together the concepts of coupling and the coupled Banach contraction. Next we establish a strong unique coupled fixed point theorem for such coupling. We also give an illustrative example.

Definition 2.1. *Let A and B be two non-empty subsets of a complete metric space (X, d). A coupling $F : X \times X \to X$ is called a Banach type coupling with respect to A and B if it satisfies the following inequality:*

$$d(F(x,y), F(u,v)) \leq \frac{k}{2}[d(x,\ u) + d(y,\ v)] \tag{2.1}$$

where $x, v \in A,\ y, u \in B$ and $k \in (0,\ 1)$.

Theorem 2.1. *Let A and B be two non-empty closed subsets of a complete metric space (X, d). Let $F : X \times X \to X$ be a Banach type coupling with respect to A and B. Then $A \cap B \neq \phi$ and F has a unique strong coupled fixed point in $A \cap B$.*

Proof. Let $x_0 \in A$ and $y_0 \in B$ be any two elements and the sequences $\{x_n\}$ and $\{y_n\}$ be defined as

$$x_{n+1} = F(y_n,\ x_n) \text{ and } y_{n+1} = F(x_n,\ y_n) \text{ for all } n \geq 0. \tag{2.2}$$

Then, for all $n \geq 0$, $x_n \in A$ and $y_n \in B$.

Now, using (2.1) and (2.2), we have

$$\begin{aligned} d(x_1, y_2) = d(x_1,\ F(x_1, y_1)) &= d(F(y_0, x_0),\ F(x_1, y_1)) \\ &\leq \frac{k}{2}[d(y_0,\ x_1) + d(x_0,\ y_1)] \end{aligned}$$

and

$$\begin{aligned} d(y_1,\ x_2) = d(y_1,\ F(y_1,\ x_1)) &= d(F(x_0,\ y_0), F(y_1,\ x_1)) \\ &\leq \frac{k}{2}[d(x_0,\ y_1) + d(y_0,\ x_1)]. \end{aligned}$$

From the above two inequalities, we have

$$\frac{d(x_1,\ y_2) + d(y_1,\ x_2)}{2} \leq \frac{k}{2}[d(x_0,\ y_1) + d(y_0,\ x_1)]. \tag{2.3}$$

Again, using (2.1), (2.2) and (2.3), we have

$$\begin{aligned} d(x_2, y_3) = d(x_2, F(x_2, y_2)) &= d(F(y_1,\ x_1), F(x_2,\ y_2)) \\ &\leq \frac{k}{2}[d(y_1,\ x_2) + d(x_1,\ y_2)] \\ &\leq \frac{k}{2}.2\left[\frac{k}{2}[d(x_0,\ y_1) + d(y_0,\ x_1)]\right] \\ &= \frac{k^2}{2}[d(x_0,\ y_1) + d(y_0,\ x_1)]. \end{aligned}$$

Similarly, using (2.1), (2.2) and (2.3), we have

$$\begin{aligned} d(y_2,\ x_3) = d(y_2, F(y_2,\ x_2)) &= d(F(x_1,\ y_1), F(y_2,\ x_2)) \\ &\leq \frac{k}{2}[d(x_1,\ y_2) + d(y_1,\ x_2)] \\ &\leq \frac{k}{2}.2\left[\frac{k}{2}[d(y_0,\ x_1) + d(x_0,\ y_1)]\right] \\ &= \frac{k^2}{2}[d(y_0,\ x_1) + d(x_0,\ y_1)]. \end{aligned}$$

Let, for some integer n,

$$d(x_n,\ y_{n+1}) \leq \frac{k^n}{2}[d(x_0,\ y_1) + d(y_0,\ x_1)] \tag{2.4}$$

and

$$d(y_n,\ x_{n+1}) \leq \frac{k^n}{2}[d(y_0,\ x_1) + d(x_0,\ y_1)]. \tag{2.5}$$

Then, using (2.1), (2.2), (2.4) and (2.5), we have

$$\begin{aligned} d(x_{n+1},\ y_{n+2}) &= d(x_{n+1},\ F(x_{n+1},\ y_{n+1})) \\ &= d(F(y_n,\ x_n), F(x_{n+1},\ y_{n+1})) \\ &\leq \frac{k}{2}[d(y_n,\ x_{n+1}) + d(x_n,\ y_{n+1})] \\ &\leq \frac{k}{2}\left[\frac{k^n}{2}[d(y_0,\ x_1) + d(x_0,\ y_1)] + \frac{k^n}{2}[d(x_0,\ y_1) + d(y_0,\ x_1)]\right] \\ &\leq \frac{k^{n+1}}{2}[d(x_0,\ y_1) + d(y_0,\ x_1)]. \end{aligned}$$

Similarly, using (2.1), (2.2), (2.4) and (2.5), we have

$$\begin{aligned} d(y_{n+1},\ x_{n+2}) &= d(y_{n+1},\ F(y_{n+1},\ x_{n+1})) \\ &= d(F(x_n,\ y_n), F(y_{n+1},\ x_{n+1})) \\ &\leq \frac{k}{2}[d(x_n,\ y_{n+1}) + d(y_n,\ x_{n+1})] \\ &\leq \frac{k}{2}\left[\frac{k^n}{2}[d(x_0,\ y_1) + d(y_0,\ x_1)] + \frac{k^n}{2}[d(y_0,\ x_1) + d(x_0,\ y_1)]\right] \\ &\leq \frac{k^{n+1}}{2}[d(y_0,\ x_1) + d(x_0,\ y_1)]. \end{aligned}$$

Thus (2.4) and (2.5) remain valid when n is replaced by $n+1$. But, as shown above, (2.4) and (2.5) are true for $n = 1, 2$.

Then, by induction, for all $n \geq 1$, we have that

$$d(x_n,\ y_{n+1}) = d(x_n,\ F(x_n,\ y_n)) \leq \frac{k^n}{2}[d(x_0,\ y_1) + d(y_0,\ x_1)] \tag{2.6}$$

and

$$d(y_n,\ x_{n+1}) = d(y_n, F(y_n,\ x_n)) \leq \frac{k^n}{2}[d(y_0,\ x_1) + d(x_0,\ y_1)]. \tag{2.7}$$

Again, by (2.1) and (2.2), we have

$$d(x_1, y_1) = d(F(y_0, x_0), F(x_0, y_0)) \leq \frac{k}{2}[d(y_0, x_0) + d(x_0, y_0)] = kd(x_0, y_0). \tag{2.8}$$

Then, from (2.1), (2.2) and (2.8), we have

$$\begin{aligned} d(x_2,\ y_2) &= d(F(y_1,\ x_1), F(x_1,\ y_1)) \\ &\leq \frac{k}{2}[d(y_1,\ x_1) + d(x_1,\ y_1)] \\ &= kd(x_1,\ y_1) \\ &\leq k^2 d(x_0, y_0). \end{aligned}$$

Let, for some integer n,

$$d(x_n,\ y_n) \leq k^n d(x_0, y_0). \tag{2.9}$$

Then, from (2.1), (2.2) and (2.9), we have

$$\begin{aligned} d(x_{n+1},\ y_{n+1}) &= d(F(y_n,\ x_n), F(x_n,\ y_n)) \\ &\leq \frac{k}{2}[d(y_n,\ x_n) + d(x_n,\ y_n)] \\ &\leq kd(x_n,\ y_n) \\ &\leq k^{n+1} d(x_0,\ y_0). \end{aligned}$$

Therefore, (2.9) also holds if we replace n by $n+1$. But (2.9) is true for $n = 1, 2$. Then, by induction, it follows that for all $n \geq 1$,

$$d(x_n, y_n) \leq k^n d(x_0, y_0). \tag{2.10}$$

Now, by (2.6), (2.7) and (2.10), for all $n \geq 1$, we have

$$\begin{aligned} d(x_n, x_{n+1}) + d(y_n, y_{n+1}) &\leq d(x_n, y_n) + d(y_n, x_{n+1}) + d(y_n, x_n) + d(x_n, y_{n+1}) \\ &= 2d(x_n, y_n) + [d(y_n, x_{n+1}) + d(x_n, y_{n+1})] \\ &\leq 2k^n d(x_0, y_0) + k^n [d(x_0, y_1) + d(y_0, x_1)]. \end{aligned}$$

Since $0 < k < 1$, it follows that $\Sigma d(x_n, x_{n+1}) + \Sigma d(y_n, y_{n+1}) < \infty$.

This implies that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences and hence are convergent to x and y respectively (say).

Since A and B are closed subsets, $\{x_n\} \subset A$ and $\{y_n\} \subset B$, it follows that

$$x_n \to x \in A \text{ and } \; y_n \to y \in B \text{ as } \; n \to \infty. \tag{2.11}$$

Again, from (2.10), $d(x_n, y_n) \to 0$ as $n \to \infty$.

Therefore, from (2.11),

$$x = y. \tag{2.12}$$

It then follows that $x \in A \cap B$ and hence $A \cap B \neq \phi$.

Now, from (2.1) and (2.2), for all $n \geq 1$, we have

$$\begin{aligned} d(x, F(x, y)) &\leq d(x, x_{n+1}) + d(x_{n+1}, F(x, y)) \\ &\leq d(x, x_{n+1}) + d(F(y_n, x_n), F(x, y)) \\ &\leq d(x, x_{n+1}) + \frac{k}{2}[d(y_n, x) + d(x_n, y)]. \end{aligned}$$

Taking the limit as $n \to \infty$ in the above inequality, using (2.11) and (2.12), we obtain $d(x, F(x, x)) = 0$. Then we conclude that $x = F(x, x)$, that is, x is a strong coupled fixed point of F.

If possible, let there be two strong couple fixed point of F, that is,

$$F(x,\ x) = x \text{ and } F(y,\ y) = y \text{ where } x, y \in A \cap B.$$

Then, by (2.1), we have

$$\begin{aligned} d(x, y) &= d(F(x,\ x),\ F(y,\ y)) \\ &\leq \frac{k}{2}[d(x,\ y) + d(y,\ x)] \\ &= kd(x,\ y) \end{aligned}$$

Since $0 < k < 1$, we have that $x = y$.

This shows that the strong coupled fixed point is unique. This completes the proof of the theorem. □

The following is an illustrative example of the above theorem.

Example 2.1. *Let $X = R$ with the metric defined as $d(x, y) = |x - y|$, where $x, y \in X$.*

Let $A = [0, \infty)$ and $B = (-\infty, 0]$.

Then A and B are nonempty closed subsets of X.

Let the function F be defined as $F(x, y) = -\dfrac{x - y}{4}$, where $x, y \in X$.

Then F is a coupling and it satisfies inequality (2.1). Then theorem 2.1 can be applied to this example and we have a strong coupled fixed point of F. Here $(0,0)$ is the unique strong coupled fixed point of F.

3. Chatterjea Type Coupling

In this section, by extending the idea of Chatterjea contraction and combining it with the concept of coupling we define the Chatterjea type coupling and show that these couplings have strong unique coupled fixed points in metric spaces. We also discuss an example.

Definition 3.1. *Let A and B be two non-empty subsets of a complete metric space (X,d). A coupling $F : X \times X \to X$ is called a Chatterjea type coupling with respect to A and B if it satisfies the following inequality:*

$$d(F(x,y),F(u,v)) \leq k[d(x,F(u,v)) + d(u,F(x,y))] \tag{3.1}$$

where $x,v \in A$, $y,u \in B$ and $k \in (0,\frac{1}{2})$.

Theorem 3.1. *Let A and B be two non-empty closed subsets of a complete metric space (X,d). Let $F : X \times X \to X$ be a Chatterjea type coupling with respect to A and B. Then $A \cap B \neq \phi$ and F has a unique strong coupled fixed point in $A \cap B$.*

Proof. Let $x_0 \in A$ and $y_0 \in B$ be any two elements and the sequences $\{x_n\}$ and $\{y_n\}$ be defined as

$$x_{n+1} = F(y_n,x_n) \text{ and } y_{n+1} = F(x_n,y_n) \text{ for all } n \geq 0. \tag{3.2}$$

Then, for all $n \geq 0$, $x_n \in A$ and $y_n \in B$.

Now, by (3.1) and (3.2), we have

$$\begin{aligned} d(x_1,y_2) &= d(F(y_0,x_0),F(x_1,y_1)) \\ &\leq k[d(y_0,F(x_1,y_1)) + d(x_1,F(y_0,x_0))] \\ &= k[d(y_0,y_2) + d(x_1,x_1)] \\ &= kd(y_0,y_2) \\ &\leq k[d(y_0,x_1) + d(x_1,y_2)] \end{aligned}$$

or

$$d(x_1,y_2) \leq \frac{k}{1-k} d(y_0,x_1),$$

that is,

$$d(x_1,y_2) \leq td(y_0,x_1), \tag{3.3}$$

where

$$0 \leq t = \frac{k}{1-k} \leq 1. \tag{3.4}$$

Again, by (3.1) and (3.2), we have

$$\begin{aligned} d(y_1,x_2) &= d(F(x_0,y_0),F(y_1,x_1)) \\ &\leq k[d(x_0,F(y_1,x_1))) + d(y_1,F(x_0,y_0))] \\ &= k[d(x_0,x_2) + d(y_1,y_1)] \\ &= kd(x_0,x_2) \\ &\leq k[d(x_0,y_1) + d(y_1,x_2)] \end{aligned}$$

or,

$$d(y_1, x_2) \leq \frac{k}{1-k} d(x_0, y_1),$$

that is,

$$d(y_1, x_2) \leq t d(x_0, y_1), \tag{3.5}$$

where t is the same as in (3.4).

Again, by (3.1) and (3.2), we have

$$\begin{aligned} d(x_2, y_3) &= d(F(y_1, x_1), F(x_2, y_2)) \\ &\leq k[d(y_1, F(x_2, y_2)) + d(x_2, F(y_1, x_1))] \\ &= k[d(y_1, y_3) + d(x_2, x_2)] \\ &= kd(y_1, y_3) \\ &\leq k[d(y_1, x_2) + d(x_2, y_3)] \end{aligned}$$

or, by (3.4) and (3.5),

$$d(x_2, y_3) \leq \frac{k}{1-k} d(y_1, x_2) \leq \frac{k}{1-k}[td(x_0, y_1)] = t^2 d(x_0, y_1) \tag{3.6}$$

and by (3.1) and (3.2), we have

$$\begin{aligned} d(y_2, x_3) &= d(F(x_1, y_1), F(y_2, x_2)) \\ &\leq k[d(x_1, F(y_2, x_2)) + d(y_2, F(x_1, y_1))] \\ &= k[d(x_1, x_3) + d(y_2, y_2)] \\ &= kd(x_1, x_3) \\ &\leq k[d(x_1, y_2) + d(y_2, x_3)] \end{aligned}$$

or, by (3.3) and (3.4), we have

$$d(y_2, x_3) \leq \frac{k}{1-k} d(x_1, y_2) \leq \frac{k}{1-k}[td(y_0, x_1)] = t^2 d(y_0, x_1). \tag{3.7}$$

Again, by (3.1) and (3.2) , we have

$$\begin{aligned} d(x_3, y_4) &= d(F(y_2, x_2), F(x_3, y_3)) \\ &\leq k[d(y_2, F(x_3, y_3)) + d(x_3, F(y_2, x_2))] \\ &= k[d(y_2, y_4) + d(x_3, x_3)] \\ &= kd(y_2, y_4) \\ &\leq k[d(y_2, x_3) + d(x_3, y_4)] \end{aligned}$$

or, by (3.4) and (3.7), we have

$$d(x_3, y_4) \leq \frac{k}{1-k} d(y_2, x_3) \leq \frac{k}{1-k}[t^2 d(y_0, x_1)] = t^3 d(y_0, x_1)]. \tag{3.8}$$

and by (3.1) and (3.2), we have

$$\begin{aligned} d(y_3, x_4) &= d(F(x_2, y_2), F(y_3, x_3)) \\ &\leq k[d(x_2, F(y_3, x_3)) + d(y_3, F(x_2, y_2))] \\ &= k[d(x_2, x_4) + d(y_3, y_3)] \\ &= kd(x_2, x_4) \\ &\leq k[d(x_2, y_3) + d(y_3, x_4)] \end{aligned}$$

or, by (3.4) and (3.6), we have

$$d(y_3, x_4) \leq \frac{k}{1-k} d(x_2, y_3) \leq \frac{k}{1-k}[t^2 d(x_0, y_1)] = t^3 d(x_0, y_1)]. \tag{3.9}$$

Let m be any integer. Let us assume that

$$d(x_n, y_{n+1}) = d(x_n, F(x_n, y_n)) \leq t^n d(y_0, x_1), \tag{3.10}$$

$$d(y_n,\ x_{n+1}) = d(y_n, F(y_n,\ x_n)) \leq t^n d(x_0,\ y_1) \tag{3.11}$$

for all $n \leq m$ where n is odd and

$$d(x_n, y_{n+1}) = d(x_n,\ F(x_n, y_n)) \leq t^n d(x_0,\ y_1), \tag{3.12}$$

$$d(y_n, x_{n+1}) = d(y_n,\ F(y_n,\ x_n)) \leq t^n d(y_0,\ x_1) \tag{3.13}$$

for all $n \leq m$ where n is even.

Let m be even. Then $(m+1)$ is odd.

Then, by (3.1) and (3.2), we have

$$\begin{aligned} d(x_{m+1},\ y_{m+2}) &= d(F(y_m,\ x_m), F(x_{m+1},\ y_{m+1})) \\ &\leq k[d(y_m,\ F(x_{m+1},\ y_{m+1})) + d(x_{m+1}, F(y_m,\ x_m))] \\ &= k[d(y_m,\ y_{m+2}) + d(x_{m+1},\ x_{m+1})] \\ &= kd(y_m,\ y_{m+2}) \\ &\leq k[d(y_m,\ x_{m+1}) + d(x_{m+1},\ y_{m+2})] \end{aligned}$$

or, by (3.4) and (3.13), we have

$$d(x_{m+1},\ y_{m+2}) \leq \frac{k}{1-k} d(y_m,\ x_{m+1}) \leq \frac{k}{1-k}[t^m d(y_0,\ x_1)] = t^{m+1} d(y_0,\ x_1)$$

and by (3.1) and (3.2) , we have

$$\begin{aligned} d(y_{m+1},\ x_{m+2}) &= d(F(x_m,\ y_m), F(y_{m+1},\ x_{m+1})) \\ &\leq k[d(x_m,\ F(y_{m+1},\ x_{m+1})) + d(y_{m+1},\ F(x_m,\ y_m))] \\ &= k[d(x_m,\ x_{m+2}) + d(y_{m+1},\ y_{m+1})] \\ &= kd(x_m,\ x_{m+2}) \\ &\leq k[d(x_m,\ y_{m+1}) + d(y_{m+1},\ x_{m+2})] \end{aligned}$$

or, by (3.4) and (3.12), we have

$$d(y_{m+1},\ x_{m+2}) \leq \frac{k}{1-k} d(x_m,\ y_{m+1}) \leq \frac{k}{1-k}[t^m d(x_0,\ y_1)] = t^{m+1} d(x_0,\ y_1).$$

Again, let m be odd. Then $(m+1)$ is even.

Then, by (3.1) and (3.2), we have

$$\begin{aligned} d(x_{m+1},\ y_{m+2}) &= d(F(y_m,\ x_m), F(x_{m+1},\ y_{m+1})) \\ &\leq k[d(y_m,\ F(x_{m+1}, y_{m+1})) + d(x_{m+1},\ F(y_m,\ x_m))] \\ &= k[d(y_m,\ y_{m+2}) + d(x_{m+1},\ x_{m+1})] \\ &= kd(y_m,\ y_{m+2}) \\ &\leq k[d(y_m,\ x_{m+1}) + d(x_{m+1},\ y_{m+2})] \end{aligned}$$

or, by (3.4) and (3.11), we have

$$d(x_{m+1},\ y_{m+2}) \leq \frac{k}{1-k} d(y_m,\ x_{m+1}) \leq \frac{k}{1-k}[t^m d(x_0,\ y_1)] \leq t^{m+1} d(x_0,\ y_1)$$

and by (3.1) and (3.2), we have

$$\begin{aligned} d(y_{m+1},\ x_{m+2}) &= d(F(x_m,\ y_m), F(y_{m+1},\ x_{m+1})) \\ &\leq k[d(x_m,\ F(y_{m+1},\ x_{m+1})) + d(y_{m+1},\ F(x_m, y_m))] \\ &= k[d(x_m,\ x_{m+2}) + d(y_{m+1},\ y_{m+1})] \\ &= kd(x_m,\ x_{m+2}) \\ &\leq k[d(x_m,\ y_{m+1}) + d(y_{m+1},\ x_{m+2})] \end{aligned}$$

or, by (3.4) and (3.10), we have

$$d(y_{m+1},\ x_{m+2}) \leq \frac{k}{1-k} d(x_m,\ y_{m+1}) \leq \frac{k}{1-k}[t^m d(y_0,\ x_1)] \leq t^{m+1} d(y_0,\ x_1).$$

Thus we can conclude that (3.10), (3.11), (3.12) and (3.13) are valid for $(m+1)$ also. But we have shown in (3.3), (3.5), (3.6) and (3.7) that (3.10)-(3.13) are valid for $m = 1, 2, 3$. Then by induction, we can conclude that (3.10), (3.11), (3.12) and (3.13) are valid for all $n \geq 1$.

From the above we conclude that for all odd integer n, we have

$$d(x_n,\ y_{n+1}) = d(x_n,\ F(x_n,\ y_n)) \leq t^n d(y_0,\ x_1), \tag{3.14}$$

$$d(y_n,\ x_{n+1}) = d(y_n,\ F(y_n,\ x_n)) \leq t^n d(x_0,\ y_1) \tag{3.15}$$

and for all even integer n, we have

$$d(x_n,\ y_{n+1}) = d(x_n,\ F(x_n, y_n)) \leq t^n d(x_0,\ y_1), \tag{3.16}$$

$$d(y_n,\ x_{n+1}) = d(y_n,\ F(y_n,\ x_n)) \leq t^n d(y_0,\ x_1). \tag{3.17}$$

Now when n is odd. Then, by (3.1), (3.2), (3.14) and (3.15), we have

$$\begin{aligned} d(x_{n+1},\ y_{n+1}) &= d(F(y_n,\ x_n),\ F(x_n,\ y_n)) \\ &\leq k[d(y_n,\ F(x_n,\ y_n)) + d(x_n,\ F(y_n,\ x_n))] \\ &= k[d(y_n,\ y_{n+1}) + d(x_n,\ x_{n+1})] \\ &\leq k[d(y_n,\ x_{n+1}) + d(x_{n+1},\ y_{n+1}) + d(x_n,\ y_{n+1}) + d(y_{n+1},\ x_{n+1})] \\ &= k[d(y_n,\ x_{n+1}) + d(x_n,\ y_{n+1}) + 2d(x_{n+1},\ y_{n+1})] \\ &\leq kt^n[d(x_0,\ y_1) + d(y_0,\ x_1)] + 2kd(x_{n+1},\ y_{n+1}), \end{aligned}$$

that is,

$$d(x_{n+1},\ y_{n+1}) \leq \frac{k}{1-2k} t^n [d(y_0,\ x_1) + d(x_0,\ y_1)] = \frac{t^{n+1}}{1-t}[d(y_0,\ x_1) + d(x_0,\ y_1)]. \tag{3.18}$$

where t is the same as in (3.4) and $\dfrac{k}{1-2k} = \dfrac{t}{1-t}$.

Again, when n is even. Then, by (3.1), (3.2), (3.16) and (3.17), we have

$$\begin{aligned} d(x_{n+1},\ y_{n+1}) &= d(F(y_n,\ x_n), F(x_n,\ y_n)) \\ &\leq k[d(y_n,\ F(x_n, y_n)) + d(x_n,\ F(y_n,\ x_n))] \\ &= k[d(y_n,\ y_{n+1}) + d(x_n,\ x_{n+1})] \\ &\leq k[d(y_n,\ x_{n+1}) + d(x_{n+1},\ y_{n+1}) + d(x_n,\ y_{n+1}) + d(y_{n+1},\ x_{n+1})] \\ &= k[d(y_n,\ x_{n+1}) + d(x_n,\ y_{n+1}) + 2d(x_{n+1},\ y_{n+1})] \\ &\leq kt^n[d(y_0,\ x_1) + d(x_0,\ y_1)] + 2kd(x_{n+1},\ y_{n+1}), \end{aligned}$$

that is,

$$d(x_{n+1},\ y_{n+1}) \leq \frac{k}{1-2k} t^n [d(y_0,\ x_1) + d(x_0,\ y_1)] = \frac{t^{n+1}}{1-t} [d(y_0,\ x_1) + d(x_0,\ y_1)]. \quad (3.19)$$

where t is the same as in (3.4) and $\dfrac{k}{1-2k} = \dfrac{t}{1-t}$.

Therefore, combining (3.18) and (3.19), for all n, we have

$$d(x_n,\ y_n) \leq \frac{t^n}{1-t} [d(y_0,\ x_1) + d(x_0,\ y_1)]. \quad (3.20)$$

It follows that

$$\begin{aligned} \Sigma d(x_n,\ x_{n+1}) + \Sigma d(y_n,\ y_{n+1}) &\leq \Sigma(d(x_n,\ y_n) + d(y_n,\ x_{n+1})) + \Sigma(d(y_n,\ x_n) + d(x_n,\ y_{n+1})) \\ &= \Sigma[d(x_n,\ y_n) + d(y_n,\ x_n)] + \Sigma[d(y_n,\ x_{n+1}) + d(x_n,\ y_{n+1})] \\ &\leq \Sigma[2\frac{t^n}{1-t}[d(y_0,\ x_1) + d(x_0,\ y_1)]] + \Sigma[t^n[d(y_0,\ x_1) + d(x_0,\ y_1)]]. \end{aligned}$$

Since $0 < t < 1$, it follows that $\Sigma d(x_n, x_{n+1}) + \Sigma d(y_n, y_{n+1}) < \infty$.

This implies that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences and hence are convergent to x and y respectively (say).

Since A and B are closed subsets, $\{x_n\} \subset A$ and $\{y_n\} \subset B$, it follows that

$$x_n \to x \in A \text{ and } y_n \to y \in B \text{ as } n \to \infty. \quad (3.21)$$

Again, from (3.20), since $0 < t < 1$, $d(x_n, y_n) \to 0$ as $n \to \infty$.

Therefore, from (3.21),

$$x = y. \quad (3.22)$$

It then follows that $x \in A \cap B$ and hence $A \cap B \neq \phi$.

Now, from (3.1) and (3.2),

$$\begin{aligned} d(x, F(x,\ y)) &\leq d(x,\ x_{n+1}) + d(x_{n+1},\ F(x,y)) \\ &= d(x,\ x_{n+1}) + d(F(y_n,\ x_n),\ F(x,\ y)) \\ &\leq d(x,\ x_{n+1}) + k[d(y_n,\ F(x,\ y)) + d(x, F(y_n,\ x_n))] \\ &= d(x,\ x_{n+1}) + k[d(y_n,\ F(x,\ y)) + d(x,\ x_{n+1})]. \end{aligned}$$

Taking the limit as $n \to \infty$ in the above inequality, using (3.21) and (3.22), we obtain $d(x, F(x,x)) = 0$. So, we conclude that $x = F(x,\ x)$, that is, x is a strong coupled fixed point of F.

If possible, let there be two strong coupled fixed points of F, that is,

$$F(x,\ x) = x \text{ and } F(y,\ y) = y \text{ where } x, y \in A \cap B.$$

Then, by (3.1), we have

$$\begin{aligned} d(x,y) &= d(F(x,\ x),\ F(y,\ y)) \\ &\leq k[d(x,\ F(y,\ y)) + d(y,\ F(x,\ x)] \\ &= k[d(x,\ y) + d(y,\ x)] \\ &= 2kd(x,\ y) \end{aligned}$$

Since $0 < k < \frac{1}{2}$, we have that $x = y$.

This shows that the strong coupled fixed point is unique.

This completes the proof of the theorem. □

Example 3.1. *Let $X = R$ with the metric defined as*

$$d(x,\ y) = |x - y|.$$

Let $A = [-\pi,\ 0]$ and $B = [0,\ \pi]$.

Then A and B are nonempty closed subsets of X.

Let $F : X \times X \to X$ be defined as

$$F(x,\ y) = \begin{cases} -\frac{1}{3} \mid y sin\frac{1}{y} \mid, & if\ \ (x,\ y) \in B \times A, \\ 0, \ \ if\ (x,\ y) \in A \times B, & \\ 2, & otherwise. \end{cases}$$

Then F is a coupling. Let $k = \frac{1}{3}$.

Then all the conditions of the theorem 3.1 are satisfied. By an application of theorem 3.1, there is a strong coupled fixed point of F. Here $(0,\ 0)$ is the unique strong coupled fixed point of F.

4. Conclusions

The concept of coupling arises by a combination of coupled mappings and cyclic mappings. Both categories of mappings are well studied in fixed point theory. It may be of interest to investigate fixed point and related properties for couplings satisfying other types of inequalities as well. This may also be treated as an open problem.

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