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EXTENDED RICCATI SUB-ODE METHOD FOR NONLINEAR DIFFERENTIAL-DIFFERENCE EQUATIONS

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In this paper, we propose an extended Riccati sub-ODE method to establish new exact solutions for nonlinear differential-difference equations. As a result, new exact solutions including hyperbolic function solutions, trigonometric function solutions and rational solutions are obtained for two nonlinear differential-difference equations, and some of them are generalizations of some known results in the literature.

Keywords: Nonlinear differential-difference equations; Riccati sub-ODE method; Exact solutions; Traveling wave solutions

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1. Introduction

Nonlinear differential-difference equations (NDDEs) can find their applications in many aspects of mathematical physics such as condensed matter physics, biophysics, atomic chains, molecular crystals and quantum physics and so on. Since the work of Fermi, Pasta and Ulam in the 1960s [1], NDDEs have been the focus of many studies for nonlinear phenomena, and much attention have been paid to the research of the theory of NDDEs during the last decades (for example, see [2-10] and the references therein). Among these research works, the investigation of exact solutions of nonlinear differential-difference equations plays an important role in the study of nonlinear physical phenomena. As we all know, it is hard to generalize one method for nonlinear differential equations to solve NDDEs due to the difficulty to search for iterative relations from indices n to $n \pm 1$. Recently, the extensions of some effective methods have been presented and applied for solving some NDDEs successfully in the literature. For example, these methods include the known (G'/G)-expansion method [11-14], the exp-function method [15], the exponential function rational expansion method [16-17], the Jacobi elliptic function method [18-19], Hirota's bilinear method [20], the extended simplest equation method [21], the tanh function method [22] and so on.

In this paper, we propose an extended Riccati sub-ODE method for solving NDDEs, in which the iterative relations from indices n to $n \pm 1$ are established. In

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Section 2, we give the description of the proposed method. Then in Section 3 and 4 we apply the method to solve two nonlinear differential-difference equations: the discrete m-KdV lattice equation [12] and the Toda lattice system [17]. Some Conclusions are presented at the end of the paper.

2. Description of the extended Riccati sub-ODE method

The main steps of the extended Riccati sub-ODE method for solving NDDEs are summarized as follows:

Step 1. Consider a system of M polynomial NDDEs in the form

$$P(u_{n+p_1}(x),...,u_{n+p_k}(x),...,u'_{n+p_1}(x),...,u'_{n+p_k}(x),...,u^{(r)}_{n+p_1}(x),...,u^{(r)}_{n+p_k}(x))=0\,, \quad (2.1)$$

where the dependent variable u has M components u_i, the continuous variable x has N components x_j, the discrete variable n has Q components n_i, the k shift vectors $p_s \in Z^Q$ have Q components p_{sj}, and $u^{(r)}(x)$ denotes the collection of mixed derivative terms of order r.

Step 2. Using a wave transformation

$$u_{n+p_s}(x)=U_{n+p_s}(\xi_{n+p_s}), \xi_n=\sum_{i=1}^{Q} d_i n_i+\sum_{j=1}^{n} c_j x_j+\zeta, \xi_{n+p_s}=\sum_{i=1}^{Q} d_i(n_i+p_{si})+\sum_{j=1}^{n} c_j x_j+\zeta,$$

where d_i, c_j, ζ are all constants, we can rewrite Eq. (2.1) in the following nonlinear ODE:

$$P(U_{n+p_1}(\xi_{n+p_1}),...U_{n+p_k}(\xi_{n+p_k}),...U'_{n+p_1}(\xi_{n+p_1}),...U'_{n+p_k}(\xi_{n+p_k}),...U^{(r)}_{n+p_1}(\xi_{n+p_1}),...U^{(r)}_{n+p_k}(\xi_{n+p_k}))=0 \quad (2.2)$$

Step 3: Suppose the solutions of Eq. (2.2) can be denoted by

$$U_n(\xi_n)=\sum_{i=0}^{l} a_i \phi^i(\xi_n) \quad (2.3)$$

where $\phi^0(\xi_n)=1$, a_i are constants to be determined later, l is a positive integer that can be determined by balancing the highest order linear term with the nonlinear terms in Eq. (2.2), and $\phi(\xi_n)$ satisfies the known Riccati equation:

$$\phi'(\xi_n)=\sigma+\phi^2(\xi_n) \quad (2.4)$$

Step 4: We present some special solutions $\phi_1,...,\phi_6$ for Eq. (2.4):

When $\sigma<0$:

$$\begin{cases} \phi_1(\xi_n)=-\sqrt{-\sigma}\tanh(\sqrt{-\sigma}\xi_n+c_0), \phi_1(\xi_n)=-\sqrt{-\sigma}\coth(\sqrt{-\sigma}\xi_n+c_0) \\ \phi_{1,2}(\xi_{n+p_s})=\dfrac{\phi_{1,2}(\xi_n)-\sqrt{-\sigma}\tanh(\sqrt{-\sigma}\sum_{i=1}^{Q} d_i p_{si})}{1-\dfrac{\phi_{1,2}(\xi_n)}{\sqrt{-\sigma}}\tanh(\sqrt{-\sigma}\sum_{i=1}^{Q} d_i p_{si})} \end{cases}, \quad (2.5)$$

where c_0 is an arbitrary constant.

When $\sigma > 0$:

$$\begin{cases} \phi_3(\xi_n) = \sqrt{\sigma}\tan(\sqrt{\sigma}\xi_n + c_0), \phi_4(\xi_n) = -\sqrt{\sigma}\cot(\sqrt{\sigma}\xi_n + c_0) \\ \phi_{3,4}(\xi_{n+p_s}) = \dfrac{\phi_{3,4}(\xi_n) + \sqrt{\sigma}\tan(\sqrt{\sigma}\sum_{i=1}^{Q} d_i p_{si})}{1 - \dfrac{\phi_{3,4}(\xi_n)}{\sqrt{\sigma}}\tan(\sqrt{\sigma}\sum_{i=1}^{Q} d_i p_{si})} \end{cases}, \quad (2.6)$$

and

$$\begin{cases} \phi_5(\xi_n) = \sqrt{\sigma}[\tan(2\sqrt{\sigma}\xi_n + c_0) + |\sec(2\sqrt{\sigma}\xi_n + c_0)|] \\ \phi_5(\xi_{n+p_s}) = \dfrac{\phi_5^{(1)}(\xi_n) + \sqrt{\sigma}\tan(2\sqrt{\sigma}\sum_{i=1}^{Q} d_i p_{si})}{1 - \dfrac{\phi_5^{(1)}(\xi_n)}{\sqrt{\sigma}}\tan(2\sqrt{\sigma}\sum_{i=1}^{Q} d_i p_{si})} + \dfrac{\phi_5^{(2)}(\xi_n)\sec(2\sqrt{\sigma}\sum_{i=1}^{Q} d_i p_{si})}{1 - \dfrac{\phi_5^{(1)}(\xi_n)}{\sqrt{\sigma}}\tan(2\sqrt{\sigma}\sum_{i=1}^{Q} d_i p_{si})} \end{cases}, \quad (2.7)$$

where $\phi_5^{(1)}(\xi_n) = \sqrt{\sigma}\tan(2\sqrt{\sigma}\xi_n + c_0), \phi_5^{(2)}(\xi_n) = \sqrt{\sigma}\,|\sec(2\sqrt{\sigma}\xi_n + c_0)|$, and c_0 is an arbitrary constant.

When $\sigma = 0$:

$$\begin{cases} \phi_6(\xi_n) = -\dfrac{1}{\xi_n + c_0} \\ \phi_6(\xi_{n+p_s}) = \dfrac{\phi_6(\xi_n)}{1 - \phi_6(\xi_n)\sum_{i=1}^{Q} d_i p_{si}} \end{cases}, \quad (2.8)$$

where c_0 is an arbitrary constant.

Step 5: Substituting (2.3) into Eq. (2.2), by use of Eqs. (2.4)-(2.8), the left hand side of Eq. (2.2) can be converted into a polynomial in $\phi(\xi_n)$. Equating each coeffcient of $\phi^i(\xi_n)$ to zero, yields a set of algebraic equations. Solving these equations, we can obtain the values of a_i, d_i, c_j .

Step 6: Substituting the values of a_i into (2.3), and combining with the various solutions of Eq. (2.4), we can obtain a variety of exact solutions for Eq. (2.1).

2. Application of the extended Riccati sub-ODE method to the discrete m-KdV lattice equation

In this section, we will apply the extended Riccati sub-ODE method to the discrete m-KdV lattice equation [12]:

$$\dot{u}_n(t)=(\alpha-u_n^2)(u_{n+1}-u_{n-1}), \tag{3.1}$$

where $u_n=u_n(t), n\in Z$.

Using a wave transformation

$$u_n=U_n(\xi_n), \xi_n=d_1n+c_1t+\zeta, \tag{3.2}$$

where d_1, c_1, ζ are all constants, Eq. (3.1) can be rewritten in the following ODE:

$$c_1U_n^{'}-(\alpha-U_n^2)(U_{n+1}-U_{n-1})=0. \tag{3.3}$$

Suppose the solutions $U_n(\xi_n)$ for Eq. (3.3) can be denoted by

$$U_n(\xi_n)=\sum_{i=0}^{l}a_i\phi^i(\xi_n), \tag{3.4}$$

where $\phi(\xi_n)$ satisfies Eq. (2.4). Balancing the order of $U_n^{'}$ and U_n^2 in Eq. (3.3) we obtain $l+1=2l$, and then $l=1$. So we have

$$U_n(\xi_n)=a_0+a_1\phi(\xi_n). \tag{3.5}$$

We will proceed to solve Eq. (3.3) in several cases.

Case 1: If $\sigma<0$, and assume (2.4) and (2.5) hold, then substituting (3.5), (2.4) and (2.5) into Eq. (3.3), collecting the coefficients of $\phi_{1,2}^i(\xi_n)$ and equating them to zero, we obtain a series of algebraic equations:

$$(a1): c_1\tanh^2(\sqrt{-\sigma}d_1)-2a_1^2\sqrt{-\sigma}\tanh(\sqrt{-\sigma}d_1)=0,$$

$$(a2): -4\sqrt{-\sigma}\tanh(\sqrt{-\sigma}d_1)a_1a_0=0,$$

$$(a3): c_1\sigma\tanh^2(\sqrt{-\sigma}d_1)+c_1\sigma+2\sqrt{-\sigma}\tanh(\sqrt{-\sigma}d_1)\alpha-2(-\sigma)^{\frac{3}{2}}\tanh(\sqrt{-\sigma}d_1)a_1^2$$
$$-2\sqrt{-\sigma}\tanh(\sqrt{-\sigma}d_1)a_0^2=0,$$

$$(a4): 4(-\sigma)^{\frac{3}{2}}\tanh(\sqrt{-\sigma}d_1)a_1a_0=0,$$

$$(a5): \frac{\sigma(c_1\sigma\cosh(\sqrt{-\sigma}d_1)+2\sqrt{-\sigma}\sinh(\sqrt{-\sigma}d_1)\alpha-2\sqrt{-\sigma}\sinh(\sqrt{-\sigma}d_1)a_0^2)}{\cosh(\sqrt{-\sigma}d_1)}=0.$$

Solving these equations, yields

$$a_1=\pm\sqrt{-\frac{\alpha}{\sigma}\tanh(\sqrt{-\sigma}d_1)}, a_0=0, d_1=d_1, c_1=\frac{2\alpha}{\sqrt{-\sigma}}\tanh(\sqrt{-\sigma}d_1).$$

So we obtain the following solitary wave solutions:

$$u_n(t)=\pm\sqrt{\alpha}\tanh(\sqrt{-\sigma}d_1)\tanh[\sqrt{-\sigma}(d_1 n+\frac{2\alpha}{\sqrt{-\sigma}}\tanh(\sqrt{-\sigma}d_1)t+\zeta)+c_0],\quad (3.6)$$

and

$$u_n(t)=\pm\sqrt{\alpha}\tanh(\sqrt{-\sigma}d_1)\coth[\sqrt{-\sigma}(d_1 n+\frac{2\alpha}{\sqrt{-\sigma}}\tanh(\sqrt{-\sigma}d_1)t+\zeta)+c_0],\quad (3.7)$$

where d_1, c_0 are arbitrary constants.

Case 2: If $\sigma>0$, and assume (2.4) and (2.6) hold, then substituting (3.5), (2.4) and (2.6) into Eq. (3.3), collecting the coefficients of $\phi_{3,4}^i(\xi_n)$ and equating them to zero, we obtain a series of algebraic equations:

$$(b1): -c_1\tan^2(\sqrt{\sigma}d_1)+2\sqrt{\sigma}\tan(\sqrt{\sigma}d_1)a_1^2=0,$$

$$(b2): 4\sqrt{\sigma}\tan(\sqrt{\sigma}d_1)a_1a_0=0,$$

$$(b3): -c_1\sigma\tan^2(\sqrt{\sigma}d_1)+c_1\sigma-2\sqrt{\sigma}\tan(\sqrt{\sigma}d_1)\alpha+2\sigma^{\frac{3}{2}}\tan(\sqrt{\sigma}d_1)a_1^2$$
$$+2\sqrt{\sigma}\tan(\sqrt{\sigma}d_1)a_0^2=0,$$

$$(b4): 4\sigma^{\frac{3}{2}}\tan(\sqrt{\sigma}d_1)a_1a_0=0,$$

$$(b5): \frac{\sigma(c_1\sigma\cos(\sqrt{\sigma}d_1)-2\sqrt{\sigma}\sin(\sqrt{\sigma}d_1)\alpha+2\sqrt{\sigma}\sinh(\sqrt{\sigma}d_1)a_0^2)}{\cos(\sqrt{\sigma}d_1)}=0.$$

Solving these equations, yields

$$a_1=\pm\sqrt{\frac{\alpha}{\sigma}}\tan(\sqrt{\sigma}d_1), a_0=0, d_1=d_1, c_1=\frac{2\alpha}{\sqrt{\sigma}}\tan(\sqrt{\sigma}d_1).$$

Then we have the following trigonometric function solutions:

$$u_n(t)=\pm\sqrt{\alpha}\tan(\sqrt{\sigma}d_1)\tan[\sqrt{\sigma}(d_1 n+\frac{2\alpha}{\sqrt{\sigma}}\tan(\sqrt{\sigma}d_1)t+\zeta)+c_0],\quad (3.8)$$

and

$$u_n(t)=\pm\sqrt{\alpha}\tan(\sqrt{\sigma}d_1)\cot[\sqrt{\sigma}(d_1 n+\frac{2\alpha}{\sqrt{\sigma}}\tan(\sqrt{\sigma}d_1)t+\zeta)+c_0],\quad (3.9)$$

where d_1, c_0 are arbitrary constants.

In [12, Eqs. (32) and (36)], Ayhan and Bekir presented some exact solutions for m-KdV lattice equation by the (G'/G)-expansion method as follows:

$$u_n=\pm\sqrt{\alpha}\tanh(\frac{\sqrt{\lambda^2-4\mu}}{2}d_1)(\frac{C_1\sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_n)+C_2\cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_n)}{C_1\cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_n)+C_2\sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_n)}),\quad (3.10)$$

where $\xi_n = d_1 n + [\frac{4\alpha}{\sqrt{\lambda^2-4\mu}}\tanh(\frac{\sqrt{\lambda^2-4\mu}}{2}d_1)]t+\zeta$, and

$$u_n = \pm\sqrt{\alpha}\tan(\frac{\sqrt{4\mu-\lambda^2}}{2}d_1)(\frac{-C_1\sin(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_n)+C_2\cos(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_n)}{C_1\cos(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_n)+C_2\sin(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_n)}), \quad (3.11)$$

where $\xi_n = d_1 n + [\frac{4\alpha}{\sqrt{4\mu-\lambda^2}}\tan(\frac{\sqrt{4\mu-\lambda^2}}{2}d_1)]t+\zeta$.

We note that our results (3.6) and (3.8) are solutions of more general forms than Eqs. (3.10) and (3.11). In fact, if we let $c_0 = arth(\frac{C_2}{C_1}), \sigma = \frac{4\mu-\lambda^2}{4}$ or $c_0 = arcoth(\frac{C_1}{C_2}), \sigma = \frac{4\mu-\lambda^2}{4}$, then our result (3.6) reduces to (3.10). If we let $c_0 = \arctan(-\frac{C_2}{C_1}), \sigma = \frac{4\mu-\lambda^2}{4}$ or $c_0 = arccot(-\frac{C_1}{C_2}), \sigma = \frac{4\mu-\lambda^2}{4}$, then our result (3.8) reduces to (3.11).

Case 3: If $\sigma > 0$, and assume (2.4) and (2.7) hold, then substituting (3.6), (2.4) and (2.7) into Eq. (3.4), using $[\phi_5^{(2)}(\xi_n)]^2 = \sigma + [\phi_5^{(1)}(\xi_n)]^2$, collecting the coeffici-ents of $[\phi_5^{(1)}(\xi_n)]^i[\phi_5^{(2)}(\xi_n)]^j$ and equating them to zero, we obtain a series of alge-braic equations:

$$(c1): -2c_1 + 4\sqrt{\sigma}\sin(2\sqrt{\sigma}d_1)a_1^2 + 2c_1\cos^2(2\sqrt{\sigma}d_1) + 4\sqrt{\sigma}\sin(2\sqrt{\sigma}d_1)a_1^2\cos(2\sqrt{\sigma}d_1) = 0,$$

$$(c2): 4\sqrt{\sigma}\sin(2\sqrt{\sigma}d_1)a_1a_0\cos(2\sqrt{\sigma}d_1) + 4\sqrt{\sigma}\sin(2\sqrt{\sigma}d_1)a_1a_0 = 0,$$

$$(c3): 4\sigma^{\frac{3}{2}}\sin(2\sqrt{\sigma}d_1)a_1a_0\cos(2\sqrt{\sigma}d_1) + 4\sigma^{\frac{3}{2}}\sin(2\sqrt{\sigma}d_1)a_1a_0 = 0,$$

$$\begin{aligned}(c4): &-2\sqrt{\sigma}\alpha\sin(2\sqrt{\sigma}d_1)\cos(2\sqrt{\sigma}d_1) + (-c_1 + 4\sqrt{\sigma}a_1^2\sin(2\sqrt{\sigma}d_1) + c_1\cos^2(2\sqrt{\sigma}d_1)\\ &+2\sqrt{\sigma}\sin(2\sqrt{\sigma}d_1)a_1^2\cos(2\sqrt{\sigma}d_1))\sigma + 4\sigma^{\frac{3}{2}}\sin(2\sqrt{\sigma}d_1)a_1^2\cos(2\sqrt{\sigma}d_1))\\ &+3c_1\sigma\cos^2(2\sqrt{\sigma}d_1) - c_1\sigma + 2\sqrt{\sigma}\sin(2\sqrt{\sigma}d_1)a_0^2\cos(2\sqrt{\sigma}d_1)) = 0,\end{aligned}$$

$$\begin{aligned}(c5): &2\sigma^{\frac{3}{2}}\sin(2\sqrt{\sigma}d_1)a_1^2 + 2\sqrt{\sigma}\sin(2\sqrt{\sigma}d_1)a_0^2 - 2\sqrt{\sigma}\sin(2\sqrt{\sigma}d_1)\alpha\\ &+4\sigma^{\frac{3}{2}}\sin(2\sqrt{\sigma}d_1)a_1^2\cos(2\sqrt{\sigma}d_1) + 2c_1\sigma\cos^2(2\sqrt{\sigma}d_1) = 0,\end{aligned}$$

$$\begin{aligned}(c6): &c_1\sigma^2\cos^2(2\sqrt{\sigma}d_1) + 2\sigma^{\frac{3}{2}}\sin(2\sqrt{\sigma}d_1)a_0^2\cos(2\sqrt{\sigma}d_1) - 2\sigma^{\frac{3}{2}}\sin(2\sqrt{\sigma}d_1)\alpha\cos(2\sqrt{\sigma}d_1)\\ &+(2\sigma^{\frac{3}{2}}\sin(2\sqrt{\sigma}d_1)a_1^2\cos(2\sqrt{\sigma}d_1) + c_1\sigma\cos^2(2\sqrt{\sigma}d_1))\sigma = 0\end{aligned}$$

$$(c7): 4\sigma^{\frac{3}{2}}\sin(2\sqrt{\sigma}d_1)a_1a_0\cos(2\sqrt{\sigma}d_1)=0.$$

Solving these equations, we get three families of values as follows:

$$a_1=\pm\sqrt{\frac{\alpha}{\sigma}}, a_0=0, d_1=-\frac{\pi}{4\sqrt{\sigma}}, c_1=-\frac{2\alpha}{\sqrt{\sigma}},$$

$$a_1=\pm\sqrt{\frac{\alpha}{\sigma}}, a_0=0, d_1=\frac{\pi}{4\sqrt{\sigma}}, c_1=\frac{2\alpha}{\sqrt{\sigma}},$$

or

$$a_1=\pm\frac{\sqrt{2\alpha-\alpha\sin^2(2\sqrt{\sigma}d_1)-2\alpha\cos(2\sqrt{\sigma}d_1)}}{\sqrt{\sigma}\sin(2\sqrt{\sigma}d_1)}, a_0=0, d_1=d_1, c_1=-\frac{2\alpha(\cos(2\sqrt{\sigma}d_1)-1)}{\sqrt{\sigma}\sin(2\sqrt{\sigma}d_1)},$$

So we obtain the following trigonometric function solutions:

$$u_n(t)=\pm\sqrt{\alpha}\{\tan[2\sqrt{\sigma}(\frac{\pi}{4\sqrt{\sigma}}n+\frac{2\alpha}{\sqrt{\sigma}}t+\zeta)+c_0]+|\sec[2\sqrt{\sigma}(\frac{\pi}{4\sqrt{\sigma}}n+\frac{2\alpha}{\sqrt{\sigma}}t+\zeta)+c_0]|\}, \quad (3.12)$$

$$u_n(t)=\pm\sqrt{\alpha}\{\tan[2\sqrt{\sigma}(-\frac{\pi}{4\sqrt{\sigma}}n-\frac{2\alpha}{\sqrt{\sigma}}t+\zeta)+c_0]+|\sec[2\sqrt{\sigma}(-\frac{\pi}{4\sqrt{\sigma}}n-\frac{2\alpha}{\sqrt{\sigma}}t+\zeta)+c_0]|\}, \quad (3.13)$$

where c_0 is an arbitrary constant, and

$$u_n(t)=\pm\frac{\sqrt{2\alpha-\alpha\sin^2(2\sqrt{\sigma}d_1)-2\alpha\cos(2\sqrt{\sigma}d_1)}}{\sqrt{\sigma}\sin(2\sqrt{\sigma}d_1)}\{\tan[2\sqrt{\sigma}(d_1n-\frac{2\alpha(\cos(2\sqrt{\sigma}d_1)-1)}{\sqrt{\sigma}\sin(2\sqrt{\sigma}d_1)}t+\zeta)+c_0]$$

$$+|\sec[2\sqrt{\sigma}(d_1n-\frac{2\alpha(\cos(2\sqrt{\sigma}d_1)-1)}{\sqrt{\sigma}\sin(2\sqrt{\sigma}d_1)}t+\zeta)+c_0]|\}, \quad (3.14)$$

where d_1, c_0 are arbitrary constants.

Case 4: If $\sigma=0$, and assume (2.4) and (2.8) hold, then substituting (3.5), (2.4) and (2.8) into Eq. (3.3), collecting the coefficients of $\phi_6^i(\xi_n)$ and equating them to zero, we obtain a series of algebraic equations. Solving these equations, yields

$$a_1=\pm\sqrt{\alpha}d_1, a_0=0, d_1=d_1, c_1=2d_1\alpha.$$

Then we obtain the following rational solution:

$$u_n(t)=\pm\frac{\sqrt{\alpha}d_1}{d_1n+2d_1\alpha t+\zeta+c_0}, \quad (3.15)$$

where d_1, c_0 are arbitrary constants.

Remark 1. We have obtained some exact solutions with more general forms than the known (G'/G)-expansion method for the discrete m-KdV lattice equation. In fact, in the (G'/G)-expansion method, the solution $U_n(\xi_n)$ is denoted by a polynomial in $(\frac{G'(\xi_n)}{G(\xi_n)})$, and $G = G(\xi_n)$ satisfies

$$G'' + \lambda G' + \mu G = 0, \tag{3.16}$$

where λ, μ are constants. If we let in Eq. (3.16) $\frac{G'(\xi_n)}{G(\xi_n)} = -\phi(\xi_n) - \frac{\lambda}{2}, \frac{4\mu - \lambda^2}{4} = \sigma,$ then Eq. (3.16) can be turned into $\phi'(\xi_n) = \sigma + \phi^2(\xi_n)$, which is the Riccati equation (2.4). So $(\frac{G'(\xi_n)}{G(\xi_n)})$ can be expressed in $\phi(\xi_n)$, and the solutions by the (G'/G)-expansion method can be expressed in those by the extended Riccati sub-ODE method, which is to some extent in accordance with the analysis results in [23].

Remark 2. Our results (3.12)-(3.15) have not been reported by other authors so far to our best knowledge.

4. Application of the extended Riccati sub-ODE method to the Toda lattice equation

In this section, we will apply the extended Riccati sub-ODE method to the relativistic Toda lattice system [17]:

$$\begin{cases} \dot{u}_n = (1 + \alpha u_n)(v_n - v_{n-1}) \\ \dot{v}_n = v_n(u_{n+1} - u_n + \alpha v_{n+1} - \alpha v_{n-1}) \end{cases}, \tag{4.1}$$

where $u_n = u_n(t), v_n = v_n(t), n \in Z$.

Using a wave transformation

$$u_n = U_n(\xi_n), v_n = V_n(\xi_n), \xi_n = d_1 n + c_1 t + \zeta,$$

where d_1, c_1, ζ are all constants, the system (4.1) can be rewritten as the following form:

$$\begin{cases} c_1 U_n' = (1 + \alpha U_n)(V_n - V_{n-1}) \\ c_1 V_n' = V_n(U_{n+1} - U_n + \alpha V_{n+1} - \alpha V_{n-1}) \end{cases} \tag{4.2}$$

Suppose the solutions for (4.2) can be denoted by

$$U_n(\xi_n) = \sum_{i=1}^{l_1} a_i \phi^i(\xi_n), \tag{4.3}$$

$$V_n(\xi_n) = \sum_{i=1}^{l_2} b_i \phi^i(\xi_n), \tag{4.4}$$

where $\phi(\xi_n)$ satisfies Eq. (2.4). Balancing the order of $U_n^{'}$ and $U_n V_n$ in Eq. (4.3), and the order of $V_n^{'}$ and $V_n U_n$ in Eq. (4.4), we obtain $l_1 = l_2 = 1$. So we have

$$U_n(\xi_n) = a_0 + a_1\phi(\xi_n). \tag{4.5}$$

$$V_n(\xi_n) = b_0 + b_1\phi(\xi_n). \tag{4.6}$$

Similar to Section 3, we will also proceed to solve Eqs. (4.2) in several cases.

Case 1: If $\sigma < 0$, and assume (2.4) and (2.5) hold, then substituting (4.5), (4.6), (2.4) and (2.5) into (4.2), collecting the coefficients of $\phi_{1,2}^i(\xi_n)$ and equating them to zero, we obtain a series of algebraic equations. Solving these equations, yields

$$a_1 = \frac{\alpha b_0\sqrt{-\sigma}\tanh(\sqrt{-\sigma}d_1)}{\sigma}, a_0 = -\frac{b_0\alpha^2+1}{\alpha}, c_1 = -\frac{\alpha b_0\sqrt{-\sigma}\tanh(\sqrt{-\sigma}d_1)}{\sigma},$$

$$b_1 = -\frac{b_0\sqrt{-\sigma}\tanh(\sqrt{-\sigma}d_1)}{\sigma}, b_0 = b_0, d_1 = d_1.$$

So we obtain the following solitary wave solutions:

$$\begin{cases} u_n(t) = \alpha b_0\tanh(\sqrt{-\sigma}d_1)\tanh[\sqrt{-\sigma}(d_1 n - \frac{\alpha b_0\sqrt{-\sigma}\tanh(\sqrt{-\sigma}d_1)}{\sigma}t + \zeta) + c_0] - \frac{b_0\alpha^2+1}{\alpha} \\ v_n(t) = -b_0\tanh(\sqrt{-\sigma}d_1)\tanh[\sqrt{-\sigma}(d_1 n - \frac{\alpha b_0\sqrt{-\sigma}\tanh(\sqrt{-\sigma}d_1)}{\sigma}t + \zeta) + c_0] + b_0 \end{cases}, \tag{4.7}$$

or

$$\begin{cases} u_n(t) = \alpha b_0\tanh(\sqrt{-\sigma}d_1)\coth[\sqrt{-\sigma}(d_1 n - \frac{\alpha b_0\sqrt{-\sigma}\tanh(\sqrt{-\sigma}d_1)}{\sigma}t + \zeta) + c_0] - \frac{b_0\alpha^2+1}{\alpha} \\ v_n(t) = -b_0\tanh(\sqrt{-\sigma}d_1)\coth[\sqrt{-\sigma}(d_1 n - \frac{\alpha b_0\sqrt{-\sigma}\tanh(\sqrt{-\sigma}d_1)}{\sigma}t + \zeta) + c_0] + b_0 \end{cases}, \tag{4.8}$$

where d_1, c_0, b_0 are arbitrary constants.

Case 2: If $\sigma > 0$, and assume (2.4) and (2.6) hold, then substituting (4.5), (4.6), (2.4) and (2.6) into (4.2), collecting the coefficients of $\phi_{3,4}^i(\xi_n)$ and equating them to zero, we obtain a series of algebraic equations. Solving these equations, yields

$$a_1 = -\frac{\alpha b_0\tan(\sqrt{\sigma}d_1)}{\sqrt{\sigma}}, a_0 = -\frac{b_0\alpha^2+1}{\alpha}, c_1 = \frac{\alpha b_0\tan(\sqrt{\sigma}d_1)}{\sqrt{\sigma}}, \quad b_1 = \frac{b_0\tan(\sqrt{\sigma}d_1)}{\sqrt{\sigma}}, b_0 = b_0, d_1 = d_1.$$

So we obtain the following solitary wave solutions:

$$\begin{cases} u_n(t) = -\alpha b_0 \tan(\sqrt{\sigma} d_1)\tan[\sqrt{\sigma}(d_1 n + \dfrac{\alpha b_0 \tanh(\sqrt{-\sigma} d_1)}{\sqrt{\sigma}} t + \zeta) + c_0] - \dfrac{b_0 \alpha^2 + 1}{\alpha} \\ v_n(t) = b_0 \tan(\sqrt{\sigma} d_1)\tan[\sqrt{-\sigma}(d_1 n + \dfrac{\alpha b_0 \tanh(\sqrt{-\sigma} d_1)}{\sqrt{\sigma}} t + \zeta) + c_0] + b_0 \end{cases}, \quad (4.9)$$

or

$$\begin{cases} u_n(t) = \alpha b_0 \tan(\sqrt{\sigma} d_1)\cot[\sqrt{\sigma}(d_1 n + \dfrac{\alpha b_0 \tanh(\sqrt{-\sigma} d_1)}{\sqrt{\sigma}} t + \zeta) + c_0] - \dfrac{b_0 \alpha^2 + 1}{\alpha} \\ v_n(t) = -b_0 \tan(\sqrt{\sigma} d_1)\cot[\sqrt{-\sigma}(d_1 n + \dfrac{\alpha b_0 \tanh(\sqrt{-\sigma} d_1)}{\sqrt{\sigma}} t + \zeta) + c_0] + b_0 \end{cases}, \quad (4.10)$$

where d_1, c_0, b_0 are arbitrary constants.

Case 3: If $\sigma > 0$, and assume (2.4) and (2.7) hold, then substituting (4.5), (4.6), (2.4) and (2.7) into (4.2), using $[\phi_5^{(2)}(\xi_n)]^2 = \sigma + [\phi_5^{(1)}(\xi_n)]^2$, collecting the coefficients of $[\phi_5^{(1)}(\xi_n)]^i [\phi_5^{(2)}(\xi_n)]^j$ and equating them to zero, we obtain a series of algebraic equations. Solving these equations, we get three families of values as follows:

$$a_1 = -b_1\alpha, a_0 = -\frac{1}{\alpha}, b_1 = b_1, b_0 = 0, d_1 = \frac{\pi}{2\sqrt{\sigma}}, c_1 = b_1\alpha,$$

$$a_1 = -b_1\alpha, a_0 = \mp\frac{\alpha^2 b_1 \sigma + \sqrt{\sigma}}{\sqrt{\sigma}\alpha}, b_1 = b_1, b_0 = \pm b_1\sqrt{\sigma}, c_1 = b_1\alpha, d_1 = \pm\frac{\pi}{4\sqrt{\sigma}},$$

or

$$a_1 = \frac{1}{2\sqrt{\sigma}}\arcsin(\frac{2b_1 b_0\sqrt{\sigma}}{b_0^2 + b_1^2\sigma}), a_0 = -\frac{\alpha^2 b_0 + 1}{\alpha}, b_1 = b_1, b_0 = b_0, c_1 = b_1\alpha, d_1 = \frac{\pi}{2\sqrt{\sigma}}.$$

So we obtain the following trigonometric function solutions:

$$\begin{cases} u_n(t) = -b_1\alpha\sqrt{\sigma}\{\tan[2\sqrt{\sigma}(\dfrac{\pi}{2\sqrt{\sigma}} n + b_1\alpha t + \zeta) + c_0] + |\sec[2\sqrt{\sigma}(\dfrac{\pi}{2\sqrt{\sigma}} n + b_1\alpha t + \zeta) + c_0]|\} - \dfrac{1}{\alpha} \\ v_n(t) = b_1\sqrt{\sigma}\{\tan[2\sqrt{\sigma}(\dfrac{\pi}{2\sqrt{\sigma}} n + b_1\alpha t + \zeta) + c_0] + |\sec[2\sqrt{\sigma}(\dfrac{\pi}{2\sqrt{\sigma}} n + b_1\alpha t + \zeta) + c_0]|\} \end{cases}, \quad (4.11)$$

$$\begin{cases} u_n(t) = -b_1\alpha\sqrt{\sigma}\{\tan[2\sqrt{\sigma}(\pm\dfrac{\pi}{4\sqrt{\sigma}} n + b_1\alpha t + \zeta) + c_0] + |\sec[2\sqrt{\sigma}(\pm\dfrac{\pi}{4\sqrt{\sigma}} n + b_1\alpha t + \zeta) + c_0]|\} \mp \dfrac{\alpha^2 b_1\sigma + \sqrt{\sigma}}{\sqrt{\sigma}\alpha} \\ v_n(t) = b_1\sqrt{\sigma}\{\tan[2\sqrt{\sigma}(\pm\dfrac{\pi}{4\sqrt{\sigma}} n + b_1\alpha t + \zeta) + c_0] + |\sec[2\sqrt{\sigma}(\pm\dfrac{\pi}{4\sqrt{\sigma}} n + b_1\alpha t + \zeta) + c_0]|\} \pm b_1\sqrt{\sigma} \end{cases}, \quad (4.12)$$

where b_1, c_0 are arbitrary constants, and

$$\begin{cases} u_n(t) = \frac{1}{2\sqrt{\sigma}}\arcsin(\frac{2b_1 b_0\sqrt{\sigma}}{b_0^2 + b_1^2\sigma})\{\tan[2\sqrt{\sigma}(\frac{\pi}{2\sqrt{\sigma}}n + b_1\alpha t + \zeta) + c_0] \\ + |\sec[2\sqrt{\sigma}(\frac{\pi}{2\sqrt{\sigma}}n + b_1\alpha t + \zeta) + c_0]|\} - \frac{\alpha^2 b_0 + 1}{\alpha} \\ v_n(t) = b_1\{\tan[2\sqrt{\sigma}(\frac{\pi}{2\sqrt{\sigma}}n + b_1\alpha t + \zeta) + c_0] \\ + |\sec[2\sqrt{\sigma}(\frac{\pi}{2\sqrt{\sigma}}n + b_1\alpha t + \zeta) + c_0]|\} + b_0 \end{cases}, \tag{4.13}$$

where b_1, c_0, b_0 are arbitrary constants.

Case 4: If $\sigma = 0$, and assume (2.4) and (2.8) hold, then substituting (4.5), (4.6), (2.4) and (2.8) into (4.2), collecting the coefficients of $\phi_6^i(\xi_n)$ and equating them to zero, we obtain a series of algebraic equations. Solving these equations, yields

$$a_1 = -\alpha b_0 d_1, a_0 = -\frac{b_0\alpha^2 + 1}{\alpha}, b_1 = d_1 b_0, b_0 = b_0, d_1 = d_1, c_1 = \alpha b_0 d_1.$$

Then we obtain the following rational solutions:

$$\begin{cases} u_n(t) = \frac{\alpha b_0 d_1}{d_1 n + \alpha b_0 d_1 t + \zeta + c_0} - \frac{b_0\alpha^2 + 1}{\alpha} \\ v_n(t) = \frac{-b_0 d_1}{d_1 n + \alpha b_0 d_1 t + \zeta + c_0} + b_0 \end{cases}, \tag{4.14}$$

Remark 3. Our results (4.7)-(4.14) are new exact solutions for the Toda lattice system, and have not been reported by other authors so far to our best knowledge.

5. Conclusions

We have proposed an extended Riccati sub-ODE method for solving nonlinear differential-difference equations, and applied it to find exact solutions of the discrete m-KdV lattice equation and the Toda lattice system. As a result, some generalized exact solutions and solitary wave solutions for them have been successfully found. For the discrete m-KdV lattice equation, we have also compared this method with the known (G'/G)-expansion method. Comparison results show that more exact solutions are obtained by the proposed method than by the (G'/G)-expansion method.

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