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MULTIVALUED TYPES OF KRASNOSELSKII'S FIXED POINT THEOREM FOR WEAK TOPOLOGY

Cesim Temel[1]

In this paper, we aim to prove some new results of Krasnoselskii's fixed point theorem for multivalued operators under weak topology acting in Banach spaces. In particular, the existence of fixed points of multivalued operator $L+S$ is discussed, where L is based on the generalized D-Lipschitzian, S has weakly sequentially closed graph, and $I-L$ may not be injective.

Keywords: Banach fixed point theorem, Krasnoselskii's fixed point theorem, weakly sequentially continuous operator, multivalued operator.
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1. Introduction

Recently, many authors have been interested in the various types of Krasnoselskii's fixed point theorem in Banach spaces [1-5, 9, 13, 14, 16, 18]. Using the combining of Banach and Schauder fixed point theorems, Krasnoselskii [16] proved that if operators L and S of a convex subset U of a Banach space into itself satisfy the following conditions:

(i) S is a continuous and compact operator,

(ii) L is a contraction operator,

(iii) $Lu+Sv\in U$ for each $u,v\in U$,

then equation $u=Lu+Su$ has a solution u in U.

Burton [9] improved the condition (iii) and proved that if $u=Lu+Sv$ for each $v\in U$, then $u\in U$. This result provides significant convenience in the applications of functional differential equation, integral equation and stability theory. The existence of the solutions of these nonlinear equations for weak topology was presented by the works [1, 2, 5, 14, 15]. In [12, 13], Dage introduced the multivalued version of Schauder's fixed point theorem and the fixed point theorems of multivalued operators in Banach algebras. In particular in [18], Liu and Li gave some results of Krasnoselskii's fixed point theorem in Banach spaces by approaching the operator with multivalued operators which are much more useful in finding out the inverse of operator.

We introduce the solutions of nonlinear operator equations under weak topology of the form

[1] Department of Mathematics, Van Yuzuncu Yil University, Faculty of Sciences, 65080, Van, Turkey, e-mail: cesimtemel@yyu.edu.tr

$$u = L(u) + S(u), \quad u \in U, \tag{1}$$

where L and S are weakly sequentially continuous operators.

We also establish the existence of fixed points of inclusions of Krasnoselskii-type multivalued operator for the weak topology of the form

$$u \in L(u) + S(u), \quad u \in U, \tag{2}$$

where L and S have weakly sequentially closed graph.

It is well known that the single-valued and multivalued versions of $L + S$ for the weak topology in Banach spaces may not be solved by the combination of the classic Schauder theorem and Banach fixed point theorem (see [1, 2, 5, 13-15, 18]). In particular, such equations in the transport equation (or the growing cell population) may not be solved under weak topology in Banach spaces. So, we should establish suitable conditions to guarantee the existence of fixed points of the single-valued and multivalued types of the operator $L + S$ under weak topology on Banach spaces.

This work aims to present some new results of Liu and Li [18] for the weak topology. The work also establish the existence of fixed points of multivalued operator $L + S$, where L is based on the generalized *D*-Lipschitzian and S has weakly sequentially closed graph.

For this, the injectivity of $I - L$ plays crucial role in the solutions of operator equation $L + S$. Especially, it is very difficult to find the solutions of such equations in the weak topology when $I - L$ is not injective. As it is, the inverse of $I - L$ could be seen as a multivalued operator as mentioned in Avramescu [4]. In order to investigate the solutions of the single-valued nonlinear equation (1) according to the weak topology, we approach the single-valued operator equation with the multivalued operator, and then improve the results of Liu and Li [18] for the weak topology. In order to detect the existence of fixed points of the inclusion (2) relative to the weak topology, we introduce the concept of the generalized *D*-Lipschitzian and prove a result for the fixed points of the multivalued operators.

2. Preliminaries

Let A be a Banach space with zero element θ. Assume $L : U \subseteq A \to 2^A$ is a multivalued operator which assigns to each element $u \in U$ a subset $L(u) \subseteq A$, where 2^A denotes the class of all subsets of A. Set $L(U)$ is identified by

$$L(U) = \bigcup_{u \in U} L(u).$$

For every subset $V \subset A$, we put

$$L^{-1}(V) = \{u \in U : L(u) \cap V \neq \emptyset\}.$$

The graph of L is determined by

$$G(L) = \{(u, v) \in U \times A : u \in U,\ v \in L(u)\}.$$

Let $d_H(.,.)$ denote the Hausdorff metric such that

$$d_H : 2^A \times 2^A \to \Re^+ \cup \{\infty\}$$

defined by $d_H(U,V) = \max\{\sup_{u\in U} d(u,V),\ \sup_{v\in V} d(U,v)\}$
for each $(U,V) \in 2^A \times 2^A$, where

$$d(u,V) = \inf_{v\in V}\|u-v\| \text{ and } d(U,v) = \inf_{u\in U}\|u-v\|.$$

Hausdorff space with the metric $d_H(.,.)$ is a metric space.

Recall that an operator L of A into itself is called weakly compact if the closure of $L(V)$ is weakly compact for each bounded subset $V \subset A$. An operator L of A into itself is called weakly sequentially continuous (w.s.c., for short) if for each weakly convergent sequence (u_n) in A with $u_n \xrightarrow{w} u$, there exists $L(u_n) \xrightarrow{w} L(u)$, where $\xrightarrow{w}$ denotes weak convergence. Let $\mathrm{cl}L(U)$ denote the the closure of $L(U)$ and by $\mathrm{co}L(U)$ the closed convex hull of $L(U)$.

The multivalued operator $L : U \to 2^U$ is called weakly sequentially upper-semicontinuous (w.s.u.sco., for short) if $L^{-1}(V)$ is sequentially closed for weak topology in U for any weakly closed subset V of U. The operator L is said to have weakly sequentially closed graph (w.s.c.g., for short) if for each $(u_n) \subset U$, $u_n \xrightarrow{w} u$ in U and for each (v_n) with $(v_n) \in L(u_n)$, $v_n \xrightarrow{w} v$ in A implies $v \in L(u)$.

Remark 2.1. *Note that every single-valued operator* $L_0 = U \subseteq A \to A$ *can be identified with a multivalued operator* $L : U \subseteq A \to 2^A$ *by setting* $L(u) = \{L_0(u)\}$ *for each* $u \in U$ (*see* [20, pp. 447]).

It is seen that if for a weakly compact subset U of A, a multivalued operator $L : U \to 2^U$ is w.s.u.sco., then the operator L is a weakly upper-semicontinuous operator (see [1, Theorem 2.2]).

We shall require the following theorem (see[1, Theorem 2.3]).

Theorem 2.2. *Let* U *be a closed convex subset of Banach space* A. *Assume* $L : U \to 2^U$ *is a weakly compact and w.s.u.sco. multivalued operator. Then* L *has a fixed point in* U.

3. Krasnoselskii's fixed point theorem for weak topology by means of multivalued operators

Here we present the existence of nonlinear operator equation (1) for weak topology in Banach spaces. In particular, we establish the weak topological version of the results of Lui and Li [18].

As the some main results of our work we give the following theorems.

Lemma 3.1. *Let* U *be a weakly closed convex subset of Banach space* A. *Let* $L: U \to 2^U$ *be a multivalued operator such that* L *has w.s.c.g, and* $L(U)$ *is relatively weakly compact. Then there is an* $u \in L(u)$ *for some* $u \in U$.

Proof. Since L has a w.s.c.g. and $L(U)$ is relatively weakly compact, then L is a w.s.u.sco. multivalued operator on U. Therefore for any weakly closed subset V of U, $L^{-1}(V)$ is sequentially closed for the weak topology on U. Hence $L^{-1}(V)$ is weakly compact by [10, 13.1]. So, $L^{-1}(V)$ is a weakly closed set. Hence the multivalued operator L is weakly upper-semicontinuous. Therefore if Arino et al. [2, Theorem 1] and Theorem 2.2 are applied to the multivalued operator L on U, then there is an $u \in L(u)$ for some $u \in U$. □

Theorem 3.2. *Let* U *be a weakly closed and bounded convex subset of Banach space* A. *Assume operators* $L, S: U \to A$ *are w.s.c. such that*

(i) $S(U) \subset (I-L)(U)$,

(ii) $S(U)$ *is relatively weakly compact in* U

(iii) *If* $(I-L)u_n \xrightarrow{w} v$, *then there is a weakly convergent subsequence* (u_{k_n}) *of* (u_n).

(iv) *For all* $v \in (I-L)(U)$,

$$W_v = \{u \in U : (I-L)u = v\}$$

is weakly closed convex.

Then $v = Lv + Sv$ *has a fixed point in* U.

Proof. If $(I-L)^{-1}$ exists on $S(U)$, then we obtain that $(I-L)^{-1}$ is a w.s.c. operator using assumption (iii). Hence, using assumption (i), the operator $(I-L)^{-1}$ of the space $S(U)$ into itself is defined by

$$v \to (I-L)^{-1}(u) = S(v)$$

for each $v \in U$. Therefore if Boyd and Wong [7, Theorem 1] is used, then there exists an $u = u_v$ in A such that for each $v \in U$,

$$(I-L)^{-1}(u_v) = S(v)$$

This means that $$L(u_v) + S(v) = u_v.$$

Hence we obtain that $u_v \in U$ using assumption (iv). It follows that we have

$$(I-L)^{-1} S(U) \subset U.$$

Since $(I-L)^{-1}$ and S are w.s.c. operators, then $(I-L)^{-1}S$ is a w.s.c. operator. Moreover, since $(I-L)^{-1}$ is a w.s.c. operator and $S(U)$ is a relatively weakly compact set in U, then

$$\operatorname{co}\left(\operatorname{cl}(I-L)^{-1}S(U)\right)$$

is a weakly compact set using the Krein-Smulian Theorem [10, 13.4]. Thus the operator $(I-L)^{-1}S$ is weakly continuous, and then using Arino et al. [2, Theorem 1], we obtain that the operator $(I-L)^{-1}S$ has a fixed point $u=u_v$ in U.

If $I-L$ is not invertible, then $(I-L)^{-1}$ could be seen as a multivalued operator as mentioned in Avramescu [4]. Thus multivalued operator $F:U\to 2^U$ can be defined by

$$F(v):=(I-L)^{-1}S(v)$$

for any $v\in U$. By assumption (i), the multivalued operator F is well defined. It suffices to show that F is a w.s.u.sco. multivalued operator on U. For this, it will be shown that F has w.s.c.g. Let $u\in U$ and sequence $(u_n)\subset U$ such that $u_n \xrightarrow{w} u$. Let $v_n\in F(u_n)$ with $v_n\xrightarrow{w} v$. If the definition of F is taken into account, then we obtain that $(I-L)(v_n)=S(u_n)$. Since $I-L$ and S are w.s.c., then we have

$$(I-L)(v_n)\xrightarrow{w}(I-L)(v) \quad \text{and} \quad S(u_n)\xrightarrow{w}S(u).$$

Hence it turns out that $(I-L)(v)=S(u)$. Therefore we have $v\in(I-L)^{-1}S(u)$. It follows that the graph of F is sequentially closed for the weak topology on U. Thus the multivalued operator F has w.s.c.g. Thus, using assumption (iv), $F(u)$ is a nonempty weakly closed convex set for each $u\in U$. It will also be demonstrated that $F(U)$ is relatively weakly compact. Let $(v_n)\subset F(U)$ and $(u_n)\subset U$ such that $v_n\in F(u_n)$. By the definition of F, we have

$$(I-L)(v_n)=S(u_n).$$

Since the operator S are w.s.c., then for $(u_n)\subset U$ with $u_n\xrightarrow{w}u$ in U,

$$S(u_n)\xrightarrow{w}S(u).$$

By assumption (ii), there is a subsequence (w_{n_k}) of $(w_n)\subset S(u_n)$ such that (w_{n_k}) converges weakly to $w\in S(u)$. In addition, since the operator $(I-L)^{-1}$ is w.s.c., then we get

$$(I-L)^{-1}(w_{n_k})\xrightarrow{w}(I-L)^{-1}(w)\in F(u).$$

Hence there is a subsequence $(v_{n_k})\subset(v_n)$ such that converges weakly to an element v of 2^U. Hence $F(U)$ is relatively weakly compact. Since $F(U)$ is relatively weakly compact and F has w.s.c.g., then F is a w.s.u.sco. multivalued operator on U. Using Lemma 3.1, we obtain that $u\in F(u)$ for some $u\in U$. This shows that $u\in L(u)+S(u)$ for some $u\in U$. □

Similarly, we can verify the proof of the following two results using the methods used in Theorem 3.2.

Theorem 3.3. *Let A be a Banach space. Assume operators $L, S : A \to A$ are w.s.c. such that*

(i) $S(A) \subset (I - L)(A)$,

(ii) *S is a weakly compact operator,*

(iii) *If $(I - L)u_n \xrightarrow{w} v$, then there is a weakly convergent subsequence (u_{k_n}) of (u_n).*

(iv) *For all $v \in (I - L)(U)$,*

$$W_v = \{u \in A : (I - L)u = v\}$$

is weakly compact convex.

Then $v = Lv + Sv$ has a fixed point in U.

Theorem 3.4. *Let U be a weakly closed and bounded convex subset of Banach space A. Assume operators $L, S : U \to A$ are w.s.c. such that*

(i) $(I - S)(U) \subset L(U)$,

(ii) *$(I - S)(U)$ is contained in a weakly compact set of U,*

(iii) *If $Lu_n \xrightarrow{w} v$, then there is a weakly convergent subsequence (u_{k_n}) of (u_n).*

(iv) *For each $v \in L(U)$,*

$$W_v = \{u \in A : Lu = v\}$$

is weakly compact convex.

Then $v = Lv + Sv$ has a fixed point in U.

4. The fixed points of the sum of multivalued operators for weak topology

Now we aim to give some results for the existence of the fixed points of the inclusion (2) of the multivalued operators in Banach spaces for weak topology case. This inclusion is based on the generalized *D*-Lipschitzian and the weak sequential closed graph of multivalued operators.

For this, we need to the following definition and a result of multivalued operators with generalized *D*-Lipschitzian contraction.

The multivalued operator $L : U \subseteq A \to 2^U$ is said to be the generalized contraction if $d_H(L(u), L(v)) \le k\|u - v\|$
for each $u, v \in U$ and fixed $k \in [0,1[$.

Definition 4.1. *The multivalued operator* $L: U \subseteq A \to 2^U$ *is said to be the generalized D-Lipschitzian if there is a continuous nondecreasing function* $\tau_L : \Re^+ \to \Re^+$ *such that* $\tau_L(0) = 0$, *for any* $r > 0$, $\tau_L(r) < r$ *and*

$$d(u, L(v)) \le d_H(L(u), L(v)) \le \tau_L(\|u - v\|) \text{ for all } u, v \in U.$$

The generalized *k*-contraction is a generalized *D*-Lipschitzian, but the converse is generally not true. Therefore the class of the generalized *k*-contraction is a subclass of the class of the generalized *D*-Lipschitzian.

Example 4.2. *Let* $A = C(\Re, [0,1])$ *and assume that* $L: A \to 2^A$ *and* $h: \Re \times [0,1] \to [0,1]$ *are defined by*

$$(L(u))(s) = \{h(s, u(s))\} = \{u(s) - u^3(s) + \sin^2(s)\}$$

for every $u \in A$ *and* $s \in \Re$.

The operator L *is a generalized D-Lipschitzian, but it is not a generalized k-contraction: Assume for each* $u, v \in A$ *and any* $s \in \Re$,

$$|u(s) - v(s)|^2 = u^2(s) + v^2(s) - 2u(s)v(s) \le u^2(s) + v^2(s) + u(s)v(s).$$

From this inequality, we have

$$\begin{aligned} |Lu(s) - Lv(s)| &= |u(s) - u^3(s) - v(s) + v^3(s)| \\ &= |u(s) - v(s)|\left|1 - \left(u^2(s) + v^2(s) + u(s)v(s)\right)\right| \\ &\le |u(s) - v(s)|\left(1 - |u(s) - v(s)|^2\right). \end{aligned}$$

Hence it follows that

$$d_H(L(u), L(v)) \le \|u - v\|\left(1 - \|u - v\|^2\right) = \tau_L(\|u - v\|)$$

by taking sup over s. *This shows that the multivalued operator* L *is a generalized D-Lipschitzian.*

Now for the rest of the assertion, suppose that the operator L is a generalized k-contraction. Therefore there is a $k \in [0,1[$ *such that*

$$d_H(L(u), L(v)) \le k\|u - v\|$$

for each $u, v \in U$. *Now let* $0 < \varepsilon < 1 - k$ *and* $W \subset A \times A$ *is identified by*

$$W = \{(u(s), v(s)) : u^2(s) + v^2(s) + u(s)v(s) = 1 - k - \varepsilon\}.$$

Since

$$d_H(L(u), L(v)) \le k\|u - v\|$$

for all $u, v \in U$, *it is clear that* $W \neq \emptyset$.

For given two constant valued functions u_0 *and* v_0 *with* $(u_0, v_0) \in W$,

$$d_H(L(u_0), L(v_0)) = (k + \varepsilon)\|u_0 - v_0\| > k\|u_0 - v_0\|.$$

This is a contradiction. For this reason, the multivalued operator L *does not have to be a generalized k-contraction.*

As the some main results of our work we give the following theorems.

Lemma 4.3. *Let U be a nonempty weakly closed convex subset of Banach space A. Suppose that the multivalued operator $L:U\to 2^U$ is a generalized D-Lipschitzian and has w.s.c.g. Then there is an $u\in L(u)$ for some $u\in U$.*

Proof. Since L is a generalized *D*-Lipschitzian, there is a continuous nondecreasing function τ_L such that $\tau_L(0)=0$, for any $r>0$, $\tau_L(r)<r$,

$$d(u,L(v))\le d_H(L(u),L(v))\le \tau_L(\|u-v\|)$$

for all $u,v\in U$. Hence, let $u_0\in U$ and choose $u_1\in L(u_0)$ with $r=\|u_0-u_1\|>0$ and $\tau_L(r)<r$,

$$d(u_1,L(u_1))\le d_H(L(u_0),L(u_1))\le \tau_L(\|u_0-u_1\|)$$

so that there exists an $u_2\in L(u_1)$ such that $d(u_1,u_2)\le \tau_L(\|u_1-u_2\|)$. If this process is continued, it can be constructed a sequence (u_n) such that for any $n\in\aleph$,

$$\tau_L(\|u_n-u_{n+1}\|)<\|u_n-u_{n+1}\|$$

and

$$u_{n+1}\in L(u_n),\quad d(u_n,u_{n+1})\le \tau_L(\|u_n-u_{n+1}\|).$$

Since L has w.s.c.g., then $L(u)$ is a nonempty weakly closed set for all $u\in U$. Therefore if the continuity of function τ_L with $\tau_L(0)=0$, and Boyd and Wong [7, Theorem 1] are applied to the last inequality, then we get

$$d(u_{n+1},L(u))\le d_H(L(u_n),L(u))\le \tau_L(\|u_n-u\|)\to \tau_L(0)=0$$

as $n\to\infty$. It follows that there is an $u\in L(u)$ for some $u\in U$. □

Theorem 4.4. *Let U be a nonempty weakly closed convex subset of Banach space A. Assume $L:U\to 2^U$ and $S:U\to 2^U$ have w.s.c.g. such that the multivalued operators L and S satisfy*

(i) *L is a generalized D-Lipschitzian,*

(ii) *$S(U)$ is relatively weakly compact,*

(iii) *For each $v\in U$, $u\in L(u)+S(v)$ implies $u\in U$.*

Then there is an $u\in L(u)+S(u)$ for some $u\in U$.

Proof. By assumption (i), there exists an inverse of $I-L$ such that it is continuous on $S(U)$. Moreover, by Lemma 4.3, we can define multivalued operator T on U by

$$\begin{cases} T:U\to 2^U, \\ \quad v\to T(v)=(I-L)^{-1}S(v). \end{cases}$$

Thus the multivalued operator T is well defined on U. Now let's show that $T(U)\subset U$: Let $u\in U$ and choose $w\in(I-L)^{-1}S(u)$, then there is $v\in S(u)$. It follows that $w\in(I-L)^{-1}(v)$. Hence we have $w\in L(w)+v\subseteq L(w)+S(u)$.
As a result of assumption (iii), we can easily see that $w\in U$. Thus we obtain $T(U)\subset U$.

Now it will be shown that it is enough to prove that T has w.s.c.g, and the set $T(U)$ is relatively weakly compact: To see this, first we will verify that T has a w.s.c.g. Let $u\in U$ and $(u_n)\subset U$ such that $u_n \xrightarrow{w} u$ in U and $v_n\in T(u_n)$ such that $v_n \xrightarrow{w} v$ in U. By the definition of T, we get $(I-L)(v_n)\in S(u_n)$.
Since L has a w.s.c.g, and $S(u)$ is relatively weakly compact for every $u\in U$, then $(I-L)(v)\in S(u)$. Using Lemma 4.3, we have $v\in T(u)=(I-L)^{-1}S(u)$.
This implies that T is a weakly closed multivalued operator. It follows from the definition of $I-L$ and the weak closeness of the operator T that $T(u)$ is a nonempty weakly closed convex set for each $u\in U$.

Then, we demonstrate that $T(U)$ is a relatively weakly compact set. Let $(v_n)\subset T(U)$ such that for each $u\in U$, $v_n\in T(u)$. Using the definition of T, we get a sequence $(z_n)\subset S(u)$ such that $(I-L)(v_n)=z_n$.
By taking into account assumption (ii), sequence (z_n) has a subsequence which converges weakly to $z\in S(u)$. So, $(I-L)^{-1}(z_n)$ converges weakly to

$$(I-L)^{-1}(z)\in T(u).$$

Thus (v_n) has a subsequence $(v_{n_k})\subset(v_n)$ such that $v_{n_k}\xrightarrow{w} v_0$ in 2^U. Hence $T(U)$ is relatively weakly compact. Therefore if T has w.s.c.g, and $T(U)$ is relatively weakly compact, then using Lemma 3.1, we have an $u\in T(u)$ for some $u\in U$. This means that $u\in L(u)+S(u)$ for some $u\in U$. □

5. Conclusions

It is well known that the single-valued operator equation (1) and the inclusion (2) cannot have a solution without the certain conditions upon operators in Banach spaces. In particular, these types of nonlinear operator equations and inclusions in the transport equation (the growing cell population or the hereditary systems) may not be solved on Banach spaces. That’s why suitable conditions were established to guarantee the existence of fixed points of the single-valued operator equation (1) and the inclusion (2) on Banach spaces. For these, in order to find the solutions of the single-valued nonlinear equation (1) according to the weak topology, we approached the single-valued operator equation with the multivalued operator. Then we improved the results of Liu and Li [18] for the weak topology.

Then, in order to detect the existence of fixed points of the inclusion (2) with respect to the weak topology, the definition of the generalized D-Lipschitzian was presented for multivalued operators. Then we proved new multivalued types of Krasnoselskii's fixed point theorem using the generalized *D*-Lipschitzian and the weak sequential closed graph of multivalued operators.

REFERENCES

[1]. *R. P. Agarwal, D. O'Regan*, Fixed-point theory for weakly sequentially upper-semicontinuous maps with applications to differential inclusions, Nonlinear Oscillations, **3**(2002), No. 5, 277-286.

[2]. *O. Arino, S. Gautier, J. P. Pento*, A fixed point theorem for sequentially continuous mapping with application to ordinary differential equations, Functional Ekvac., **27**(1984), No. 3, 273-279.

[3]. *C. Avramescu*, A fixed points theorem for multivalued mapping, Electronic J. Qualitative Theory of Differential Equations, **17**(2004), 1-10.

[4]. *C. Avramescu*, *C. Vladimirescu*, A fixed points theorem of Krasnoselskii type in a space of continuous functions, Fixed Point Theory, **5**(2004), 1-11.

[5]. *C. S. Barroso*, Krasnoselskii's fixed point theorem for weakly continuous maps, Nonlinear Anal., **55**(2003), 25-31.

[6]. *C. S. Barroso*, *E. V. Teixeira*, A topological and geometric approach to fixed points results for sum of operators and applications, Nonlinear Anal., **60**(2005), 625-650.

[7]. *D. W. Boyd, J. S. W. Wong*, On nonlinear contractions, Proc. Amer. Math. Soc., **20**(1969), 458-464.

[8]. *T. A. Burton, C. Kirk*, A fixed point theorem of Krasnoselskii-Schaefer type. Math. Nachr., **189**(1998), 23-31.

[9]. *T. A. Burton*, A fixed point theorem of Krasnoselskii, Appl. Math. Lett., **11**(1998), 85-88.

[10]. *J. B. Conway*, A course in functional analysis, Springer-Verlag, New York, 1990.

[11]. *I. Dobrakov*, On representation of linear operators on $C_0(T;X)$, Czechoslovak Math. J., **21**(1971), 13-30.

[12]. *B. C. Dage*, Multivalued operators and fixed point theorems in Banach algebras, I. Taiwanese J. Math., **10**(2006), 1025-1045.

[13]. *B. C. Dage*, Some generalization of multi-valued version of Schauder's fixed point theorem and applications, Cubo, **12**(2010), 139-151.

[14]. *J. Garcia-Falset, K. Latrach*, Krasnoselskii-type fixed-point theorems for weakly sequentially continuous mappings, Bull. Lond. Math. Soc., **44**(2012), 25-38.

[15]. *J. R. Graef, J. Henderson, A. Ouahab*, Multivalued versions of a Krasnosel'skii-type fixed point theorem, J. Fixed Point Theory Appl., **19**(2017), 1059-1082.

[16]. *M. A. Krasnoselskii*, Some problems of nonlinear analysis, Amer. Math. Soc. Trans., **10**(1958), No. 2, 345-409.

[17]. *Y. C. Liu, Z. X. Li*, Schaefer type theorem and periodic solutions of evolution equations, J. Math. Anal. Appl., **136**(2006), No. 1, 237-255.

[18]. *Y. C. Liu, Z. X. Li*, Krasnoselskii type fixed point theorems and applications, Proc. Amer. Math. Soc., **136**(2008), No. 4, 1213-1220.

[19]. *A. C. Thompson*, On certain contraction mappings in a partially ordered vector space, Proc. Amer. Math. Soc., **14**(1963), 438-443.

[20]. *E. Zeidler*, Nonlinear functional analysis and its applications, I: Fixed point theorems, Springer-Verlag, New York, 1986.