AN ITERATIVE ALGORITHM BASED ON SIMPSON METHODS FOR SOLVING FIXED POINT PROBLEM OF NONEXPANSIVE MAPPINGS

Long He[1], Li-Jun Zhu[2], Yuanmin Fu[3]

Fixed point methods have been studied extensively by the Scholars. The purpose of this paper is to suggest an iterative algorithm based on Simpson methods for solving fixed point problem of nonexpansive mappings in Hilbert spaces. Weak convergence of the presented methods has been proved for finding fixed points of nonexpansive mappings.

Keywords: nonexpansive mapping, Simpson methods, weak convergence, fixed point.
MSC2010: 47J05, 47J25, 47H09.

1. Introduction

Fixed point theory has been broadly applied in many areas such as convex optimization problems, nonlinear analysis, equilibrium problems, operator theory, integral and differential equations and so on ([5, 6],[18]-[40]). The famous Banach fixed point theorem says that if $T : H \to H$ is a contraction defined on a complete metric space H, then T has a unique fixed point that a point for which

$$T\bar{x} = \bar{x}.$$

The iterative method of fixed points can be used to solve practical problems with different physical engineering backgrounds such as resolution recognition and signal synthesis in signal processing, image recovery and reconstruction in image processing, power control and bandwidth allocation in CDMA data networks optical imaging problems, the image denoising problem based on wavelet transform, video coding technology, radar antenna mode synthesis problem, rocket tower control design and so on ([33, 34, 38]).

Differential equations is applied in various domain including fundamental mathematic, applied mathematics, science and technology ([7, 12, 14]). The numerical solution of differential equation has been studied extensively, and a large number of numerical methods have appeared, such as Euler method, Trapezoidal formula, Simpson formula and so on. So we are always use these method to solve differential equation. see, e.g., [3, 4, 8, 10, 22].

The fixed point iterative process is also useful for solving the ordinary differential equation. There are many methods for solving the fixed point problems that one of those methods is midpoint method ([2, 15, 16, 41]). The implicit midpoint formula is of the form

$$u_{n+1} = u_n + hf_{(n+1/2)h}\frac{u_n + u_{n+1}}{2},\ n \geq 0.$$

For the ordinary differential equation

$$x^{'} = f(t),\ x(0) = x_0, \tag{1}$$

[1]School of Mathematics and Information Science, North Minzu University, Yinchuan, 750021, China, e-mail: `helongdyx@hotmail.com`
[2]Corresponding author, The Key Laboratory of Intelligent Information and Data Processing of NingXia Province; and North Minzu University, Yinchuan 750021, China, e-mail: `zhulijun1995@yahoo.com`
[3]School of Mathematics and Information Science, North Minzu University, Yinchuan, 750021, China, e-mail: `fuyuanmin126@126.com`

the implicit midpoint rule generates a sequence $\{x_n\}$ by the recursion procedure

$$x_{n+1} = x_n + hf\left(\frac{x_n + x_{n+1}}{2}\right),\ n \geq 0, \tag{2}$$

where $h > 0$ is a stepsize. It is known that if $f : \mathbb{R}^N \to \mathbb{R}^N$ is Lipschitz continuous and sufficiently smooth, then the sequence $\{x_n\}$ generated by (2) converges to the exact solution of (1) as $h \to 0$ uniformly over $t \in [0, \bar{t}]$ for any fixed $\bar{t} > 0$.

According to (2), Alghamdi, Alghamdi, Shahzad and Xu [2] presented the following implicit midpoint rule for nonexpansive mappings:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T\left(\frac{x_n + x_{n+1}}{2}\right), n \geq 0, \tag{3}$$

where $\alpha_n \in (0, 1)$ and $T : H \to H$ is a nonexpansive mapping. They proved the weak convergence of (3) under some additional conditions on $\{\alpha_n\}$.

Consequently, the midpoint rule has been studied extensively for solving fixed point problem of nonlinear mappings, see, for instance [11, 13, 17, 23]. However, we note that Euler formula only has first-order algebraic precision.

In this paper, we will continue to consider high algebraic precision methods for solving the fixed point problem of nonexpansive mappings. Our method is based on the well-known Simpson method which has higher algebraic precision than Euler method. We suggest a new iterative algorithm with weak convergence for finding fixed points of nonexpansive mappings. This paper is arranged as follows: Section 2 is concerned with some definitions and lemmas which will be applicable in proving our results. In Section 3, we present our iterative methods of solution connected with fixed point theory. Section 4 is to discuss the properties and to give the convergence analysis of suggested algorithms, respectively.

2. Notations and Lemmas

In this part, we collect some definitions and lemmas which will be used in the following parts. Let H be a real Hilbert space with inner project $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. Let C be a nonempty closed convex subset of H. It is well-known that the following equality holds in Hilbert spaces, which can be verified directly.

$$\|tx + (1 - t)y\|^2 = t\|x\|^2 + (1 - t)\|y\|^2 - t(1 - t)\|x - y\|^2 \tag{4}$$

for all x,y in H and t in $[0, 1]$, the generalization of this equality is as follows

$$\| \sum_{i=1}^{s} \alpha_i x_i \|^2 = \sum_{i=1}^{s} \alpha_i \|x_i\|^2 - \frac{1}{2} \sum_{i,j=1}^{s} \alpha_i \alpha_j \|x_i - x_j\|^2 \tag{5}$$

for all $x_i (i = 1, \cdots, s)$, $x_j (j = 1, \cdots, s)$ in H and $\{\alpha_i\}_{i=1}^{s} \subset [0, 1]$ with $\sum_{i=1}^{s} \alpha_i = 1$ ([4]).

Definition 2.1. *Recall that a mapping $T : C \to C$ is said to be nonexpansive if*

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in C$.

We will use the following expressions:

(i) $Fix(T)$ stands for the set of fixed points of T. If T is nonexpansive, then $Fix(T)$ is closed and convex.

(ii) " $\rightharpoonup$ " stands for the weak convergence and " $\to$ " stands for the strong convergence.

(iii) $\omega_w(u_n) = \{u^\dagger : \exists \{u_{n_i}\} \subset \{u_n\}$ such that $u_{n_i} \rightharpoonup u^\dagger (i \to \infty)\}$.

Lemma 2.1 ([9]). *Let C be a nonempty closed convex subset of a Hilbert space H, and let $T : C \to C$ be a nonexpansive mapping with $Fix(T) \neq 0$. Assume that $\{y_n\}$ is a sequence in C such that $y_n \rightharpoonup x^*$ and $(I - T)y_n \to 0$, then $x^* \in Fix(T)$.*

Lemma 2.2 ([1]). *Let C be a nonempty closed convex subset of a Hilbert space H, and let $\{y_n\}$ be a bounded sequence in H. Assume that*
(i) $\lim_{n\to\infty} \|y_n - \tilde{x}\|$ *exists for all $\tilde{x}$ in C;*
(ii) $\omega_\omega(y_n) \subseteq C$.
Then the sequence $\{y_n\}$ weakly converges to a point in C.

3. **Iterative method**

For the differential equation, we often use the Euler methods to get the numerical solution, but it has first-order algebraic precision. In this section, we continue to consider higher-order algebraic accuracy and use the fixed methods to construct an algorithm for solving the fixed point problem of nonexpansive mappings.

We consider the ordinary differential equation

$$y'(x) = f(x, y(x)),\ y(0) = y_0, \tag{6}$$

where $f{:}[0,x] \times R^n$.

Write $f_x(\cdot) = f(x,\cdot)$ and let $h > 0$ be a step of iterative. By applying Euler method, we have

$$y((n+1)h) = y(nh) + hf_{nh}(y(nh)). \tag{7}$$

For simplicity, we use a simple way, we have approximative plots $y_n = y(nh)$, so $f_{nh}(y(nh)) = f_n(y_n)$ and (7) is rewritten as

$$y_{n+1} = y_n + hf_n(y_n). \tag{8}$$

This is a forward Euler foemula and we have a backword Euler formula following

$$y_{n+1} = y_n + hf_n(y_{n+1}). \tag{9}$$

The sequence y_n converges to the exact solution of formula (9) as $h \to 0$ uniformly over x in any fixed finite time in interval $[0, x]$.

The combination of formula (8) and (9) is implicit midpoint rule

$$y_{n+1} = y_n + \frac{1}{2}[hf_n(y_n) + hf_n(y_{n+1})] \tag{10}$$

and the generalization of this formula is that

$$y_{n+1} = y_n + h[\alpha_{n,1}f_n(y_n) + \alpha_{n,2}f_n(y_{n+1})],\ \alpha_{n,1} + \alpha_{n,2} = 1. \tag{11}$$

For this formula, if $\alpha_{n,1} = 1,\ \alpha_{n,2} = 0$, it reduces to (8); if $\alpha_{n,1} = 0,\ \alpha_{n,2} = 1$, it reduces to (9); if $\alpha_{n,1} = 1/2,\ \alpha_{n,2} = 1/2$, it reduces to (10).

If we consider higher order algebraic precision which use Simpson method and study this case when one point divide the interval equally on a fixed length h which insert one points in this interval. The promotion based on the Simpson method is the following

$$y_{(n+1)h} = y_{nh} + \frac{h}{6}(f_{nh}(y_{nh}) + 4f_{(n+1/2)h}(y_{(n+1/2)h}) + f_{(n+1)h}(y_{(n+1)h})), n \in Z^+ \tag{12}$$

where $f{:}[0,x] \times R^n$ and $h > 0$ is a step of iterative. This is well known Simpson methods of differential equation.

If we write the function f in the form $f_x(y(x)) = g_x(y(x)) - y(x)$, then differential equation (1) becomes

$$y'(x) = f(x, y(x)) = f_x(y(x)) = g_x(y(x)) - y(x),\ y(0) = y_0. \tag{13}$$

The equilibrium problem associated with differential equation is the common fixed point problem

$$y(x) = g_x(y(x)). \tag{14}$$

This make us to transplant the iterative formula for solving of the common fixed point equation

$$y(x) = Ty(x), \tag{15}$$

where $T = g_x$ in above, and in general, is a nonlinear operator on a Hilbert space H.

Now we study Simpson methods of differential equation. For formula (12), we can get

$$\begin{aligned} y_{(n+1)} =& y_{(n)} + \frac{h}{6}(f_n(y_{(n)}) + 4f_{(n+1/2)}(y_{(n+1/2)}) + f_{(n+1)}(y_{(n+1)})) \\ =& \frac{1}{6}[y_{(n)} + hf_n(y_{(n)})] + \frac{1}{6}[y_{(n)} + hf_{(n+1)}(y_{(n+1)})] \\ &+ \frac{4}{6}[y_{(n)} + hf_{(n+1/2)}(y_{(n+1/2)})]. \end{aligned} \tag{16}$$

Set $y_{(n)} = y_n$, $y_{(n+1/2)} = \frac{y_n+y_{n+1}}{2}$ and $f_{(n+1/2)} = g_{(n+1/2)} - I$. Then, we obtain

$$\begin{aligned} y_{n+1} = y_n &+ \frac{1}{6}[hg_n(y_n) - hy_n] + \frac{1}{6}[hg_{(n+1)}(y_{n+1}) - hy_{n+1}] \\ &+ \frac{4}{6}[hg_{(n+1/2)}(\frac{y_n + y_{n+1}}{2}) - h\frac{(y_n + y_{n+1})}{2}], \end{aligned}$$

then we have

$$y_{n+1} = \frac{2-h}{2+h}y_n + \frac{h}{6+3h}[g_n(y_n) + 4g_{(n+1/2)}(\frac{y_n + y_{n+1}}{2}) + g_{(n+1)}(y_{n+1})]. \tag{17}$$

In (17), let $g_n = T$ and $\beta_n = \frac{2h}{2+h}$, we get

$$y_{n+1} = (1-\beta_n)y_n + \beta_n[\frac{1}{6}T(y_n) + \frac{4}{6}T(\frac{y_n + y_{n+1}}{2}) + \frac{1}{6}T(y_{n+1})],$$

where $\beta_n > 0$ for all $n > 0$.

Finally, the Simpson iteration we get is

$$\begin{cases} y_{n+1} = (1-\beta_n)y_n + \beta_n W(y_n),\ \beta_n \in [0,1], \\ W(y_n) = \frac{1}{6}T(y_n) + \frac{4}{6}T(\frac{y_n + y_{n+1}}{2}) + \frac{1}{6}T(y_{n+1}),\ n > 0. \end{cases}$$

4. **Main results**

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a nonexpansive mapping with nonempty fixed point set $Fix(T) \neq \emptyset$.

Algorithm 4.1. *Initialize $x_0 \in C$ arbitrarily and define the iterate*

$$\begin{cases} y_{n+1} = (1-\beta_n)y_n + \beta_n W(y_n),\ \beta_n \in (0,1), \\ W(y_n) = \frac{1}{6}T(y_n) + \frac{4}{6}T(\frac{y_n + y_{n+1}}{2}) + \frac{1}{6}T(y_{n+1}),\ n > 0. \end{cases} \tag{18}$$

Remark 4.1. *Algorithm 4.1 is well-defined.*

As a matter of fact, for each of $u \in C$ and $\beta_n \in (0,1)$, we can defined a mapping

$$x \to S_n x = (1-\beta_n)u + \beta_n[\frac{1}{6}Tu + \frac{4}{6}T(\frac{u+x}{2}) + \frac{1}{6}Tx].$$

Observe that for any $x_1, x_2 \in C$,

$$\begin{aligned}\|S_n x_1 - S_n x_2\| =& \|(1-\beta_n)u + \beta_n[\frac{1}{6}Tu + \frac{4}{6}T(\frac{u+x_1}{2}) + \frac{1}{6}Tx_1] \\ &- (1-\beta_n)u + \beta_n[\frac{1}{6}Tu + \frac{4}{6}T(\frac{u+x_2}{2}) + \frac{1}{6}Tx_2]\| \\ =& \beta_n\|\frac{4}{6}T(\frac{u+x_1}{2}) + \frac{1}{6}Tx_1 - \frac{4}{6}T(\frac{u+x_2}{2}) - \frac{1}{6}Tx_2\| \\ \le& \frac{4}{6}\beta_n\|T(\frac{u+x_1}{2}) - T(\frac{u+x_2}{2})\| + \frac{1}{6}\beta_n\|Tx_1 - Tx_2\| \\ \le& \frac{4}{6}\beta_n\|\frac{u+x_1}{2} - \frac{u+x_2}{2}\| + \frac{1}{6}\beta_n\|x_1 - x_2\| \\ \le& \frac{1}{2}\beta_n\|x_1 - x_2\|.\end{aligned}$$

This means that operator S_n is a contraction with coefficient $\frac{1}{2}\beta_n$ in $(0, 1/2)$, hence, iterative method has a unique fixed point in C, thus Algorithm 4.1 is well-defined.

Proposition 4.1. *Let $\{y_n\}$ be the sequence generated by the Algorithm 4.1. Then,*
(*a*) $\|y_{n+1} - p\| \le \|y_n - p\|$ *for all* $n \ge 0$ *and* $p \in Fix(T)$.
(*b*) $\sum_{n=1}^{\infty} \beta_n\|y_n - y_{n+1}\|^2 < \infty$.
(*c*) $\sum_{n=1}^{\infty}(1-\beta_n)\beta_n\|y_n - W(y_n))\|^2 < \infty$.

Proof. Take $p \in Fix(T)$. Form (18) and applying (4), we have

$$\begin{aligned}\|W(y_n) - p\|^2 &= \|\frac{1}{6}T(y_n) + \frac{4}{6}T(\frac{y_n + y_{n+1}}{2}) + \frac{1}{6}T(y_{n+1}) - p\|^2 \\ &\le \frac{1}{6}\|T(y_n) - p\|^2 + \frac{4}{6}\|T(\frac{y_n + y_{n+1}}{2}) - p\|^2 + \frac{1}{6}\|T(y_{n+1}) - p\|^2 \\ &\le \frac{1}{6}\|y_n - p\|^2 + \frac{4}{6}\|\frac{y_n + y_{n+1}}{2} - p\|^2 + \frac{1}{6}\|y_{n+1} - p\|^2 \\ &= \frac{1}{6}\|y_n - p\|^2 + \frac{4}{6}[\frac{1}{2}\|y_n - p\|^2 + \frac{1}{2}\|y_{n+1} - p\|^2 - \frac{1}{4}\|y_{n+1} - y_n\|^2] \\ &\quad + \frac{1}{6}\|y_{n+1} - p\|^2 \\ &= \frac{1}{2}\|y_n - p\|^2 - \frac{1}{6}\|y_{n+1} - y_n\|^2 + \frac{1}{2}\|y_{n+1} - p\|^2.\end{aligned} \tag{19}$$

According to (18), (19) and (4), we obtain

$$\begin{aligned}\|y_{n+1} - p\|^2 =& \|(1-\beta_n)y_n + \beta_n W(y_n) - p\|^2 \\ =& (1-\beta_n)\|y_n - p\|^2 + \beta_n\|W(y_n) - p\|^2 - (1-\beta_n)\beta_n\|y_n - W(y_n)\|^2 \\ \le& (1 - \frac{1}{2}\beta_n)\|y_n - p\|^2 + \frac{1}{2}\beta_n\|y_{n+1} - p\|^2 - (1-\beta_n)\beta_n\|y_n - W(y_n)\|^2 \\ &- \frac{1}{6}\beta_n\|y_n - y_{n+1}\|^2\end{aligned}$$

It follows that

$$\begin{aligned}\|y_{n+1} - p\|^2 \le& \|y_n - p\|^2 - \frac{1}{6}(1 - \frac{1}{2}\beta_n)^{-1}\beta_n\|y_n - y_{n+1}\|^2 \\ &- (1 - \frac{1}{2}\beta_n)^{-1}(1-\beta_n)\beta_n\|y_n - W(y_n)\|^2.\end{aligned} \tag{20}$$

We deduce immediately that

$$\|y_{n+1} - p\| \le \|y_n - p\|. \tag{21}$$

Note that $\beta_n \in (0,1)$. Thus, $(1-\frac{1}{2}\beta_n)^{-1} > 1$ and hence $(1-\frac{1}{2}\beta_n)^{-1}\beta_n > 0$ and $(1-\frac{1}{2}\beta_n)^{-1}(1-\beta_n)\beta_n > 0$. Further, by virtue of (20), we deduce

$$\sum_{n=1}^{\infty}\beta_n\|y_n - y_{n+1}\|^2 < \infty$$

and

$$\sum_{n=1}^{\infty}(1-\beta_n)\beta_n\|y_n - W(y_n)\|^2 < \infty.$$

□

Proposition 4.2. *Let $\{y_n\}$ be the sequence generated by Algorithm 4.1. Assume that the iterative parameter β_n satisfies the control condition $(C1) : \beta_{n+1}^2 \le a\beta_n, \forall n \ge 0$ for some $a > 0$. Then, $\lim_{n\to\infty}\|y_{n+1} - y_n\| = 0$.*

Proof. Since T is nonexpansive, we have

$$\begin{aligned}
\|y_{n+2} - y_{n+1}\| &= \beta_{n+1}\|y_{n+1} - W(y_{n+1})\| \le \beta_{n+1}\|y_{n+1} - W(y_n)\| + \beta_{n+1}\|W(y_{n+1}) - W(y_n)\| \\
&\le \beta_{n+1}(1-\beta_n)\|y_n - W(y_n)\| + \frac{1}{6}\beta_{n+1}\|T(y_{n+1}) - T(y_n)\| \\
&\quad + \frac{4}{6}\beta_{n+1}\|T(\frac{y_{n+1}+y_{n+2}}{2}) - T(\frac{y_n + y_{n+1}}{2})\| + \frac{1}{6}\beta_{n+1}\|T(y_{n+2}) - T(y_{n+1})]\| \\
&\le \beta_{n+1}(1-\beta_n)\|y_n - W(y_n)\| + \frac{1}{2}\beta_{n+1}\|y_{n+1} - y_n\| \\
&\quad + \frac{1}{2}\beta_{n+1}\|y_{n+2} - y_{n+1}\|.
\end{aligned}$$

In the light of (18), we derive

$$\begin{aligned}
\|y_{n+2} - y_{n+1}\| \le& (1-\frac{1}{2}\beta_n)^{-1}\beta_{n+1}(1-\beta_n)\|y_n - W(y_n)\| \\
&+ \frac{1}{2}(1-\frac{1}{2}\beta_n)^{-1}\beta_{n+1}\|y_{n+1} - y_n\| \\
\le& 2\beta_{n+1}(1-\beta_n)\|y_n - W(y_n)\| + \beta_{n+1}\|y_{n+1} - y_n\|.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|y_{n+2} - y_{n+1}\|^2 \le& 4\beta_{n+1}^2(1-\beta_n)^2\|y_n - W(y_n)\|^2 + \beta_{n+1}^2\|y_{n+1} - y_n\|^2 \\
&+ 4\beta_{n+1}^2(1-\beta_n)\beta_n\|y_n - W(y_n)\|^2 \\
\le& 4\beta_{n+1}^2(1-\beta_n)\|y_n - W(y_n)\|^2 + \beta_{n+1}^2\|y_{n+1} - y_n\|^2.
\end{aligned}$$

Using the condition (C1), we further derive that

$$\|y_{n+2} - y_{n+1}\|^2 \le 4a(1-\beta_n)\beta_n\|y_n - W(y_n)\|^2 + a\beta_n\|y_{n+1} - y_n\|^2.$$

This together with Proposition 4.1 implies that $\lim_{n\to\infty}\|y_{n+1} - y_{n+2}\| = 0$. □

Proposition 4.3. *Let $\{y_n\}$ be the sequence generated by Algorithm 4.1. Assume that the iterative parameter β_n satisfies the control condition $(C1)$ and another control condition $(C2) : \liminf_{n\to\infty}\beta_n > 0$, then we have $\lim_{n\to\infty}\|y_n - Ty_n\| = 0$.*

Proof. By virtue of (18), we have

$$\|w(y_n) - Ty_n\| \quad \le \tfrac{1}{6}\|Ty_{n+1} - Ty_n\| + \tfrac{4}{6}\|T(\tfrac{y_n+y_{n+1}}{2}) - Ty_n\| \le \tfrac{1}{2}\|y_{n+1} - y_n\|$$

and

$$\begin{aligned}
\|y_{n+1} - Ty_n\| &\le (1-\beta_n)\|y_n - Ty_n\| + \beta_n\|W(y_n) - Ty_n\| \\
&\le (1-\beta_n)\|y_n - Ty_n\| + \frac{\beta_n}{2}\|y_{n+1} - y_n\|.
\end{aligned}$$

Consequently,

$$\|y_n - Ty_n\| \leq \|y_{n+1} - y_n\| + \|y_{n+1} - Ty_n\|$$
$$\leq \|y_{n+1} - y_n\| + (1-\beta_n)\|y_n - Ty_n\| + \frac{\beta_n}{2}\|y_{n+1} - y_n\|.$$

It follows that $\|y_n - Ty_n\| \leq \frac{2+\beta_n}{2\beta_n}\|y_{n+1} - y_n\|$. Since $\liminf_{n\to\infty} \beta_n > 0$ (by (C2)) and $\lim_{n\to\infty} \|y_{n+1} - y_n\| = 0$ (by Proposition 4.2), we deduce that $\lim_{n\to\infty} \|y_n - Ty_n\| = 0$. □

Theorem 4.1. *Let H be a real Hilbert space and C be a nonempty closed convex subset of H. Let $T : C \to C$ be a nonexpansive mapping and $Fix(T) \neq 0$. Assume that the conditions (C1) and (C2) hold. Then the sequence $\{y_n\}$ generated by Algorithm 4.1 converges weakly to a fixed point of T.*

Proof. Thanks to (21), we deduce that the sequence $\{y_n\}$ is bounded. Pick up any $u^\dagger \in \omega_w(y_n)$. There exists a subsequence $\{y_{n_i}\} \subset \{y_n\}$ such that $y_{n_i} \rightharpoonup z^\dagger$. This together with Lemma 2.1 and Proposition 4.3 implies that $z^\dagger \in Fix(T)$. Thus, $\omega_w(y_n) \subseteq Fix(T)$. Again using (20), we have $\lim_{n\to\infty} \|y_n - z^\dagger\|$ exists for any $z^\dagger \in \omega_w(y_n)$. Consequently, all assumptions of Lemma 2.2 are satisfied. We apply Lemma 2.2 to derive the weak convergence of $\{y_n\}$ to a point in $Fix(T)$. □

Acknowledgments

Li-Jun Zhu was supported by the National Natural Science Foundation of China [grant number 11861003] , the Natural Science Foundation of Ningxia province [grant numbers NZ17015, NXYLXK2017B09].

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