AN ALGEBRAIC APPROACH TO VERY TRUE IN MONOIDAL T-NORM BASED LOGIC

Juntao Wang[1], Xiaoli He[2], Xiaolong Xin[3]

In this paper, we expand **MTL** *with an unary connective* **vt** *whose algebraic counterpart is an unary operator playing the role of very true. Then, we prove the corresponding completeness theorem and show that the expanded logic is a conservative extension of* **MTL**. *Finally, we study the algebraic semantics of this logic, very true MTL-algebras. In particular, we show that very true MTL-algebra is representable and their corresponding logic is semilinear.*

Keywords: Fuzzy Logic, MTL-algebra, very true, representation, semilinear.
MSC2020: 06D35; 06B99.

1. Introduction

Among various non-classical logic systems, monoidal t-norm based logic (**MTL** [2] for short) is one of the most significant of them. Jenei and Montagna [14] proved that **MTL** is indeed the logic of all left-continuous t-norms and their residua. The corresponding algebraic structures, MTL-algebras have interesting algebraic properties and cover all the mathematical structures that appear in t-norm based fuzzy logic framework, for example, MV-algebras, BL-algebras, Gödel algebras and NM-algebras. Therefore, MTL-algebras are important structures in which the community of fuzzy logicians have got interested.

Linguistic *hedges* interpreted by fuzzy sets were examined by Lakoff in [13]. After that, Zadeh [4] introduced a computational way to approximate human reasoning with respect to hedges, his main ideas are as follows: vague concepts, including the linguistic truth values, are interpreted as fuzzy sets taking values in the closed unit interval $[0, 1]$. In this formulation they have been represented in fuzzy logic systems (in broad sense) as functions from the set of truth values (typically the real unit interval) into itself that modify the meaning of a proposition by being applied to the membership function of the fuzzy set underlying the proposition. More specifically, in the setting of mathematical fuzzy logic, Hájek proposes in a series of papers [5, 6, 7] to understand them as truth functions of new unary connectives, a kind of modal modifiers or truth modifiers, called truth-stressing or truth-depressing hedges depending on whether they reinforce or weaken the meaning of the proposition they are applied to. The intuitive mathematical interpretation of a truth-stressing hedge on a chain of truth-values is a subdiagonal non-decreasing function preserving 0 and 1. The class of such functions will be called hedge functions from now on in this paper. The notion of hedge functions has been extended to various t-norm based fuzzy logics, such as **BL** [5], **MV** [12] so far. Although these way can be expanded the scope of hedge functions, they both have as codomain the closed unit interval [0,1]. However, logical algebras with hedge functions are

[1]School of Science, Xi'an Shiyou University, Xi'an, 710065, P. R. China, e-mail: `wjt@xsyu.edu.cn`
[2]School of Science, Xi'an Shiyou University, Xi'an, 710065, P. R. China, e-mail: `hexl@xsyu.edu.cn`
[3]School of Mathematics, Northwest University, Xi'an, 710127, P. R. China, e-mail: `xlxin@nwu.edu.cn` (X. L.Xin)

not algebras in the sense of Universal Algebra, and hence they do not automatically induce an assertional logic.

To present a unified approach to hegde functions and introduce in the many valued context a deduction apparatus able to reason by analogy in a logical and algebraic setting, a new approach to hedge functions on BL-algebras was introduced by Hájek [5], where they added an unary operation vt, called very true, to the language of BL-algebras intended to capture some basic properties of hegde functions. The resulting class of algebras were so-called *very true BL-algebras*. This approach generalizes the hegde functions, as a function on the algebra taking values in the interval [0,1] with the addition property. As a consequence, very true has been extended to other logical algebras such as MV-algebras[12], Rℓ-monoids [8], commutative basic algebras [9], equality algebras [10], effect algebras [11], BCK-algebras [18] and Quasi-pseudo-MV-algebras [19] so on.

In this paper, we will provide a more general algebraic foundation for assigning a fuzzy truth value to each variable in **MTL** and obtain the following main results:

(1) $\mathbf{MTL_{vt}}$ is semilinear, that is, every very true MTL-algebra is representable. (See **Theorems 3.3, 3.4**).

(2) $\mathbf{MTL_{vt}}$ is a conservative extension of **MTL**. (See **Theorems 3.5**).

(3) Any hegde function on standard MTL-algebra [0,1] is a sound interpretation of the very true connective in **MTL**, that is, $\mathbf{MTL_{vt}}$ accommodates most of the hedge functions of **MTL** in a broader sense. (See **Remark 3.3**).

(4) $\mathbf{MTL_{vt}}$ has the (finitely) strong $\mathbb{K}$-completeness if and only if **MTL** has the (finitely) strong $\mathbb{K}$-completeness. (See **Theorems 3.8, 3.9**).

The paper is structured as follows. In Sect.2, we overview well known results about **MTL** and its algebraic structures. In Sect.3, we study the logic $\mathbf{MTL_{vt}}$ and their algebraic version, very true MTL-algebras.

2. Preliminaries

In this section, we summarize some definitions and results about **MTL** and its corsponding algebraic semantics, which will be used in the following sections.

Definition 2.1. [2] (Monoidal t-norm based logic) **MTL** is the logic $\mathcal{L}$ given by the Hilbert-style calculus with Modus Ponens (i.e., **MP**: ϕ, $\phi \Rightarrow \psi \vdash \psi$) as its only inference rule and the following axioms:

(MTL1) $(\phi \Rightarrow \psi) \Rightarrow ((\psi \Rightarrow \chi) \Rightarrow (\phi \Rightarrow \chi))$,
(MTL2) $(\phi \& \psi) \Rightarrow \phi$,
(MTL3) $(\phi \& \psi) \Rightarrow (\psi \& \phi)$,
(MTL4) $(\phi \sqcap \psi) \Rightarrow \phi$,
(MTL5) $(\phi \sqcap \psi) \Rightarrow (\psi \sqcap \phi)$,
(MTL6) $(\phi \& (\phi \Rightarrow \psi) \Rightarrow (\phi \sqcap \psi)$,
(MTL7) $(\phi \Rightarrow (\psi \Rightarrow \chi)) \Rightarrow ((\phi \& \psi) \Rightarrow \chi)$,
(MTL8) $((\phi \& \psi) \Rightarrow \chi) \Rightarrow (\phi \Rightarrow (\psi \Rightarrow \chi))$,
(MTL9) $((\phi \Rightarrow \psi) \Rightarrow \chi) \Rightarrow (((\psi \Rightarrow \phi) \Rightarrow \chi) \Rightarrow \chi)$,
(MTL10) $\bar{0} \Rightarrow \phi$.

Other connectives can be defined from $\&, \Rightarrow, \sqcap$ as follows:

$$\bar{1} = \phi \Rightarrow \phi, \neg\phi = \phi \Rightarrow \bar{0},$$
$$\phi \sqcup \psi = ((\phi \Rightarrow \psi) \Rightarrow \psi) \sqcap (\psi \Rightarrow \phi) \Rightarrow \phi).$$

The notion of proof in **MTL** is the usual one defined from the above axioms and rule, we write $T \vdash_{\mathbf{MTL}} \varphi$ to denote that φ follows from a set of formulas T.

MTL and its axiomatic extensions are all algebraizable [15], and their corresponding semantics forms a variety $\mathbb{MTL}$ of MTL-algebras [2, 20].

Definition 2.2. [2] An algebraic structure $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ is called an *MTL-algebra* if it satisfies the following conditions:

(1) $(L, \wedge, \vee, 0, 1)$ is a bounded lattice,
(2) $(L, \odot, 1)$ is a commutative monoid,
(3) $x \odot y \leq z$ if and only if $x \leq y \rightarrow z$,
(4) $(x \rightarrow y) \vee (y \rightarrow x) = 1$, for any $x, y, z \in L$.

In what follows, by L we denote the universe of an MTL-algebra $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$. We shall define $x^n = x \odot \cdots \odot x$ and $\neg x = x \rightarrow 0$. We say that the MTL-algebra L is linearly ordered if the lattice $(L, \wedge, \vee, 0, 1)$ is linearly ordered. Every MTL-algebra is a subdirect products of the linearly ordered members. This also gives completeness of **MTL** with respect to the class of linearly ordered MTL-algebras, and hence the logic **MTL** is semilinear[1] [1, 2].

Definition 2.3. [3] Let $\mathcal{L}$ be the logic **MTL** and $\mathbb{K}$ be a class of linearly ordered MTL-algebras. We say that $\mathcal{L}$ has the (finitely) strong $\mathbb{K}$-completeness property, ((F)S$\mathbb{K}$C for short), when foe every (finite) set of formulae T and every formula φ it holds that $T \vdash \varphi$ iff $e(\varphi) = 1$ for each L-evaluation such that $e\{T\} \subseteq \{1\}$ for every MTL-algebra $L \in \mathbb{K}$.

Theorem 2.1. [3] Let $\mathcal{L}$ be the logic **MTL** and $\mathbb{K}$ a class of linearly ordered MTL-algebras. Then

(1) $\mathcal{L}$ has the S$\mathbb{K}$C if and only if every countable linearly ordered MTL-algebra is embeddable into some member of $\mathbb{K}$.

(2) If $\mathcal{L}$ is finite, then $\mathcal{L}$ has the FS$\mathbb{K}$C if and only if every countable linearly ordered MTL-algebra is partially embeddable into $\mathbb{K}$.

3. The logic $\mathbf{MTL_{vt}}$

Hájek enrich the language **BL** by the connective **vt** and define the axioms of the logic $\mathbf{BL_{vt}}$.

Definition 3.1. [5] $\mathbf{BL_{vt}}$ is the expansion of **BL** with the Generalization Rule for **vt**, $(G_{vt} : \alpha \vdash vt(\alpha))$ and the following axioms:

(VT1) $vt\alpha \Rightarrow \alpha$,
(VT2) $vt(\alpha \Rightarrow \beta) \Rightarrow (vt\alpha \Rightarrow vt\beta)$,
(VT3) $vt(\alpha \sqcup \beta) \Rightarrow (vt\alpha \sqcup vt\beta)$.

The $\mathbf{BL_{vt}}$ is a Rasiowa implicative logic and hence is algebraizable.

Definition 3.2. [5] A *very true BL-algebra* is a pair (L, σ), where L is a BL-algebra and $\sigma : L \rightarrow L$ is an unary operation on L such that the following conditions hold:

(1) $\sigma(1) = 1$,
(2) $\sigma(x) \leq x$,
(3) $\sigma(x \rightarrow y) \leq \sigma(x) \rightarrow \sigma(y)$,
(4) $\sigma(x \vee y) \leq \sigma(x) \vee \sigma(y)$, *for any* $x, y \in L$.

Hájek proves $\mathbf{BL_{vt}}$ is semilinear via representable very true BL-algebras [5].

Remark 3.1. It must be pointed that Hájek's axiomatizations have an important drawback, that is, not any linearly ordered BL-algebra expanded with hedge functions are algebraic models of the logic $\mathbf{BL_{vt}}$, or in other words, the logic $\mathbf{BL_{vt}}$ does not accommodate most of the truth hedge functions used in the literature about of fuzzy logic in a broader sense, which is due to the requirement of the axiom (VT2).

[1]The logic $\mathcal{L}$ is called *semilinear* if and only if it is strongly complete with respect to the semantics given by linearly ordered algebras.

In order to solve the above problem, we will give a simple and general axiomatizations with respect to very true in **MTL**.

Definition 3.3. $\mathbf{MTL_{vt}}$ is the expansion of **MTL** with the inference rule for **vt**,

$$(vt)^{\vee} : (\alpha \Rightarrow \beta) \vee \chi \vdash (vt\alpha \Rightarrow vt\beta) \vee \chi,$$

and the following axioms:

(RVT1) $vt(\overline{1})$,

(RVT2) $vt\alpha \Rightarrow \alpha$.

Remark 3.2. Such a proliferation of logics deserves some explanation: (RVT1) means that absolutely true is very true, which is sound for each natural interpretation in many valued logic system; (RVT2) coincides with Hájek's (VT1) and means that very true is true; $(vt)^{\vee}$ is the $\vee$-form of **vt**, which can ensure $\mathbf{MTL_{vt}}$ is semilinear.

Proposition 3.1. The following deductions are valid in $\mathbf{MTL_{vt}}$.

(1) $\vdash vt\overline{0}$,

(2) $\alpha \Rightarrow \beta \vdash vt\alpha \Rightarrow vt\beta$,

(3) $\beta \vdash vt\beta$,

(4) $vt\alpha, \alpha \Rightarrow \beta \vdash vt\beta$,

(5) $vt\phi, vt(\alpha \Rightarrow \beta) \vdash vt\beta$,

(6) $\alpha \sqcup \neg\alpha \vdash \alpha \Rightarrow vt\alpha$.

Proof. (1) It follows directly from (RVT2) that $\alpha = \overline{0}$.

(2) It follows directly from ($(vt)^{\vee}$) that $\chi = \overline{0}$.

(3) It follows directly from (2) that $\alpha = \overline{1}$ and using (RVT1).

(4) It follows from (2) and **MP**.

(5) It follows from (4), (RVT2) and **MP**.

(6) It follows from (RVT2). □

Proposition 3.1(2) shows that $\mathbf{MTL_{vt}}$ is a Rasiowa implicative logic, and hence we can define very true MTL-algebra in a natural way.

Definition 3.4. A *very true MTL-algebra* is a pair (L, σ), where L is an MTL-algebra and $\sigma : L \to L$ is an unary operation on L such that the following hold,

(1) $\sigma(1) = 1$,

(2) $\sigma(x) \leq x$,

(3) *if* $(x \to y) \vee z = 1$, *then* $(\sigma(x) \to \sigma(y)) \vee z = 1$, *for any* $x, y, z \in L$.

The class $\mathbb{VTMTL}$ of very true MTL-algebras forms a quasivariety.

Example 3.1. (1) Let L be an MTL-algebra and id_L is an identity map on L. Then (L, id_L) is a very true MTL-algebra.

(2) Any linearly ordered MTL-algebra L can expanded to a very true MTL-algebra by the following unary operator σ:

$$\sigma(x) = \begin{cases} 1, & x = 1 \\ 0, & x \neq 1 \end{cases},$$

(3) Let $L = \{0, a, b, 1\}$ with $0 \leq a \leq b \leq 1$. Consider the operations $\odot$ and $\to$ given by the following tables:

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	0	a
b	0	0	b	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	a	a	1	1
1	0	a	b	1

Then $(L, \wedge, \vee, \odot, \to, 0, 1)$ is an MTL-algebra. Now, we define σ as follow:

$$\sigma(x) = \begin{cases} 0, & x = 0 \\ a, & x = a, b \\ 1, & x = 1. \end{cases},$$

It is easily verified that (L, σ) is a very true MTL-algebra.

(4) For any L be a complete MTL-algebra $a_1, a_2, \cdots, a_k \in L$, and nonnegative integers $n_1, n_2, \cdots, n_k$. Let

$$\sigma(x) = \begin{cases} 1, & x = 1 \\ \bigvee_{i=1}^{k} a_i \odot x^{n_i}, & x < 1. \end{cases},$$

Then (L, σ) is a very true MTL-algebra.

(5) Let $[0,1]_{\text{Ł}}$ be the standard Łukasiewicz structure, and hence is an MTL-algebra, and define σ by

$$\sigma(x) = 1 - \sqrt{1 - a}$$

Then (L, σ) is a very true MTL-algebra.

Algebraizability implies completeness of $\mathbf{MTL_{vt}}$ with respect to $\mathbb{VTMTL}$.

Theorem 3.1. Let T be a theory and α be a formula over $\mathbf{MTL_{vt}}$. Then the following statements are equivalent:

(1) $T \vdash \alpha$,

(2) for each very true MTL-algebra (L, σ) and for every model e of T, $e(\alpha) = 1$,

(3) $[\alpha]_T = [\overline{1}]_T$ *in* $\mathbf{MTL_{vt}}$.

However, the global local deduction theorem does not hold in $\mathbf{MTL_{vt}}$, that is,

$$\alpha \vdash vt\beta, \text{ but} \nvdash \alpha \Rightarrow vt\beta.$$

Example 3.2. Let $[0,1]_{\text{Ł}}$ be the standard Łukasiewicz structure, and hence an MTL-algebra. Also, for each number $n \geq 2$, then n-element set

$$S_n = \{0, \tfrac{1}{n-1}, \tfrac{2}{n-1}, \cdots, \tfrac{n-2}{n-1}, 1\}$$

is a subalgebra of L. Now, we define σ as follow:

$$\sigma(x) = \begin{cases} 1, & x = 1 \\ 0, & x \neq 1 \end{cases}.$$

Then it follows from Example 3.1(2) that (L, σ) is a very true MTL-algebra. Indeed, for any evaluation e in this algebra, if $e(\alpha) = 1$, then $e(vt\alpha) = 1$, but for $e(\beta) = \frac{1}{3}$ we have $e(vt\beta) = 0$, and thus, $e(\alpha \Rightarrow vt\beta) = 0$.

$\mathbf{MTL_{vt}}$ enjoys the same form of deduction theorem holding for logics with the $\triangle$ in [6].

Theorem 3.2. $T, \alpha \vdash \beta$ if and only if $T \vdash vt\alpha \Rightarrow \beta$.

Proof. We prove by induction on every formula α_i $(1 \leq i \leq n)$ of the given derivation of β from $T \cup \alpha$ that $T \vdash vt\alpha \Rightarrow \alpha_i$.

If $\alpha_i = \alpha$, then the result follows due to (RVT2). If $\alpha_i \in T$ or is an instance of an axiom, then the result follows using modus ponens and the derivability of the schema $\alpha_i \Rightarrow (vt\alpha \Rightarrow \alpha_i)$.

If α_i comes by application of modus ponens on previous formulas in the derivation, then the result follows, because from $vt\alpha \Rightarrow \alpha_k$ and $vt\alpha \Rightarrow (\alpha_k \Rightarrow \alpha_i)$ we may derive $(vt\alpha \odot vt\alpha) \Rightarrow (\alpha_k \odot (\alpha_k \Rightarrow \alpha_i)$ and hence also $vt\alpha \Rightarrow \alpha_i$, using transitivity of $\Rightarrow$ applied to Proposition 3.1.(3) and $(\alpha_k \odot (\alpha_k \Rightarrow \alpha_i) \Rightarrow \alpha_i$.

If $\alpha_i = vt\alpha_k$ comes using Necessitation Rules from α_k, then from $vt\alpha \Rightarrow \alpha_k$, we may derive $vt\alpha \Rightarrow vt\alpha_k$ using Proposition 3.1(5).

Conversely, to the derivation given by the hypothesis add a step with α. In the next step put $vt\alpha$, which follows from the previous formula using Necessitation Rules. Finally, derive β using modus ponens. □

Applying the disjunction form of the inference rule $(vt)^{\vee}$ and the axiomatization of the least semilinear extension, we have the following important result.

Theorem 3.3. $\mathbf{MTL_{vt}}$ is semilinear, that is, it is complete with respect to the class of linearly ordered very true MTL-algebras.

Proof. This is a well known consequence of the axiomatization of the least semilinear extension using the very true MTL-algebra obtained as the free very true MTL-algebra by the disjunction form of the rule (vt). □

As an algebraic version of Theorem 3.3, the following statement hold.

Theorem 3.4. Every very true MTL-algebra is representable, that is, it is a subalgebra of the direct product of a system of linearly ordered very true MTL-algebras.

Proof. It follows from Theorems 3.1 and 3.3. □

Theorem 3.5. $\mathbf{MTL_{vt}}$ is a conservative extensions of $\mathbf{MTL}$, that is, $T \vdash_{\mathbf{MTL_{vt}}} \alpha$ if and only if $T \vdash_{\mathbf{MTL}} \alpha$.

Proof. One implication is obviously. For the other hand, let $T \vdash_{\mathbf{MTL_{vt}}} \alpha$ and $T \nvdash_{\mathbf{MTL}} \alpha$. It follows that there exists a linearly ordered MTL-algebra L and a model e of T such that $e(\alpha) \neq 1$. It follows from Example 3.1(2) that (L, σ) is a very true MTL-algebra. Hence we consider e_σ by $e_\sigma(x) = e(x)$ for any propositional variable x. It is easily prove by structure induction that $e_\sigma(\beta) = e(\beta)$ for any formula β of $\mathbf{MTL}$. Thus, $e_\sigma(T) = e(T) = 1$ and $e_\sigma(\alpha) = e(\alpha) \neq 1$. By Theorem 3.1, this a contradiction of our hypothesis, so $T \vdash_{\mathbf{MTL}} \alpha$. □

The equivalence form of linearly ordered very true MTL-algebra is given.

Theorem 3.6. Let (L, σ) be a linearly ordered very true MTL-algebra. Then the following statement are equivalent: for any $x, y, z \in L$,

(1) if $(x \to y) \vee z = 1$, then $(\sigma(x) \to \sigma(y)) \vee z = 1$,

(2) if $x \leq y$, then $\sigma(x) \leq \sigma(y)$.

Proof. (1) $\Rightarrow$ (2) Taking $z = 0$ in (1).

(2) $\Rightarrow$ (1) Let L be a linearly ordered MTL-algebra and x, y, z be three arbitrary elements of L. If $x \leq y$, then $x \to y = 1$, and hence $\sigma(x) \leq \sigma(y)$, that is $(\sigma(x) \to \sigma(y)) \vee z = 1$. If $y < x$, then $x \to y \neq 1$, and hence $z = 1$, which implies $(\sigma(y) \to \sigma(x)) \vee z = 1$. □

The condition "linearly ordered " in the above theorem is necessary.

Example 3.3. Let $L = \{0, a, b, c, d, e, 1\}$ be a set such that $0 \leq a, b; a \leq c, d; b \leq c; c, d \leq 1$. Defining operations $\odot$ and $\to$ as follows:

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	0	0	0	a
b	0	0	b	b	0	b
c	0	0	b	b	a	c
d	0	0	0	0	d	d
1	0	a	b	c	d	1

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	c	1	c	1	1	1
b	d	d	1	1	d	1
c	a	d	c	1	d	1
d	b	c	b	c	1	1
1	0	a	b	c	d	1

Then $(L, \wedge, \vee, \odot, \to, 0, 1)$ is an MTL-algebra. Now, we define σ as follow:

$$\sigma(x) = \begin{cases} 1, & x = 1 \\ 0, & x \neq 1 \end{cases},$$

It is easily verified that σ satisfies Definition 3.4 (1),(2) and Theorem 3.6(2). However, it is clear that $(1 \to c) \vee d = 1$, while $(\sigma(1) \to \sigma(c)) \vee d = d \neq 1$.

Theorem 3.7. Let L be a MTL-algebra and σ be an unary operator on L. Then (L, σ) is a very true MTL-algebra if and only if it satisfies the condition Definition 3.4(1),(2) and Theorem 3.6(2).

Proof. It follows from Definition 3.4, Theorem 3.6. □

Remark 3.3. From Theorem 3.7, any hegde function on standard MTL-algebra [0,1] is a sound interpretation of the very true connective in **MTL**, that is, $\mathbf{MTL_{vt}}$ accommodates most of the hedge functions of **MTL** in a broader sense.

Theorem 3.8. **MTL** has the FS$\mathbb{K}$C if and only if $\mathbf{MTL_{vt}}$ has the FS$\mathbb{K}$C.

Proof. The implication from right to left follows from the fact that $\mathbf{MTL_{vt}}$ is a conservative expansion of **MTL**. Assume that **MTL** has the FSRC. Taking any linearly ordered very true MTL-algebra $(L, \odot, \to, \wedge, \vee, 0, 1, \sigma)$ and let B be a finite partial subalgebra of L. We have to show that there exists a standard very true MTL-algebra $([0, 1], \wedge, \vee, *, \Rightarrow, 0, 1)$ and a mapping $f : B \to [0, 1]$ preserving the existing operations. From Theorem 1, using the necessary of the embeddability property, we know that the σ-free reduct of L is partially embeddable into a standard MTL-algebra $([0, 1], \wedge, \vee, *, \Rightarrow, 0, 1)$. Denote this embedding by f and consider any non-decreasing and subdiagonal

$$\sigma^{'} : [0, 1] \to [0, 1] \text{ satisfying } \sigma^{'}(f(x)) = f(\sigma(x))$$

for every $x \in B$ such that $\sigma(x) \in B$. There are obviously many such function $\sigma^{'}$ interpolating the set of points

$$P = \{(f(x), f(\sigma(x)))|x, \sigma(x) \in B\},$$

for instance a piecewise linear interpolant. Another interpolant can be defined as follows: let $0 = z_1 < \cdots < z_n < 1$.

be the set of elements of $[0, 1]$ such that $(z_i, x) \in P$ for some x and define $\sigma^{'}(1) = 1$ and, for all $z \in [0, 1]$, $\sigma^{'}(z) = f(\sigma(x_i))$, if $z_i \leq z < z_{i-1}$, where $x_i \in B$ is such that $z_i = f(x_i)$. In any case, $\sigma^{'}$ makes $([0, 1], \wedge, \vee, *, \Rightarrow, 0, 1, \sigma)$ an linearly ordered very true MTL-algebra and f is a paritial embedding of linearly ordered very true MTL-algebras. □

As a more general version of Theorem 3.8, we have the following corollary.

Corollary 3.1. Let $\mathcal{L}$ be the logic **MTL**, $\mathbb{K}$ a class of linearly ordered MTL-algebras whose vt-free reducts are in $\mathbb{K}$. Then **MTL** has the FS$\mathbb{K}$C, if and only if $\mathbf{MTL_{vt}}$ has the FS$\mathbb{K}\mathbb{V}\mathbb{T}$C.

As a strong real completeness, we can obtain the following result.

Theorem 3.9. **MTL** has the S$\mathbb{K}$C if and only if $\mathbf{MTL_{vt}}$ has the S$\mathbb{K}$C.

Proof. Again one implication just follows from the fact that $\mathbf{MTL_{vt}}$ is a conservation expansion of **MTL**. For the converse one assume that **MTL** has the S$\mathbb{K}$C. WE have to show that any countable linearly ordered very true MTL-algebras can be embedded into a standard linearly ordered very true MTL-algebra. Let (L, σ) be a linearly ordered very true MTL-algebra, by Theorem 2.1, we know that the vt-free reduct of (L, σ) is embeddable into a standard very true MTL-algebra $(L_1, \sigma) = ([0, 1], \odot, \to, \wedge, \vee, 0, 1)$. Denote this embedding by f and define $vt' : (L_1, \sigma) \to (L_1, \sigma)$ in the following way: for any $z \in [0, 1]$

$$vt'(z) = \sup\{f(vt(x))|x \in L, f(x) \leq z\}.$$

So defined, vt' is a non-decreasing and subdiagonal function such that $vt'(f(x)) = f(vt(x))$, for any $x \in L$ and hence (L_1, σ) expanded with vt' is a standard linearly ordered very true MTL-algebra where (L, σ) is embedded. □

4. **Conclusion**

This paper deals with the algebraic interpretation of very true in **MTL** and gives some representation of very true MTL-algebras. Since the above topics are of current interest, we suggest further directions of research:

(1) The class of very true MTL-algebras form a quasivariety. There is a natural question that under which conditions, the quasivariety become variety.

(2) Studying very true at a general logical and algebraic setting as possible. E.g. for (almost) all algebraizable logics one can define very true companion, so it makes sense to study their corresponding algebras.

Acknowledgements

The authors are extremely grateful to the editor and the referees for their valuable comments and helpful suggestions which help to improve the presentation of this paper. This study was funded by a grant of National Natural Science Foundation of China (61976244,11961016), the Natural Science Basic Research Plan in Shaanxi Province of China (2020JQ-762) and Natural Science Foundation of Education Committee of Shannxi Province (20JK0626).

REFERENCES

[1] *S. Burris, H. P. Sankappanavar*, A course in Universal Algebra, Springer, New York, 1981.

[2] *F. Esteva, L. Godo*, Monoidal t-norm based logic: towards a logic for left-continuoust-norms, Fuzzy Sets and Systems., **124**, (2001), 271-288.

[3] *F. Esteva, L. Godo, C. Noguera*, A logical approach to fuzzy truth hedges, Information Sciences., **232**, (2013), 366-385.

[4] *L. A. Zadeh*, Fuzzy logic and approximate reasoning, Synthese., **30**, (1975), 407-428.

[5] *P. Hájek*, On very true, Fuzzy Sets and Systems., **124**,(2001), 329-333.

[6] *P. Hájek*, Metamathematics of Fuzzy logic, Kluwer Academic Publishers, Dordrech., 1998.

[7] *P. Hájek, D. Harmancová*, A hedge for Gödel fuzzy logic, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems., **8**,(2000),495-498.

[8] *J. Rachunek, D. Švrček*, Truth values on generalizations of some commutative fuzzy structures, Fuzzy Sets and Systems., **157**,(2006),3157-3168.

[9] *M. Butor, F. Švrček*, Very true on CBA fuzzy logic, Mathematica Slovaca., **60**,(2010),435-446.

[10] *J. T. Wang, X. L. Xin, Y. B. Jun*, Very true operators on equality algebras, Journal of Computational Analysis and Applications., **24**,(2018),507-521.

[11] *I. Chajda, M. Kolařik*, Very true operators in effect algebras, Soft Computing., **16**,(2012), 1213-1218.

[12] *I. Leuştean*, Non-commutative Łukasiewicz propositional logic, Archive for Mathematical Logic., **45**, (2006), 191-213.

[13] *G. Lakoff*, Hedges: a study in meaning criteria and the logic of fuzzy concepts, paper presented at the 8th Regional Meeting of the Chicago Linguistic Society., 1972.

[14] *S. Jenei, F. Montagna*, A proof of standard completeness for Esteva and Godo's logic MTL, Studia Logica., **70**, (2002) 183-192.

[15] *C. Noguera*, Algebraic study of axiomatic extensions of triangular norm based fuzzy logics. Ph.D. thesis, IIIA-CSIC.

[16] *H. Rasiowa*, An algebraic approach to non-classical logic, North Holland, Amsterdam.,1974.

[17] *J. M. Zhan, W. A. Dudek*, Soft MTL-algebras based on fuzzy sets, University Politehnica of Bucharest Scientific Bulletin-Series A-Applied Mathematics and Physics., **74**, (2012), 41-56.

[18] *L. C. Ciungu*, Very true pseudo BCK-algebras, Soft Computing., **23**, (2019), 10587-10600.

[19] *G. Q. Yang, W. J. Chen, A. R. Chen*, Very True Operators on Quasi-pseudo-MV-algebras, ICNC-FSKD 2019, AISC 1074., **21**, (2020), 781-790.

[20] *J. T. Wang, A. Borumand Saeid, P. F. He*, Similarity MTL-algebras and Their corresponding logics, Journal of Multiple-Valued Logic and Soft Computing., **32**, (2019), 607-628.