# KHAN AND ĆIRIĆ CONTRACTION PRINCIPLES IN ALMOST $b$-METRIC SPACES

by Katarina Kukić, Wasfi Shatanawi* and Milanka Gardašević-Filipović

*In the presented paper we utilize the concept of almost b-metric to construct and prove fixed point results of Khan and Ćirić contraction types. Our findings extended and modified many existing results in the literature. Moreover, our results present a positive answer of some open questions proposed by N. Mlaiki et al. in [N. Mlaiki , K. Kukić, M. Gardašević-Filipović, H. Aydi, On Almost b-Metric Spaces and Related Fixed Point Results, Axioms 2019, 8, 70; doi:10.3390/axioms8020070]*

**Keywords:** Khan contraction, Ćirić contraction, almost $b$-metric space
**MSC2010:** 46T99, 47H10, 54H25

## 1. Introduction

The $b$-metric space in the sense of Bakhtin [3] is one of the most important spaces in the field of applied sciences. For some more works in this interesting space, we may suggest the readers to see the following articles: [2], [6], [7] and [13]-[15]. Many generalizations of $b$-metric spaces are still being developed and the fixed points theorems in such spaces are examined, see, for example, [1] and [12].

Recently, the concept of almost rectangular $b$-metric spaces was introduced in [10] and authors proved a theorem of Reich type contraction for that kind of space. Motivated by that approach, in [11] N. Mlaiki et al. proposed the replacement of symmetry condition in $b$-metric spaces by one or both of following postulates (bM2l) and (bM2r). It turns out that many contraction principles remain valid even without classical symmetry condition in $b$-metric spaces. Some examples showing that quasi $b$-metric, almost $b$-metric and classical $b$-metric spaces are different classes of spaces may also be seen in [11].

Contraction principles with some symmetry in contraction condition, such as Reich or Hardy-Rogers type contractions, are very easily obtained in almost $b$-metric spaces. For some other principles, for example those having maximum of some set in contraction condition or having contractive condition of rational type, the situation is a bit more delicate. In [11] we proved one such principle and left some questions concerning Ćirić type contractions open. In this article we give answers on two of those three open questions. Also, we investigate

[1]Associate Professor, Faculty for Traffic and Transport Engineering, University of Belgrade, Serbia, e-mail: `k.mijailovic@sf.bg.ac.rs`

[2]Professor, Department of Mathematics and General courses, Prince Sultan University, Riyadh, Saudi Arabia, e-mail: `wshatanawi@psu.edu.sa`

[2]Professor, Department of Medical Research, China Medical University Hospital, China Medical University, Taichung, Taiwan, e-mail: `wshatanawi@yahoo.com`

*Corresponding author

[3]Assistant Professor, School of Computing, Union University, Belgrade, Serbia, e-mail: `mgardasevic@raf.edu.rs`

two contraction principles with conditions of rational type in almost $b$-metric space, such as Khan type contraction.

## 2. Preliminaries

First, we briefly remind on the definition of almost b-metric spaces from [11].

**Definition 2.1.** *Let $X$ be a nonempty set and $s \geq 1$ be a given real number. Let $d_{ab}$ : $X \times X \to [0, +\infty)$ be a mapping and $x, y, z, x_n \in X, n \in \mathbb{N}$:*

**(bM1):** $d_{ab}(x, y) = 0$ *if and only if* $x = y$,

**(bM2l):** $d_{ab}(x_n, x) \to 0$, $n \to \infty$ *implies* $d_{ab}(x, x_n) \to 0$, $n \to \infty$ *in standard metric,*

**(bM2r):** $d_{ab}(x, x_n) \to 0$, $n \to \infty$ *implies* $d_{ab}(x_n, x) \to 0$, $n \to \infty$ *in standard metric,*

**(bM3):** $d_{ab}(x, y) \leq s(d_{ab}(x, z) + d_{ab}(z, y))$.

*Then:*

(1) $(X, d_{ab}, s)$ *is called l-almost b-metric space if (bM1), (bM2l) and (bM3) hold;*
(2) $(X, d_{ab}, s)$ *is called r-almost b-metric space if (bM1), (bM2r) and (bM3) hold;*
(3) $(X, d_{ab}, s)$ *is called almost b-metric space if (bM1), (bM2l), (bM2r) and (bM3) hold.*

The almost $b$-metric spaces represent a subclass of quasi $b$-metric spaces. From quasi $b$-metric $d_q$, we can construct $b$-metric $d_b$ such as

$$d_b(x, y) = \frac{d_q(x, y) + d_q(y, x)}{2}$$

and this method proved to be an elegant way to validate some contraction principles, such as Reich or Hardy-Rogers, in quasi $b$, and so in almost $b$-metric spaces. For some other principles, we need to apply (bM2r) or (bM2l) and in those cases we use terms left-Cauchy and right-Cauchy sequence, so we recall on the next definition:

**Definition 2.2.** [11] *Let $(X, d_{ab}, s)$ be an almost b-metric space. A sequence $\{x_n\}$ in $X$ is said to be*

**left-Cauchy:** *if and only if for each $\varepsilon > 0$ there is an $n_0 \in \mathbb{N}$ such that $d_{ab}(x_n, x_m) < \varepsilon$ for all $n \geq m > n_0$, which can be written as* $\lim\limits_{n \geq m \to \infty} d_{ab}(x_n, x_m) = 0$

**right-Cauchy:** *if and only if for each $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ so that $d_{ab}(x_n, x_m) < \varepsilon$ for all $m \geq n > n_0$, which can be written as* $\lim\limits_{m \geq n \to \infty} d(x_n, x_m) = 0$

**Cauchy:** *if and only if for each $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ so that $d_{ab}(x_n, x_m) < \varepsilon$ for all $n, m > n_0$.*

Further, notions of left or right complete quasi or almost $b$-metric spaces are common, and more on that can be seen in [11].

## 3. Main results

In [10] for almost rectangular $b$-metric and later in [11] for almost $b$-metric spaces, some contraction principles, such as Reich and Hardy-Rogers have been proved by constructing the appropriate symmetric $b$-metric from almost $b$-metric. Such a method of proving can be applied to a certain number of symmetric contractions, but not to all. In [11] we have shown that it is not possible to prove the contractions in which the conditions refer to the maximum by that simple methodology. Here we present some additional principles and prove they are valid in almost $b$-metric spaces.

Before we proceed, we refer to an important result from theory of $b$-metric spaces that sequence $\{x_n\}$ which satisfies $d(x_{n+1}, x_n) \leq \lambda d(x_n, x_{n-1})$ for some $\lambda \in (0, 1)$ and for any $n \in \mathbb{N}$ is Cauchy. That result facilitates many proofs of contraction principles in $b$-metric spaces and it is a relatively new result obtained in 2017 and presented in [16] and [8], later also in [9] and [17] in which authors separately present an elegant and shorter proof of mentioned result. Since this result is also valid in the quasi-$b$-metric spaces, here we modify the proofs from [9] and [17] for right and left Cauchy sequence in those spaces.

**Lemma 3.1.** *Let $\{x_n\}$ be a sequence in quasi-b-metric space $(X, d_q, s \geq 1)$ such that*

$$d_q(x_n, x_{n+1}) \leq \lambda \cdot d_q(x_{n-1}, x_n) \tag{3.1}$$

*for some $\lambda \in [0,1)$ and each $n \in N$. Then, $\{x_n\}$ is right-Cauchy sequence.*

*Proof.* In [11] we proved the lemma for $\lambda \in [0, \frac{1}{s})$. That part of the proof is quite straightforward, based on the inequality

$$d_q(x_n, x_{n+1}) \leq \lambda^n d_q(x_0, x_1)$$

and

$$d_q(x_n, x_m) \leq \left( \frac{s\lambda^n}{1 - s\lambda} + \frac{(s\lambda)^{m-1}}{s^n} \right) d_q\left(x_0, x_1\right) \to 0,\ m > n \to \infty.$$

Here we extend it to the case $\lambda \in [\frac{1}{s}, 1)$. This can be proved by slightly modifying the proof from [9] or from [17]. Now we present briefly the modification of the approach from [9]. The following inequality is straightforward:

$$d_q(x_n, x_{n+j}) \leq s^j \left( d_q(x_n, x_{n+1}) + d_q(x_{n+1}, x_{n+2}) + ... + d_q(x_{n+j-1}, x_{n+j}) \right). \tag{3.2}$$

If $\lambda \in [\frac{1}{s}, 1)$ one can find $n_0 \in \mathbb{N}$ such that $\lambda^{n_0} < \frac{1}{s}$, namely $n_0 > -\frac{\log s}{\log \lambda}$. Then:

(1) $\{x_{nn_0}\}$ is right-Cauchy sequence. The proof of this claim is completely analogous to the proof from [9].

$$\begin{aligned} d_q(x_{nn_0}, x_{(n+1)n_0}) &\leq s^{n_0} \left( d_q(x_{nn_0}, x_{nn_0+1}) + ... + d_q(x_{(n+1)n_0-1}, x_{(n+1)n_0}) \right) \\ &\leq s^{n_0} \lambda^{nn_0} \frac{d_q(x_0, x_1)}{1-\lambda} = \frac{s^{n_0} \cdot d_q(x_0, x_1)}{1-\lambda} \cdot (\lambda^{n_0})^n \end{aligned}$$

Since $\lambda^{n_0} < \frac{1}{s}$, as a consequence of proved case for $\lambda \in [0, \frac{1}{s})$, we conclude that $\{x_{nn_0}\}$ is right-Cauchy sequence.

(2) Since without assumption of symmetry we must keep in mind the index order in $d_q(x_n, x_m)$, recall first some basic characteristics of the integral part we use in proof. If by $[A]$ we denote the maximal integer not exceeding $A$ then $[A] \leq A < [A] + 1$ and so $\left[\frac{n}{n_0}\right] \leq \frac{n}{n_0} < \left[\frac{n}{n_0}\right] + 1$ and therefore $n_0 \left[\frac{n}{n_0}\right] \leq\ n < n_0 \left[\frac{n}{n_0}\right] + n_0$ for $n, n_0 \in \mathbb{N}$. Then, from (**3**.2) we have:

$$\begin{aligned} & d_q\left(x_{n_0\left[\frac{n}{n_0}\right]}, x_n\right) \\ & \leq s^{n - n_0\left[\frac{n}{n_0}\right]} \left( d_q(x_{n_0[\frac{n}{n_0}]}, x_{n_0[\frac{n}{n_0}]+1}) + d_q(x_{n_0[\frac{n}{n_0}]+1}, x_{n_0[\frac{n}{n_0}]+2}) + ... + d_q(x_{n-1}, x_n) \right) \\ & \leq s^{n - n_0\left[\frac{n}{n_0}\right]} \left( \lambda^{n_0\left[\frac{n}{n_0}\right]} + \lambda^{n_0\left[\frac{n}{n_0}\right]+1} + ... + \lambda^{n-1} \right) d(x_0, x_1) \\ & \leq s^{n_0} \lambda^{n_0\left[\frac{n}{n_0}\right]} \left(1 + \lambda + \lambda^2 + ...\right) d(x_0, x_1) \\ & \leq s^{n_0} \lambda^{n_0\left[\frac{n}{n_0}\right]} \frac{d(x_0, x_1)}{1-\lambda} \to 0,\ n \to \infty. \end{aligned}$$

Similarly,

$$\begin{aligned}
& d_q\left(x_n, x_{n_0(1+[\frac{n}{n_0}])}\right)\\
& \leq s^{n_0(1+[\frac{n}{n_0}])-n}\left(d_q(x_n, x_{n+1}) + ... + d_q(x_{n_0(1+[\frac{n}{n_0}])-1}, x_{n_0(1+[\frac{n}{n_0}])})\right)\\
& \leq s^{n_0(1+[\frac{n}{n_0}])-n}\left(\lambda^n + \lambda^{n+1} + ... + \lambda^{n_0(1+[\frac{n}{n_0}])-1}\right) d(x_0, x_1)\\
& \leq s^{n_0}\lambda^n\left(1 + \lambda + \lambda^2 + ...\right) d(x_0, x_1)\\
& \leq s^{n_0}\lambda^{n_0}\lambda^{n_0\left[\frac{n}{n_0}\right]}\frac{d(x_0, x_1)}{1-\lambda} \to 0,\ n \to \infty.
\end{aligned}$$

(3) $\{x_n\}$ is right-Cauchy sequence. Let $m > n$:

$$\begin{aligned}
d_q(x_n, x_m) \leq s^2(d_q(x_n, x_{n_0(1+[\frac{n}{n_0}])}) + d_q(x_{n_0(1+[\frac{n}{n_0}])}, x_{n_0[\frac{m}{n_0}]})\\
+ d_q(x_{n_0[\frac{m}{n_0}]}, x_m)) \to 0, m > n,\ n \to \infty.
\end{aligned}$$

□

Analogously we obtain the following Lemma for left-Cauchy sequence in quasi $b$-metric spaces.

**Lemma 3.2.** *Let $\{x_n\}$ be a sequence in quasi-b-metric space $(X, d_q, s \geq 1)$ such that*

$$d_q(x_{n+1}, x_n) \leq \lambda \cdot d_q(x_n, x_{n-1}) \qquad (\mathbf{3}.3)$$

*for some $\lambda \in [0, 1)$ and each $n \in N$. Then, $\{x_n\}$ is left-Cauchy sequence.*

*Proof.* Here we only present the main steps that are different for cases of the left and the right-Cauchy sequence. Starting from (**3**.3) we obtain $d_q(x_{n+1}, x_n) \leq \lambda^n d_q(x_1, x_0)$. Instead of (**3**.2), we obtain:

$$d_q(x_{n+j}, x_n) \leq s^j\left(d_q(x_{n+j}, x_{n+j-1}) + d_q(x_{n+j-1}, x_{n+j-2}) + ... + d_q(x_{n+1}, x_n)\right). \qquad (\mathbf{3}.4)$$

(1) $\{x_{nn_0}\}$ is a left-Cauchy sequence:

$$\begin{aligned}
d_q(x_{(n+1)n_0}, x_{nn_0}) &\leq s^{n_0}\left(d_q(x_{(n+1)n_0}, x_{(n+1)n_0-1}) + ... + d_q(x_{nn_0+1}, x_{nn_0})\right)\\
&\leq s^{n_0}\lambda^{nn_0}\frac{d_q(x_1, x_0)}{1-\lambda} = const \cdot (\lambda^{n_0})^n
\end{aligned}$$

(2) Similarly to the case of the right-Cauchy sequence, we get that $d_q\left(x_n, x_{n_0\left[\frac{n}{n_0}\right]}\right) \to 0,\ n \to \infty$ and $d_q\left(x_{n_0(1+[\frac{n}{n_0}])}, x_n\right) \to 0,\ n \to \infty$. Briefly:

$$\begin{aligned}
& d_q\left(x_n, x_{n_0\left[\frac{n}{n_0}\right]}\right) \leq s^{n-n_0\left[\frac{n}{n_0}\right]}\left(d_q(x_n, x_{n-1}) + ... + d_q(x_{n_0[\frac{n}{n_0}]+1}, x_{n_0[\frac{n}{n_0}]})\right)\\
& \leq s^{n_0}\lambda^{n_0\left[\frac{n}{n_0}\right]}\left(1 + \lambda + \lambda^2 + ...\right) d(x_1, x_0) \leq s^{n_0}\lambda^{n_0\left[\frac{n}{n_0}\right]}\frac{d(x_1, x_0)}{1-\lambda} \to 0,\ n \to \infty.
\end{aligned}$$

(3) $\{x_n\}$ is left-Cauchy sequence. Let $n > m$:

$$\begin{aligned}
d_q(x_n, x_m) \leq s^2(d_q(x_n, x_{n_0[\frac{n}{n_0}]}) + d_q(x_{n_0[\frac{n}{n_0}]}, x_{n_0([\frac{m}{n_0}]+1)})\\
+ d_q(x_{n_0([\frac{m}{n_0}]+1)}, x_m)) \to 0, n > m,\ m \to \infty.
\end{aligned}$$

□

Previous Lemmas are, of course, also valid in almost $b$-metric spaces and we will use them in the rest of the paper. Now, we proceed with the modified version of Khan's Theorem, [5]. Before that, we briefly recall that almost-$b$-metric space $(X, d_{ab}, s)$ is right-complete if and only if each right-Cauchy sequence $\{x_n\}$ in $X$ satisfies $\lim_{n\to\infty} d_{ab}(x, x_n) = 0$, similarly for left-completeness and complete if and only if each Cauchy sequence in $X$ is convergent.

**Theorem 3.1.** *Let $(X, d_{ab}, s)$ be a right-complete r-almost b-metric space with coefficient $s > 1$ and $T : X \to X$ be a mapping satisfying*

$$d_{ab}(Tx, Ty) \leq a_1 \cdot d_{ab}(x, y) + a_2 \frac{d_{ab}(x, Tx) \cdot d_{ab}(x, Ty) + d_{ab}(y, Ty) \cdot d_{ab}(y, Tx)}{d_{ab}(x, Ty) + d_{ab}(y, Tx)} \tag{3.5}$$

*for all $x, y \in X$, $d_{ab}(x, Ty) + d_{ab}(y, Tx) \neq 0$ where $a_1$, $a_2$ are non-negative constants such that $a_1 + a_2 < 1$. Then $T$ has a unique fixed point.*

*Proof.* At the beginning of the proof, let's consider the uniqueness of a possible fixed point. To prove that a fixed point is unique, if it exists, suppose that $T$ has two distinct fixed points $x^*, y^* \in X$. Then we get

$$\begin{aligned} d_{ab}(x^*, y^*) = d_{ab}(Tx^*, Ty^*) &\leq a_1 \cdot d_{ab}(x^*, y^*) \\ &+ a_2 \frac{d_{ab}(x^*, Tx^*) \cdot d_{ab}(x^*, Ty^*) + d_{ab}(y^*, Ty^*) \cdot d_{ab}(y^*, Tx^*)}{d_{ab}(x^*, Ty^*) + d_{ab}(y^*, Tx^*)} \\ &\leq a_1 d_{ab}(x^*, y^*) < d_{ab}(x^*, y^*) \end{aligned}$$

so we conclude that if $T$ has a fixed point, then it is a unique fixed point of $T$.
For arbitrary $x_0 \in X$, consider the sequence $x_n = Tx_{n-1} = T^n x_0$, $n \in \mathbb{N}$. If $x_n = x_{n+1}$ for some $n \in \mathbb{N}$ then $x_n$ is the unique fixed point of T. Hence, we suppose that $d_{ab}(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N}$. We start from (**3**.5) for $d_{ab}(x_n, x_{n+1})$. Then for any $n \in \mathbb{N}$ we get:

$$\begin{aligned} d_{ab}(x_n, x_{n+1}) = d_{ab}(Tx_{n-1}, Tx_n) &\leq a_1 d_{ab}(x_{n-1}, x_n) \\ &+ a_2 \frac{d_{ab}(x_{n-1}, x_n) \cdot d_{ab}(x_{n-1}, x_{n+1}) + d_{ab}(x_n, x_{n+1}) \cdot d_{ab}(x_n, x_n)}{d_{ab}(x_{n-1}, x_{n+1}) + d_{ab}(x_n, x_n)} \\ &\leq (a_1 + a_2) d_{ab}(x_{n-1}, x_n). \end{aligned}$$

Now, since $a_1 + a_2 < 1$, from Lemma **3**.1 we conclude that $\{x_n\}$ is right-Cauchy sequence. Since $(X, d_{ab}, s > 1)$ is a right-complete r-almost $b$-metric space, we get that the sequence $\{x_n\}$ right converges to the point $x^* \in X$, ie. $d_{ab}(x^*, x_n) \to 0$, $n \to \infty$ which, from (bM2r), implies $d_{ab}(x_n, x^*) \to 0$, $n \to \infty$. Hence, applying (**3**.5) on $(x_n, x^*)$, we get:

$$\begin{aligned} d_{ab}(Tx_n, Tx^*) &\leq a_1 d_{ab}(x_n, x^*) \\ &\quad + a_2 \frac{d_{ab}(x_n, Tx_n) \cdot d_{ab}(x_n, Tx^*) + d_{ab}(x^*, Tx^*) \cdot d_{ab}(x^*, Tx_n)}{d_{ab}(x_n, Tx^*) + d_{ab}(x^*, Tx_n)} \end{aligned}$$

Since $d_{ab}(x_n, Tx_n) \to 0$, $d_{ab}(x_n, x^*) \to 0$ and $d_{ab}(x^*, Tx_n) \to 0$ when $n \to \infty$, we conclude that $d_{ab}(Tx_n, Tx^*) \to 0$ when $n \to \infty$. Finally, from (bM3), we obtain

$$d_{ab}(x^*, Tx^*) \leq s\,(d_{ab}(x^*, Tx_n) + d_{ab}(Tx_n, Tx^*)) \to 0,\ n \to \infty$$

so $x^* = Tx^*$. □

**Theorem 3.2.** *Let $(X, d_{ab}, s)$ be a right-complete r-almost b-metric space with coefficient $s > 1$ and $T : X \to X$ be a mapping satisfying*

$$\begin{aligned} d_{ab}(Tx, Ty) \leq a_1 \cdot d_{ab}(x, y) + a_2 \frac{d_{ab}(y, Ty) \cdot (1 + d_{ab}(x, Tx))}{1 + d_{ab}(x, y)} + \\ + a_3 \frac{d_{ab}(y, Ty) + d_{ab}(y, Tx)}{1 + d_{ab}(y, Ty) \cdot d_{ab}(y, Tx)} \end{aligned} \tag{3.6}$$

*for all $x, y \in X$, where $a_1$, $a_2$, $a_3$ are non-negative constants such that $a_1 + a_2 + a_3 < 1$ and $a_2 + a_3 < \frac{1}{s}$. Then $T$ has a unique fixed point.*

*Proof.* Again, we start with the uniqueness of a possible fixed point. Suppose that $T$ has two distinct fixed points $x^*, y^* \in X$. Then, from (**3**.6) for $(Tx^*, Ty^*)$ we get

$$d_{ab}(Tx^*, Ty^*) \leq a_1 \cdot d_{ab}(x^*, y^*) + a_2 \frac{d_{ab}(y^*, Ty^*) \cdot (1 + d_{ab}(x^*, Tx^*))}{1 + d_{ab}(x^*, y^*)} + \\ + a_3 \frac{d_{ab}(y^*, Ty^*) + d_{ab}(y^*, Tx^*)}{1 + d_{ab}(y^*, Ty^*) \cdot d_{ab}(y^*, Tx^*)}$$

and finally

$$d_{ab}(x^*, y^*) \leq a_1 d_{ab}(x^*, y^*) + a_3 d_{ab}(y^*, x^*). \quad (\mathbf{3}.7)$$

Similar, starting from (**3**.6) for $(Ty^*, Tx^*)$, we obtain

$$d_{ab}(y^*, x^*) \leq a_1 d_{ab}(y^*, x^*) + a_3 d_{ab}(x^*, y^*). \quad (\mathbf{3}.8)$$

After summing (**3**.7) and (**3**.8), we get that

$$d_{ab}(x^*, y^*) + d_{ab}(y^*, x^*) \leq (a_1 + a_3)\left(d_{ab}(x^*, y^*) + d_{ab}(y^*, x^*)\right).$$

Since $a_1 + a_3 < 1$, we conclude that $d_{ab}(x^*, y^*) + d_{ab}(y^*, x^*) = 0$ and further from (bM1) we get that $d(x^*, y^*) = d(y^*, x^*) = 0$, so the fixed point, if it exists, is unique.

For arbitrary $x_0 \in X$, consider the sequence $x_n = Tx_{n-1} = T^n x_0$, $n \in \mathbb{N}$. If $x_n = x_{n+1}$ for some $n \in \mathbb{N}$ then $x_n$ is the unique fixed point of T. Hence, we suppose that $d_{ab}(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N}$. We start from (**3**.6) for $d_{ab}(x_n, x_{n+1})$, hence for any $n \in \mathbb{N}$ we get:

$$\begin{aligned} d_{ab}(x_n, x_{n+1}) &= d_{ab}(Tx_{n-1}, Tx_n) \\ &\leq a_1 d_{ab}(x_{n-1}, x_n) + a_2 \frac{d_{ab}(x_n, Tx_n)\left(1 + d_{ab}(x_{n-1}, Tx_{n-1})\right)}{1 + d_{ab}(x_{n-1}, x_n)} \\ &+ a_3 \frac{d_{ab}(x_n, Tx_n) + d_{ab}(x_n, Tx_{n-1})}{1 + d_{ab}(x_n, Tx_n) \cdot d(x_n, Tx_{n-1})} \\ &= a_1 d_{ab}(x_{n-1}, x_n) + a_2 d_{ab}(x_n, x_{n+1}) + a_3 d_{ab}(x_n, Tx_n). \end{aligned} \quad (\mathbf{3}.9)$$

From (**3**.9), we obtain

$$d_{ab}(x_n, x_{n+1}) \leq \frac{a_1}{1 - a_2 - a_3} d_{ab}(x_{n-1}, x_n).$$

Since $\frac{a_1}{1-a_2-a_3} < 1$, from Lemma **3**.1 we conclude that $\{x_n\}$ is a right-Cauchy sequence. Since $(X, d_{ab}, s > 1)$ is a right-complete r-almost $b$-metric space, we get that the sequence $\{x_n\}$ right converges to the point $x^* \in X$, ie. $d_{ab}(x^*, x_n) \to 0$, $n \to \infty$ what, from (bM2r) implies $d_{ab}(x_n, x^*) \to 0$, $n \to \infty$. Further, applying (**3**.6) on $(x_n, x^*)$, we obtain:

$$\begin{aligned} d_{ab}(Tx_n, Tx^*) &\leq a_1 d_{ab}(x_n, x^*) + a_2 \frac{d_{ab}(x^*, Tx^*)\left(1 + d_{ab}(x_n, Tx_n)\right)}{1 + d_{ab}(x_n, x^*)} \\ &+ a_3 \frac{d_{ab}(x^*, Tx^*) + d_{ab}(x^*, Tx_n)}{1 + d_{ab}(x^*, Tx^*) d_{ab}(x^*, Tx_n)} \\ &\leq (a_2 + a_3) d_{ab}(x^*, Tx^*), \text{ when } n \to \infty. \end{aligned}$$

Finally, starting from (bM3), using previous inequality, we get:

$$\begin{aligned} d_{ab}(x^*, Tx^*) &\leq s\left(d_{ab}(x^*, Tx_n) + d_{ab}(Tx_n, Tx^*)\right) \\ &\leq s(a_2 + a_3) d_{ab}(x^*, Tx^*). \end{aligned}$$

Since $a_2 + a_3 < \frac{1}{s}$, from previous relation we can conclude that $d_{ab}(x^*, Tx^*) = 0$, so $x^*$is fixed point of $T$. □

In [11] we left some open questions concerning Ćirić type contractions. In the sequel we give answers on two of those three questions. First we prove the generalized Ćirić type contraction of the first order.

**Theorem 3.3.** *Let $(X, d_{ab}, s \geq 1)$ be a right-complete $r$-almost $b$-metric space and $T : X \to X$ be a mapping satisfying*

$$d_{ab}(Tx, Ty) \leq k \cdot \max \left\{ d_{ab}(x, y), \frac{d_{ab}(x, Tx) + d_{ab}(y, Ty)}{2s}, \frac{d_{ab}(x, Ty) + d_{ab}(y, Tx)}{2s} \right\} \tag{3.10}$$

*for all $x, y \in X$ where $0 \leq k < \min\{1, \frac{2}{s}\}$. Then $T$ has a unique fixed point.*

*Proof.* At the beginning of the proof, let's consider the uniqueness of a possible fixed point. To prove that a fixed point is unique, if it exists, suppose that $T$ has two distinct fixed points $x^*, y^* \in X$. Then, from (**3**.10) we get

$$\begin{aligned} &d_{ab}(x^*, y^*) = d_{ab}(Tx^*, Ty^*) \\ &\leq k \cdot \max\{d_{ab}(x^*, y^*), \frac{d_{ab}(x^*, Tx^*) + d_{ab}(y^*, Ty^*)}{2s}, \frac{d_{ab}(x^*, Ty^*) + d_{ab}(y^*, Tx^*)}{2s}\} \\ &= k \cdot \max\{d_{ab}(x^*, y^*), \frac{d_{ab}(x^*, y^*) + d_{ab}(y^*, x^*)}{2s}\}. \end{aligned}$$

The first case

$$\max\{d_{ab}(x^*, y^*), \frac{d_{ab}(x^*, y^*) + d_{ab}(y^*, x^*)}{2s}\} = d_{ab}(x^*, y^*)$$

immediately leads to the contradiction

$$d_{ab}(x^*, y^*) \leq k \cdot d_{ab}(x^*, y^*) < d_{ab}(x^*, y^*).$$

The other case

$$\max\{d_{ab}(x^*, y^*), \frac{d_{ab}(x^*, y^*) + d_{ab}(y^*, x^*)}{2s}\} = \frac{d_{ab}(x^*, y^*) + d_{ab}(y^*, x^*)}{2s}$$

is equivalent to $d_{ab}(y^*, x^*) \geq (2s - 1)d_{ab}(x^*, y^*)$ and further in this case we obtain that

$$d_{ab}(y^*, x^*) \geq \frac{d_{ab}(x^*, y^*) + d_{ab}(y^*, x^*)}{2s} \geq d_{ab}(x^*, y^*). \tag{3.11}$$

Starting from condition (**3**.10) applied to $d_{ab}(Ty^*, Tx^*)$ and keeping in mind (**3**.11), we again come to the contradiction:

$$\begin{aligned} &d_{ab}(y^*, x^*) = d_{ab}(Ty^*, Tx^*) \\ &\leq k \cdot \max\{d_{ab}(y^*, x^*), \frac{d_{ab}(y^*, Ty^*) + d_{ab}(x^*, Tx^*)}{2s}, \frac{d_{ab}(y^*, Tx^*) + d_{ab}(x^*, Ty^*)}{2s}\} \\ &= k \cdot \max\{d_{ab}(y^*, x^*), \frac{d_{ab}(y^*, x^*) + d_{ab}(x^*, y^*)}{2s}\} \\ &= k \cdot d_{ab}(y^*, x^*) < d_{ab}(y^*, x^*). \end{aligned}$$

Finally, we conclude that the fixed point, if it exists, is unique.
For arbitrary $x_0 \in X$, consider the sequence $x_n = Tx_{n-1} = T^n x_0$, $n \in \mathbb{N}$. If $x_n = x_{n+1}$ for some $n \in \mathbb{N}$ then $x_n$ is the unique fixed point of T. Hence, we suppose that $d_{ab}(x_n, x_{n+1}) > 0$

for all $n \in \mathbb{N}$. We start from (**3**.10) for $d_{ab}(x_n, x_{n+1})$. Then for any $n \in \mathbb{N}$ we get:

$$\begin{aligned} d_{ab}(x_n, x_{n+1}) &= d_{ab}(Tx_{n-1}, Tx_n) \\ &\leq k \cdot \max\{d_{ab}(x_{n-1}, x_n), \frac{d_{ab}(x_{n-1}, Tx_{n-1}) + d_{ab}(x_n, Tx_n)}{2s}, \\ &\frac{d_{ab}(x_{n-1}, Tx_n) + d_{ab}(x_n, Tx_{n-1})}{2s}\} \\ &= k \cdot \max\{d_{ab}(x_{n-1}, x_n), \frac{d_{ab}(x_{n-1}, x_n) + d_{ab}(x_n, x_{n+1})}{2s}, \\ &\frac{d_{ab}(x_{n-1}, x_{n+1}) + d_{ab}(x_n, x_n)}{2s}\} \\ &\leq k \cdot \max\{d_{ab}(x_{n-1}, x_n), \frac{d_{ab}(x_{n-1}, x_n) + d_{ab}(x_n, x_{n+1})}{2s}, \\ &\frac{s(d_{ab}(x_{n-1}, x_n) + d_{ab}(x_n, x_{n+1})}{2s}\} \\ &= k \cdot \max\{d_{ab}(x_{n-1}, x_n), \frac{d_{ab}(x_{n-1}, x_n) + d_{ab}(x_n, x_{n+1})}{2}\}. \end{aligned} \tag{3.12}$$

If we suppose that $d_{ab}(x_{n-1}, x_n) < d_{ab}(x_n, x_{n+1})$, from (**3**.12) we get the contradiction

$$d_{ab}(x_n, x_{n+1}) \leq k \cdot d_{ab}(x_n, x_{n+1}) < d_{ab}(x_n, x_{n+1}),$$

so it must be $d_{ab}(x_n, x_{n+1}) \leq d_{ab}(x_{n-1}, x_n)$. Then, from (**3**.12) we obtain:

$$d_{ab}(x_n, x_{n+1}) \leq k \cdot d_{ab}(x_{n-1}, x_n)$$

and from Lemma **3**.1 we conclude that $\{x_n\}$ is a right Cauchy sequence. Since $(X, d_{ab}, s > 1)$ is a right-complete r-almost $b$-metric space, we get that the sequence $\{x_n\}$ right converges to some $x^* \in X$, i.e., $d_{ab}(x^*, x_n) \to 0$, $n \to \infty$ what, from (bM2r) implies $d_{ab}(x_n, x^*) \to 0$, $n \to \infty$.
Further, from (**3**.10) applied to $(x_n, x^*)$, we obtain:

$$\begin{aligned} d_{ab}(Tx_n, Tx^*) &\leq k \cdot \max\{d_{ab}(x_n, x^*), \frac{d_{ab}(x_n, Tx_n) + d_{ab}(x^*, Tx^*)}{2s}, \\ &\quad \frac{d_{ab}(x_n, Tx^*) + d_{ab}(x^*, Tx_n)}{2s}\} \\ &\leq k \cdot \max\{d_{ab}(x_n, x^*), \frac{d_{ab}(x_n, Tx_n) + d_{ab}(x^*, Tx^*)}{2s}, \\ &\quad \frac{d_{ab}(x_n, Tx^*) + d_{ab}(x^*, Tx_n)}{2s}\} \\ &\leq k \cdot \max\{d_{ab}(x_n, x^*), \frac{d_{ab}(x_n, Tx_n) + d_{ab}(x^*, Tx^*)}{2s}, \\ &\quad \frac{s(d_{ab}(x_n, x^*) + d_{ab}(x^*, Tx^*)) + d_{ab}(x^*, Tx_n)}{2s}\}. \end{aligned} \tag{3.13}$$

Apart from the transformations shown on the right, in order to estimate $d_{ab}(x^*, Tx^*)$ we have to write the left side in a more convenient form and then to use (**3**.13):

$$\begin{aligned} &\frac{1}{s} d_{ab}(x^*, Tx^*) \leq d_{ab}(x^*, Tx_n) + d_{ab}(Tx_n, Tx^*) \\ &\leq d_{ab}(x^*, Tx_n) + k \cdot \max\{d_{ab}(x_n, x^*), \frac{d_{ab}(x_n, Tx_n) + d_{ab}(x^*, Tx^*)}{2s}, \\ &\frac{s(d_{ab}(x_n, x^*) + d_{ab}(x^*, Tx^*)) + d_{ab}(x^*, Tx_n)}{2s}\}. \end{aligned}$$

Previous inequality with $n \to \infty$ becomes

$$\frac{1}{s} d_{ab}(x^*, Tx^*) \leq k \cdot \frac{d_{ab}(x^*, Tx^*)}{2}$$

and finally we obtain the inequality

$$d_{ab}(x^*, Tx^*) \leq \frac{k \cdot s}{2} d_{ab}(x^*, Tx^*)$$

that can be satisfied only with $d_{ab}(x^*, Tx^*) = 0$, which means that $x^*$ is fixed point of $T$. □

We proceed with Ćirić type contractions and we consider Ćirić type contraction of second order in r-almost $b$-metric space.

**Theorem 3.4.** *Let $(X, d_{ab}, s \geq 1)$ be a right-complete r-almost b-metric space and $T : X \to X$ be a mapping satisfying*

$$d_{ab}(Tx, Ty) \leq k \cdot \max \left\{ d_{ab}(x, y), d_{ab}(x, Tx), d_{ab}(y, Ty), \frac{d_{ab}(x, Ty) + d_{ab}(y, Tx)}{2s} \right\} \tag{3.14}$$

*for all $x, y \in X$ where $k$ is non-negative constant such that $k < \frac{1}{s}$. Then $T$ has a unique fixed point.*

*Proof.* We start with the proof that a fixed point is unique, if it exists, which is almost the same as in the previous Theorem, so here we only state the part that differs. Suppose that $T$ has two distinct fixed points $x^*, y^* \in X$. Then, from (**3**.14) we get

$$\begin{aligned} d_{ab}(x^*, y^*) &= d_{ab}(Tx^*, Ty^*) \\ &\leq k \cdot \max\{d_{ab}(x^*, y^*), d_{ab}(x^*, Tx^*), d_{ab}(y^*, Ty^*), \frac{d_{ab}(x^*, Ty^*) + d_{ab}(y^*, Tx^*)}{2s}\} \\ &\leq k \cdot \max\{d_{ab}(x^*, y^*), \frac{d_{ab}(x^*, y^*) + d_{ab}(y^*, x^*)}{2s}\}. \end{aligned}$$

The rest of the proof for uniqueness is the same as in Theorem **3**.3.
For arbitrary $x_0 \in X$, consider the sequence $x_n = Tx_{n-1} = T^n x_0$, $n \in \mathbb{N}$. If $x_n = x_{n+1}$ for some $n \in \mathbb{N}$ then $x_n$ is the unique fixed point of T. Hence, we suppose that $d_{ab}(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N}$. We start from (**3**.14) for $d_{ab}(x_n, x_{n+1})$. Then for any $n \in \mathbb{N}$ we get:

$$\begin{aligned} d_{ab}(x_n, x_{n+1}) =& d_{ab}(Tx_{n-1}, Tx_n) \\ & \leq k \cdot \max\{d_{ab}(x_{n-1}, x_n), d_{ab}(x_{n-1}, Tx_{n-1}), d_{ab}(x_n, Tx_n), \\ & \frac{d_{ab}(x_{n-1}, Tx_n) + d_{ab}(x_n, Tx_{n-1})}{2s}\} \\ & \leq k \cdot \max\{d_{ab}(x_{n-1}, x_n), d_{ab}(x_{n-1}, x_n), d_{ab}(x_n, x_{n+1}), \\ & \frac{d_{ab}(x_{n-1}, x_{n+1}) + d_{ab}(x_n, x_n)}{2s}\} \\ & \leq k \cdot \max\{d_{ab}(x_{n-1}, x_n), d_{ab}(x_{n-1}, x_n), d_{ab}(x_n, x_{n+1}), \\ & \frac{s(d_{ab}(x_{n-1}, x_n) + d_{ab}(x_n, x_{n+1}))}{2s}\} \\ & \leq k \cdot \max\{d_{ab}(x_{n-1}, x_n), d_{ab}(x_n, x_{n+1}), \\ & \frac{d_{ab}(x_{n-1}, x_n) + d_{ab}(x_n, x_{n+1})}{2}\}. \end{aligned} \tag{3.15}$$

If $d_{ab}(x_{n-1}, x_n) < d_{ab}(x_n, x_{n+1})$, then from (**3**.15) we get the contradiction

$$d_{ab}(x_n, x_{n+1}) < k \cdot d_{ab}(x_n, x_{n+1}).$$

Finally, we conclude that must be satisfied $d_{ab}(x_{n-1}, x_n) \geq d_{ab}(x_n, x_{n+1})$, so from (**3**.15) we get

$$d_{ab}(x_n, x_{n+1}) \leq k \cdot d_{ab}(x_{n-1}, x_n).$$

The same reasoning as in Theorem **3**.3 gives us the conclusion that the sequence $\{x_n\}$ right converges to some $x^* \in X$, ie. $d_{ab}(x^*, x_n) \to 0$, $n \to \infty$ which, from (bM2r) implies $d_{ab}(x_n, x^*) \to 0$, $n \to \infty$.
Again, similarly as in Theorem **3**.3 from (**3**.14), we obtain:

$$\begin{aligned}
&\frac{1}{s} d_{ab}(x^*, Tx^*) \leq d_{ab}(x^*, Tx_n) + d_{ab}(Tx_n, Tx^*) \\
&\leq d_{ab}(x^*, Tx_n) + k \cdot \max\{d_{ab}(x_n, x^*), d_{ab}(x_n, Tx_n), d_{ab}(x^*, Tx^*), \\
&\frac{d_{ab}(x_n, Tx^*) + d_{ab}(x^*, Tx_n)}{2s}\} \\
&\leq d_{ab}(x^*, Tx_n) + k \cdot \max\{d_{ab}(x_n, x^*), d_{ab}(x_n, Tx_n), d_{ab}(x^*, Tx^*), \\
&\frac{s(d_{ab}(x_n, x^*) + d_{ab}(x^*, Tx^*)) + d_{ab}(x^*, Tx_n)}{2s}\}
\end{aligned}$$

And when $n \to \infty$, finally we get

$$d_{ab}(x^*, Tx^*) \leq s \cdot k \cdot d_{ab}(x^*, Tx^*)$$

so we conclude that $x^*$ is the fixed point of $T$. □

Note here that "left variants" of the preceding theorems can be proved analogously.

**Remark 3.1.** *It might looks like the Theorem* **3**.*3 is a consequence of Theorem* **3**.*4, but the condition for contraction in Theorem* **3**.*3 is* $0 \leq k < \min\{1, \frac{2}{s}\}$, *unlike in the Theorem* **3**.*4 where we demand* $0 \leq k < \frac{1}{s}$. *If in the Theorem* **3**.*3 we demanded* $0 \leq k < \frac{1}{s}$ *then it would be the consequence of the Theorem* **3**.*4 since*

$$\begin{aligned}
&\frac{d_{ab}(x, Tx) + d_{ab}(y, Ty)}{2s} \leq \frac{1}{s} \cdot \max\{d_{ab}(x, Tx), d_{ab}(y, Ty)\} \\
&\leq \max\{d_{ab}(x, Tx), d_{ab}(y, Ty)\}.
\end{aligned}$$

*Under stated conditions, it is not the case and we find this result interesting and not observed in the case of b-metric spaces, as far as we know.*

**Remark 3.2.** *Bannach, Kannan, Chaterjea and Reich type contraction principles are direct consequences of Theorems* **3**.*3 and* **3**.*4.*

At the end, we leave one open problem:

**Problem (Quasicontraction of Ćirić type)** Let $(X, d_{ab}, s \geq 1)$ be a right-complete $r$-almost $b$-metric space and $T : X \to X$ be such that

$$d_{ab}(Tx, Ty) \leq k \max\{d_{ab}(x, y), d_{ab}(x, Tx), d_{ab}(y, Ty), d_{ab}(x, Ty), d_{ab}(y, Tx)\}$$

for all $x, y \in X$ where $0 \leq k < \frac{1}{s}$. Then $T$ has a unique fixed point.

## 4. Conclusions

We studied contraction principles in almost $b$-metric spaces and obtained that many principles are valid even without symmetry condition for $b$-metric. In almost $b$-metric spaces we replaced symmetry with weaker conditions (bM2l) and (bM2r). As a sequel of paper [11], here we expanded the set of contraction principles valid in almost $b$-metric space.

All results obtained in the paper, may further generalize results obtained in [4] for metric type spaces. For example, we formulate Theorem **3**.1 in this manner, and emphasize that the same can be done for all other results in this paper:
Let $(X, d_{ab}, s)$ be right-complete r-almost $b$-metric space with coefficient $s > 1$ and $T, S : X \to X$ be two mappings such that $TX \in SX$ and one of these subsets of $X$ is right-complete. Suppose that

$$\begin{aligned} d_{ab}(Tx, Ty) \leq\ & a_1 \cdot d_{ab}(Sx, Sy) \\ & + a_2 \frac{d_{ab}(Sx, Tx) \cdot d_{ab}(Sx, Ty) + d_{ab}(Sy, Ty) \cdot d_{ab}(Sy, Tx)}{d_{ab}(Sx, Ty) + d_{ab}(Sy, Tx)} \end{aligned} \tag{\textbf{4}.1}$$

for all $x, y \in X$, $d_{ab}(Sx, Ty) + d_{ab}(Sy, Tx) \neq 0$ where $a_1$, $a_2$ are non-negative constants such that $a_1 + a_2 < 1$. Then $T$ and $S$ have a unique point of coincidence. If moreover, the pair $(T, S)$ is weakly compatible, then $T$ and $S$ have a unique common fixed point.

All of the above, together with the examples given in the paper [11], confirms that it is useful to consider the almost $b$-metric spaces, as well as that there are still many open questions and topics for further research.

**Acknowledgments:** The research of the first author was partially supported by the Serbian Ministry of Science and Technological Development, Project TR36002. The second author would like to thank Prince Sultan University for funding this work through research group Nonlinear Analysis Methods in Applied Mathematics (NAMAM) group number RG-DES-2017-01-17.

## REFERENCES

[1] *M.U. Ali, Fahimudin and M. Postolache*, Generalized Prešić type mappings in order-b-metric spaces, J Math. Anal., **10**(2019), No. 3, 1-13.

[2] *M.U. Ali, T. Kamran, and M. Postolache*, Solution of Volterra integral inclusion in b-metric spaces via new fixed point theorem, Nonlinear Analysis: Modelling and Control, **22** (2017), No. 1, 17–30.

[3] *I.A. Bakhtin*, The contraction mapping principle in quasi metric spaces, Funct. Anal. Ulianowsk Gos. Ped. Inst.,**30**(1989), 26-37.

[4] *M. Jovanović, Z. Kadelburg and S. Radenović*, Common fixed point results in metric type spaces, J. Fixed Point Theory Appl., (2010), Article ID 978121, 15 pages.

[5] *M. S. Khan*, A Fixed Point Theorem For Metric Spaces, Rend. Inst. Math. Univ. di Trieste, **8** (1976), 69-72.

[6] *T. Kamran, M. Postolache, M.U. Ali, and Q. Kiran*, Feng and Liu type F-contraction in b-metric spaces with application to integral equations, J. Math. Anal. **7**(2016), No. 5, 18-27.

[7] *T. Kamran, M. Postolache, A. Ghiura, S. Batul and R. Ali*, The Banach contraction principle in C*-algebra-valued b-metric spaces with application, Fixed Point Theory and Applications **1** (2016),No. 10, 1–7.

[8] *R. Miculescu, A. Mihail*, New fixed points theorems for set-valued contractions in b-metric spaces, J. Fixed Point Theory Appl., **19**(2017), 2153-2163.

[9] *Z. D. Mitrović*, A note on the result of Suzuki, Miculescu and Mihail, J. Fixed Point Theory Appl. (2019) 21: 24, https://doi.org/10.1007/s11784-019-0663-5.

[10] *Z. D. Mitrović, R. George and N. Hussain*, Some remarks on contraction mappings in rectangular b-metric spaces, in press in BSPM,(2018), doi:10.5269/bspm.41754.

[11] *N. Mlaiki, K. Kukić, M. Gardasević-Filipović and H. Aydi*, On almost $b$-metric spaces and related fixed points results, Axioms,**8**(2019), No. 70, 12 pages, doi:10.3390/axioms8020070

[12] *M. Samreen, T. Kamran and M. Postolache*, Extended b-metric space, extended b-comparison function and nonlinear contractions, UPB Scientific Bulletin, Series A: Applied Mathematics and Physics, **80**(2018), No. 4 , 21–28.

[13] *W. Shatanawi, A. Pitea and R. Lazović*, Contraction conditions using comparison functions on b-metric spaces, Fixed Point Theory and Applications, (2014), Art. No. 135, 11 pages, doi:10.1186/1687-1812-2014-135.

[14] *W. Shatanawi and M. Postolache*, Coincidence and fixed point results for generalized weak contractions in the sense of Berinde on partial metric spaces. Fixed Point Theory Appl., (2013), Art. No. 54.

[15] *W. Shatanawi and M. Postolache*, Common fixed point results of mappings for nonlinear contractions of cyclic form in ordered metric spaces, Fixed Point Theory Appl.,(2013), Art. No. 60.

[16] *T. Suzuki*, Basic inequality on a b-metric space and its applications, J. Inequal. Appl., (2017), Art. No. 256, 11 pages

[17] *J. Vujaković, H. Aydi, S. Radenović and A. Mukheimer*, Some remarks and new results in ordered partial b-metric spaces, Mathematics, **7**(2019), No. 334, doi: 10.3390/math7040334.