# THE CORRESPONDENCE OF FUSION FRAMES AND FRAMES IN A HILBERT $C^*$-MODULE: COUNSTRUCTION AND APPLICATION IN MODULAR FUSION FRAMES

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*In this article, we show that a fusion frame in an infinite dimensional separable Hilbert space $\mathcal{H}$ with finite dimensional subspaces whose dimensions are at most $m$, corresponds to a frame in the Hilbert $\mathcal{B}(\mathbb{C}^m)$-module $\mathcal{H}^m$ and present they share some properties. We also show that a Reisz fusion basis in $\mathcal{H}$ is correspondent with a Reisz basis in $\mathcal{H}^m$. Finally, we introduce a new notion, modular fusion frame, using modular frames in the Hilbert $\mathcal{B}(\mathbb{C}^m)$-module $\mathcal{H}^m$.*

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## 1. Introduction

Fusion frame was originally called frame of subspaces introduced by Casazza and Kutyniok [9]. Fusion frame is a generalization of frame theory and it is useful for robust and stable representation of a signal. In other words, a signal is represented by the magnitude of projection of a signal onto frame vectors in frame theory, while in fusion frame, a signal is represented by a collection of vectors, in which their entries are equal to the inner product of the signal and orthogonal bases of subspaces of the fusion frame. Fusion frames have wide range of applications including sampling theory [14], data quantization [7], coding [6], image processing [8], time frequency analysis [12], and speech recognition [5].

On the other hand, many mathematicians generalized the notion of frame in a Hilbert space to frame in a Hilbert $C^*$-module and achieved significant results. Standard frames in Hilbert $C^*$-modules over unital $C^*$-algebras, were first defined by Frank and Larson in 1998 [18]. However, the case of Hilbert $C^*$-module over non-unital $C^*$-algebra has been investigated in [28] as well as in [4]. The most significant advantages of frame in Hilbert $C^*$-module to frame in Hilbert space are the additional degree of freedom coming from the $C^*$-algebra of coefficients and therefore, similar procedures used for frames in Hilbert spaces are applied to deal with frames in Hilbert $C^*$-modules.

In this article, we show that every fusion frame in a separable Hilbert space $\mathcal{H}$ corresponds to a frame in the Hilbert $\mathcal{B}(\mathbb{C}^m)$-module $\mathcal{H}^m$. Thus, they share similar properties and results in different interpretations. We also define modular fusion frame using modular frames in the Hilbert $\mathcal{B}(\mathbb{C}^m)$-module $\mathcal{H}^m$.

This paper is structured as follows: Section 2 starts with preliminaries about fusion frames and frames in Hilbert $C^*$-modules and some of their characterisation. In section 3, we present the equivalence of fusion frames and frames in Hilbert $C^*$-modules. Then, we discuss some properties of these two notions and show that they share similar properties.

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Finally, section 4 is devoted to introducing of the new notion, modular fusion frames, using modular frames in the Hilbert $\mathcal{B}(\mathbb{C}^m)$-module $\mathcal{H}^m$ and it explores some of their properties.

## 2. Preliminaries and Notations

We recall that a fusion frame for a separable infinite dimensional Hilbert space $\mathcal{H}$ is a family of $m$ dimensional subspaces $\{W_i\}_{i\in I}$ in $\mathcal{H}$ and a family of positive weights $\{\omega_i\}_{i\in I}$ where $I$ is a countable index set and for every $f \in \mathcal{H}$ there exist two positive constants $0 < A \le B < \infty$ such that for every $f \in \mathcal{H}$

$$A\|f\|^2 \le \sum_{i\in I} \omega_i^2 \|P_{W_i} f\|^2 \le B\|f\|^2, \tag{1}$$

where $P_{W_i}$ is the orthogonal projection onto $W_i$. The constants $A$ and $B$ are called the fusion frame bounds. Furthermore, the fusion frame is tight when $A = B$ and it is a Bessel sequence if the right hand side of (1) holds. A fusion frame $\{(W_i, \omega_i)\}_{i\in I}$ is said to be an orthonormal fusion basis if $\mathcal{H} = \oplus_{i\in I} W_i$ and it is a Riesz decomposition of $\mathcal{H}$ if for every $f \in \mathcal{H}$, there exists a unique choice of $f_i \in W_i$ such that $f = \sum_{i\in I} \omega_i f_i$. Moreover, a family of subspaces $\{W_i,\}_{i\in I}$ of $\mathcal{H}$ is called a Riesz fusion basis whenever it is complete for $\mathcal{H}$ and there exist two positive constants $0 < C \le D < \infty$ such that for any arbitrary vector $f_i \in W_i$ we have

$$C\sum_{i\in I} \|f_i\|^2 \le \|\sum_{i\in I} \omega_i f_i\|^2 \le D\sum_{i\in I} \|f_i\|^2.$$

It is clear that any Riesz fusion basis is a fusion frame and also a fusion frame is a Riesz basis if and only if it is a Riesz decomposition for $\mathcal{H}$ [9].

To define the operators associated with a fusion frame we consider the Hilbert space

$$\sum_{i\in I} \oplus W_i = \left\{\{f_i\}_{i\in I} : f_i \in W_i, \text{and}\{\|f_i\|\}_{i\in I} \in \ell^2(I)\right\},$$

with the inner product

$$\langle \{f_i\}_{i\in I}, \{g_i\}_{i\in I}\rangle = \sum_{i\in I} \langle f_i, g_i\rangle.$$

The analysis operator $T_W : \mathcal{H} \to \sum_{i\in I} \oplus W_i$ for any $f \in \mathcal{H}$ is defined as

$$T_W(f) = \{\omega_i P_{W_i} f\}_{i\in I}.$$

The adjoint of analysis operator $T_W^* : \sum_{i\in I} \oplus W_i \to \mathcal{H}$ is called the synthesis operator and it is given by

$$T_W^*(\{f_i\}_{i\in I}) = \sum_{i\in I} \omega_i f_i.$$

Let $\{(W_i, \omega_i)\}_{i\in I}$ be a fusion frame. The fusion frame operator $S_W : \mathcal{H} \to \mathcal{H}$ is defined by

$$S_W f = \sum_{i\in I} \omega_i^2 P_{W_i} f,$$

which is a bounded, invertible and positive operator. Consequently, we have

$$f = \sum_{i\in I} \omega_i^2 S_W^{-1} P_{W_i} f.$$

The family $\{\omega_i^2 S_W^{-1} W_i\}_{i\in I}$ is called the cannonical dual fusion frame associated to $\{(W_i, \omega_i)\}_{i\in I}$. Moreover, a Bessel fusion sequence $\{(V_i, \nu_i)\}_{i\in I}$ is an alternate dual of $\{(W_i, \omega_i)\}_{i\in I}$ if and only if [21]:

$$T_V \phi_{VW} T_W^* = I_{\mathcal{H}}, \tag{2}$$

where $\phi_{VW} : \sum_{i\in I} \oplus W_i \to \sum_{i\in I} \oplus V_i$ is a bounded operator. If we assume

$$\phi_{VW}(\{f_i\}_{i\in I}) = \{P_{V_i} A f_i\}_{i\in I}, \tag{3}$$

then, (2) will be equivalent to:

$$f = \sum_{i \in I} \nu_i P_{V_i} A \omega_i P_{W_i} f. \tag{4}$$

A Hilbert $C^*$-module over a $C^*$-algebra $\mathcal{A}$ is a left $\mathcal{A}$-module $E$ with an $\mathcal{A}$-valued inner product $\langle .,. \rangle : E \times E \to \mathcal{A}$ such that $E$ is a Banach space with respect to the norm $\|f\| = \|\langle f,f \rangle\|^{\frac{1}{2}}$. Recall that the inner product on $E$ has the properties

- $\langle f,f \rangle \geq 0$,
- $\langle f,f \rangle = 0 \Leftrightarrow x = 0$,
- $\langle f, g+h \rangle = \langle f,g \rangle + \langle f,h \rangle$,
- $\langle af, g \rangle = a\langle f,g \rangle$,
- $\langle f,g \rangle^* = \langle g,f \rangle$,

where $f,g,h \in E$ and $a \in \mathcal{A}$. We say that $E$ is countably generated if there exists a sequence $\{f_i\}_{i \in I}$ in $E$ such that the closed linear span of the set $\{f_i a : i \in I, a \in \mathcal{A}\}$ is equal to $E$. It is clear that $\mathcal{B}(\mathbb{C}^m)$ is a unital $C^*$-algebra.

Let $E$ be a Hilbert $C^*$-module. A sequence $\{f_i\}_{i \in I}$ in $E$ is called a frame for $E$ if there exist two positive constants $A$ and $B$ such that for every $f \in E$:

$$A\langle f,f \rangle \leq \sum_{i \in I} \langle f, f_i \rangle \langle f_i, f \rangle \leq B\langle f,f \rangle,$$

which is identical to the following inequalities:

$$A\|f\|^2 \leq \sum_{i \in I} \|\langle f, f_i \rangle\|^2 \leq B\|f\|^2. \tag{5}$$

If only the second inequality of (5) is satisfied, we say that $\{f_i\}_{i \in I}$ is a Bessel sequence. The constants $A$ and $B$ are called frame bounds. If $A = B = 1$, i.e. if for every $f \in E$

$$\sum_{i \in I} \langle f, f_i \rangle \langle f_i, f \rangle = \langle f,f \rangle,$$

the sequence $\{f_i\}_{i \in I}$ is called a Parseval frame for $E$.

## 3. The Correspondence between Fusion Frames in $\mathcal{H}$ and Frames In Hilbert $\mathcal{B}(\mathbb{C}^m)$-Module $\mathcal{H}^m$

In this section we present the relationship of fusion frames in a seperable infinite dimensional Hilbert space with frames in a Hilbert $C^*$-module. In this paper, we assume that subspaces of fusion frames are finite dimensional and the dimension of each subspace is at most $m \in \mathbb{N}$. Moreover, as we can embed each subspace in the $m$-dimensional subspace, we may consider the dimension of each subspace is equal to $m$. We know that $\mathcal{B}(\mathbb{C}^m)$ is a $C^*$-algebra with multiplication as the matrix multiplication and we can consider the elements of $\mathcal{B}(\mathbb{C}^m)$ as a square matrix of dimension $m \times m$. It is obvious that $\mathcal{H}^m$ is a Hilbert $C^*$-module on the $C^*$-algebra $\mathcal{B}(\mathbb{C}^m)$ [3] with the inner product defined for every $F = (f_1, \cdots, f_m), G = (g_1, \cdots, g_m) \in \mathcal{H}^m$ as

$$\langle F, G \rangle = \left(\langle f_i, g_j \rangle\right)_{1 \leq i,j \leq m},$$

where $\langle f_i, g_j \rangle$ is the inner product defined on the Hilbert space $\mathcal{H}$.

In order to show that fusion frame is related a frame in a Hilbert $C^*$-module, we discuss how to represent fusion frame elements in the Hilbert $C^*$-module $\mathcal{H}^m$. To do this, we assume that $W$ is a subspace of $\mathcal{H}$ and the dimension of $W$ is equal to $t \leq m$. We represent $W$ by a matrix $U_W$ as follows:

$$U_W = [e_1, \cdots, e_m],$$

where $\{e_1, \cdots, e_t\}$ is an orthonormal basis for $W$ and $e_i = 0$ for $i = t+1, \cdots, m$. We also define a map $\tilde{}^{i} : \mathcal{H} \to \mathcal{H}^m$ by transfering $f \in \mathcal{H}$ to $\tilde{F}^i = (0, \cdots, 0, f, 0, \cdots, 0) \in \mathcal{H}^m$ where the $i$-th component of $\tilde{F}^i$ is equal to $f$ and other components are equal to zero. By this process, the inner product of a signal and subspaces as fusion frame elements are well defined.

We will show the relationship of fusion frames in the Hilbert space $\mathcal{H}$ with finite dimensional subspaces and frames in the Hilbert $C^*$-module $\mathcal{H}^m$ on the $C^*$-algebra $\mathcal{B}(\mathbb{C}^m)$ are investigated in the next theorem.

**Theorem 3.1.** *Let $\{W_i\}_{i\in I}$ be a family of subspaces of $\mathcal{H}$ with the dimension is at most $m$ and $\{\omega_i\}_{i\in I}$ be a set of positive weights. Then, the following statements are equivalent.*

(i) *$\{(W_i, \omega_i)\}_{i\in I}$ is a fusion frame for $\mathcal{H}$.*

(ii) *$\{\omega_i U_{W_i}\}_{i\in I}$ is a frame for $\mathcal{H}^m$.*

*Proof.* $i \to ii$ Consider $\{(W_i, \omega_i)\}_{i\in I}$ as a fusion frame for $\mathcal{H}$. Then there exist two constants $0 < A \le B < \infty$ such that for every $f \in \mathcal{H}$ we have:

$$A\|f\|^2 \le \sum_{i\in I} \omega_i^2 \|P_{W_i} f\|^2 \le B\|f\|^2. \tag{6}$$

On the other hand:

$$\|P_{W_i} f\|^2 = \sum_{j=1}^{m} |\langle f, e_i^j\rangle|^2 = \|\langle \tilde{F}^1, U_{W_i}\rangle\|_{\mathcal{F}r},$$

where $\{e_i^j\}_{j=1}^m$ is an orthonormal basis for $W_i$ for every $i \in I$ and $\|.\|_{\mathcal{F}r}$ is the Frobenius norm. Moreover,

$$\|\tilde{F}^1\|_{\mathcal{F}r}^2 = \|f\|_2^2.$$

Therefore, we have:

$$A\|\tilde{F}^1\|_{\mathcal{F}r} \le \sum_{i\in I} \omega_i^2 \|\langle \tilde{F}^1, U_{W_i}\rangle\|_{\mathcal{F}r} \le B\|\tilde{F}^1\|_{\mathcal{F}r}.$$

Suppose that $G = (g_1, \cdots, g_m) \in \mathcal{H}^m$ is given. So we have:

$$\begin{aligned} \|G\|_{\mathcal{F}r}^2 &= \sum_{i,j=1}^{m} |\langle g_i, g_j\rangle| = \sum_{i=1}^{m} \|g_i\|^2 + \sum_{i,j=1, i\neq j}^{m} |\langle g_i, g_j\rangle| \\ &\le \sum_{i=1}^{m} \|g_i\|^2 + \sum_{i,j=1, i\neq j}^{m} \|g_i\|\|g_j\| \le \sum_{i=1}^{m} \|g_i\|^2 + \sum_{i,j=1, i\neq j}^{m} \frac{1}{2}\left(\|g_i\|^2 + \|g_j\|^2\right) \\ &\le \sum_{i=1}^{m} \|g_i\|^2 + \sum_{i=1}^{m}\sum_{j=i+1}^{m} \left(\|g_i\|^2 + \|g_j\|^2\right) = \sum_{i=1}^{m} \|g_i\|^2 + (m-1)\sum_{i=1}^{m} \|g_i\|^2 \\ &= m\sum_{i=1}^{m} \|g_i\|^2. \end{aligned} \tag{7}$$

On the other hand,

$$\sum_{j=1}^{m} \|g_j\|^2 \le \|G\|_{\mathcal{F}r}^2 = \sum_{j=1}^{m} \|g_j\|^2 + \sum_{i,j=1, i\neq j}^{m} |\langle g_i, g_j\rangle| \tag{8}$$

Moreover, it is obvious that:

$$\|\langle G, \omega_j^2 U_{W_i}\rangle\|_{\mathcal{F}r}^2 = \sum_{j=1}^{m} \omega_j^2 \|\langle \tilde{G_j}^{j}, U_{W_i}\rangle\|_{\mathcal{F}r}^2 = \sum_{j=1}^{m} \omega_j^2 \|P_{W_i} g_j\|^2. \tag{9}$$

So (6), (7), (8), and (9) lead to

$$\begin{aligned}A\frac{1}{m}\|G\|^2 = \sum_{j=1}^{m} A\|g_j\|^2 \leq \sum_{i\in I} \|\langle G, \omega_i U_{W_i}\rangle\|^2_{\mathcal{F}r} = \sum_{i\in I}\sum_{j=1}^{m} \omega_j^2 \|P_{W_i} g_j\|^2 \\ \leq \sum_{j=1}^{m} B\|g_j\|^2 = B\|G\|^2.\end{aligned} \tag{10}$$

By the fact that $\mathcal{B}(\mathbb{C}^m)$ is a finite dimensional $C^*$-algebra, all norms on it are equivalent. Therefore, by the inequality (10) $\{\omega_i U_{W_i}\}_{i\in I}$ is a frame in the Hilbert $\mathcal{B}(\mathbb{C}^m)$-module $\mathcal{H}^m$.

$ii \to i$ Now assume $\{\omega_i U_{W_i}\}_{i\in I}$ is a frame in the Hilbert $C^*$-module $\mathcal{H}^m$. As all norm on $\mathcal{B}(\mathbb{C}^m)$ are equivalent, we have the following inequalities:

$$A\|F\|^2_{\mathcal{F}r} \leq \sum_{i\in I} \|\langle F, \omega_i U_{W_i}\rangle\|^2_{\mathcal{F}r} \leq B\|F\|^2_{\mathcal{F}r}.$$

By the fact $\|\langle \tilde{F}^1, U_{W_i}\rangle\|^2_{\mathcal{F}r} = \|P_{W_i} f\|^2$, it is easily achieved that $\{(W_i, \omega_i)\}_{i=1}^N$ is a fusion frame for $\mathcal{H}$.

□

Now we study the relation between the analysis and synthesis operators of a fusion frame in the Hilbert space $\mathcal{H}$ and the corresponding frame in the Hilbert $\mathcal{B}(\mathbb{C}^m)$-module $\mathcal{H}^m$. The analysis operator for $\{U_{W_i}\}_{i\in I}$ is defined as:

$$\begin{aligned}T :& \mathcal{H}^m \to \ell^2(\mathcal{B}(\mathbb{C}^m)) \\ & F \to \{\langle F, \omega_i U_{W_i}\rangle\}_{i\in I},\end{aligned}$$

and the analysis operator for $\tilde{F}^1 = (f, 0, \cdots, 0)$ is

$$T\left(\tilde{F}^1\right) = \{\langle \tilde{F}^1, U_{W_i}\rangle\}_{i\in I} = \{\widetilde{\omega_i P_{W_i} f}^1\}_{i\in I}. \tag{11}$$

By (11) we have the following equation which shows the relationship of the analysis operator of a fusion frame in $\mathcal{H}$ and the analysis operator of the corresponding frame in the Hilbert $\mathcal{B}(\mathbb{C}^m)$-module.

$$\{\widetilde{P_{W_i} f}^1\}_{i\in I} = T\left(\tilde{F}^1\right)\{U_{W_i}\}_{i\in I}.$$

where $\{\widetilde{P_{W_i} f}^1\}_{i\in I}$ is equivalent to the analysis operator of the fusion frame $T_W(f)$. Moreover, the multiplication between $T\left(\tilde{F}^1\right)$ as an element of $\mathcal{B}(\mathbb{C}^m)$ and $\{U_{W_i}\}_{i\in I}$ as an element of $\mathcal{H}^m$ is the multiplication of an element of a $C^*$-algebra and $C^*$-Hilbert module which results in $\mathcal{H}^m$.

The synthesis operator for $\{X_i\}_{i\in I} \subset \mathcal{B}(\mathbb{C}^m)$ is achieved as

$$T^*(\{X_i\}_{i\in I}) = \sum_{i\in I} X_i \omega_i U_i = \sum_{i\in I} \omega_i X_i U_i. \tag{12}$$

Now we are looking for $F_i \in \mathcal{H}^m$ such that $\langle F_i, U_i\rangle = \left(\langle f_i^j, e_i^k\rangle\right)_{1\leq j,k\leq m} = X_i$. Consider $f_i^j = X_i(j,k)\, e_i^k$. Then, for $F_i = \left(f_i^1, f_i^2, \cdots, f_i^m\right)$, we have $\langle F_i, U_i\rangle = X_i$. So, we can rewrite (12) as

$$T^*(\{X_i\}_{i\in I}) = \sum_{i\in I} \omega_i \langle F_i, U_i\rangle U_i.$$

Assume $f_i \in W_i$ for $i \in I$ and $\tilde{F_i}^1$ is its associated matrix. Therefore

$$T^*\left(\{\langle \tilde{F_i}^1, U_i\rangle\}_{i\in I}\right) = \sum_{i\in I} \omega_i \langle \tilde{F_i}^1, U_i\rangle U_i = \sum_{i\in I} \omega_i \widetilde{P_{W_i} f_i}^1.$$

As a result, the synthesis operator of $\{\omega_i U_{W_i}\}_{i=1}^N$ in the Hilbert $\mathcal{B}(\mathbb{C}^m)$-module $\mathcal{H}^m$ is equivalent to the synthesis operator of the fusion frame $\{W_i\}_{i\in I}$ in the Hilbert space $\mathcal{H}$.

The frame operator of $\{\omega_i U_{W_i}\}_{i\in I}$ in the Hilbert $\mathcal{B}(\mathbb{C}^m)$-module $\mathcal{H}^m$ is concluded as the combination of the synthesis and analysis operators which is equal to

$$S(F) = \sum_{i\in I} \langle F, \omega_i U_i\rangle \omega_i U_i = \sum_{i\in I} \omega_i^2 \langle F, U_i\rangle U_i,$$

for every $F \in \mathcal{H}^m$. Therefore, for $f \in \mathcal{H}$ we have

$$S\left(\tilde{F}^1\right) = \sum_{i\in I} \omega_i^2 \langle \tilde{F}^1, U_{W_i}\rangle U_{W_i} = \sum_{i\in I} \omega_i^2 \widetilde{P_{W_i} f}.$$

On the other hand the corresponding fusion frame operator for $f \in \mathcal{H}$ is equal to

$$\begin{aligned} S_W\left(\tilde{F}^1\right) &= \sum_{i\in I} \langle \langle \tilde{F}^1, \omega_i U_{W_i}\rangle U_{W_i}, U_{W_i}\omega_i\rangle U_{W_i} \\ &= \sum_{i\in I} \omega_i^2 \langle \tilde{F}^1, U_{W_i}\rangle \langle U_{W_i}, U_{W_i}\rangle U_{W_i} = \sum_{i\in I} \omega_i^2 \langle \tilde{F}^1, U_{W_i}\rangle U_{W_i}, \end{aligned}$$

which is equal to the frame operator of $\{U_{W_i}\} i \in I$ in $\mathcal{H}^m$.

One of the most favorite type of frames is tight frames which atracts attentions of many researchers. Next theorem shows that the fusion frame $\{(W_i, \omega_i)\}_{i\in I}$ is tight if and only if $\{\omega_i U_{W_i}\}_{i\in I}$ is a tight frame in the Hilbert $\mathcal{B}(\mathbb{C}^m)$-module $\mathcal{H}^m$.

**Theorem 3.2.** *Assume $\{W_i\}_{i\in I}$ is a sequence of finite dimensional subspaces in $\mathcal{H}$ with dimensions equal to $m$. Then the following statements are equivalent.*

(*i*) *$\{(W_i, \omega_i)\}_{i\in I}$ is a $A$-tight fusion frame in $\mathcal{H}$.*

(*ii*) *$\{\omega_i U_{W_i}\}_{i\in I}$ is a $A$-tight frame in the Hilbert $\mathcal{B}(\mathbb{C}^m)$-module $\mathcal{H}^m$.*

*Proof.* $i \to ii$ Since $\{(W_i, \omega_i)\}_{i\in I}$ is a $A$-tight fusion frame, so for every $f \in \mathcal{H}$, we have

$$f = \frac{1}{A} \sum_{i\in I} \omega_i^2 P_{W_i} f, \tag{13}$$

or

$$\tilde{F}^1 = \frac{1}{A} \sum_{i\in I} \omega_i^2 \langle \tilde{F}^1, U_{W_i}\rangle U_{W_i}. \tag{14}$$

Consider $F = (f_1|\cdots|f_m) \in \mathcal{H}^m$ as $\sum_{j=1}^m \tilde{F}_j^j$ where each $\tilde{F}_j^j$ contains the $j$-th component of $F$ and other components are equal to zero. Since (13) holds for every $f_i$, (14) is valid for every $F \in \mathcal{H}^m$. Thus, $\{\omega_i U_{W_i}\}_{i\in I}$ is a $A$-tight frame in the Hilbert $C^*$-module $\mathcal{H}^m$.

$ii \to i$ Assume $\{\omega_i U_{W_i}\}_{i\in I}$ is a $A$-tight frame so for every $F \in \mathcal{H}^m$ we have

$$F = \frac{1}{A} \sum_{i\in I} \omega_i^2 \langle F, U_{W_i}\rangle U_{W_i}.$$

As a result for any $f \in \mathcal{H}$

$$\tilde{F}^1 = \frac{1}{A} \sum_{i\in I} \omega_i^2 \langle \tilde{F}^1, U_{W_i}\rangle U_{W_i} = \frac{1}{A} \sum_{i\in I} \omega_i^2 \widetilde{P_{W_i} f}^1.$$

Therefore, $\{(W_i, \omega_i)\}_{i\in I}$ is a $A$-tight frame.

□

We recall that a sequence $\{F\}_{i\in I}$ is a dual frame of $\{G_i\}_{i\in I}$ in the Hilbert $\mathcal{B}(\mathbb{C}^m)$-module $\mathcal{H}^m$ if [22]

$$F = \sum_{i\in I}\langle F, G_i\rangle F_i,$$

for all $F \in \mathcal{H}^m$.

On the other hand, by (4) we have a sequence of subspaces $\{(V_i, \nu_i)\}_{i\in I}$ in $\mathcal{H}$ is a dual of the fusion frame $\{(W_i, \omega_i)\}_{i\in I}$ if for all $f \in \mathcal{H}$

$$\tilde{F}^1 = \sum_{i\in I} A\langle \tilde{F}^1, \omega_i U_{W_i}\rangle\langle U_{W_i}, U_{V_i}\rangle \nu_i U_{V_i} = \sum_{i\in I}\omega_i\nu_i A\langle \tilde{F}^1, U_{W_i}\rangle\langle U_{W_i}, U_{V_i}\rangle U_{V_i}. \tag{15}$$

Based on (15) it is much easier to work with the dual of the corresponding frame in the Hilbert $\mathcal{B}(\mathbb{C}^m)$-module instead of working with the dual of fusion frames [27].

Now we introduce the notion of Riesz fusion bases using Reisz bases in Hilbert $C^*$-module and show that it coincides with Reisz decomposition of fusion frame. We recall that every frame $\{F_i\}_{i\in I}$ is a Reisz basis if and only if $F_i \neq 0$ for each $i \in I$ and if $\sum_{i\in I} A_i F_i = 0$ for some sequence $\{A_i\}_{i\in I} \in \ell^2(\mathcal{B}(\mathbb{C}^m))$, then $A_i F_i = 0$ for each $i \in I$ [22]. Next theorem shows that every fusion Riesz basis in $\mathcal{H}$ corresponds to a Riesz basis in the Hilbert $C^*$-module $\mathcal{H}^m$ and vice versa.

**Theorem 3.3.** *Assume $\{(W_i, \omega_i)\}_{i\in I}$ is a fusion frame in $\mathcal{H}$. The following statements are equivalent.*

- *(i)* $\{\omega_i U_{W_i}\}_{i\in I}$ *is a Reisz basis in* $\mathcal{H}^m$.
- *(ii)* $\{(W_i, \omega_i)\}_{i\in I}$ *is a Reisz fusion basis in* $\mathcal{H}$.

*Proof.* $i \to ii$ Assume $\sum_{i\in I} f_i = 0$. So, $\sum_{i\in I}\omega_i P_{W_i} f_i = 0$ and $\sum_{i\in I}\langle \tilde{F}_i^1, U_{W_i}\rangle\omega_i U_{W_i} = 0$. Since $\{\omega_i U_{W_i}\}_{i=\in I}$ is a Riesz basis in $\mathcal{H}^m$, we have $\widetilde{\omega_i P_{W_i} f}^1 = \langle \tilde{F}_i^1, U_{W_i}\rangle\omega_i U_{W_i} = 0$ for each $i \in I$ or $P_{W_i} f_i = f_i = 0$ which means that $\{(W_i, \omega_i)\}_{i\in I}$ is a Riesz fusion basis in $\mathcal{H}$.

$ii \to i$ Consider $\{(W_i, \omega_i)\}_{i\in I}$ as a Riesz fusion basis, so every $f \in \mathcal{H}$ has a unique representation based on $\{W_i\}_{i\in I}$, which means that if $\sum_{i\in I}\omega_i P_{W_i} f_i = 0$, then $P_{W_i} f_i = 0$ for all $i \in I$ and all $\{f_i\}_{i\in I} \in \oplus_{i\in I} W_i$. Now consider $\sum_{i\in I} X_i\omega_i U_{W_i} = 0$. We can rewrite it as $\sum_{i\in I}\sum_{j=1}^m X_i^j\omega_i U_{W_i} = 0$ where $X_i^j$ is the matrix with $j$-th row is equal to the $j$-th row of $X_i$ and other rows are equal to zero. By the fact $\sum_{i\in I} X_i^j\omega_i U_{W_i}$ has the $j$-th element equal to nonzero and other elements are equal to zero, $\sum_{i\in I} X_i^j\omega_i U_{W_i}$ for $j = 1, \cdots, m$ are linear independent. So for each $i \in I$ we have $\sum_{i\in I} X_i^j\omega_i U_{W_i} = 0$. By the same process which is done for the equivalence of synthesis operators, there exists $\tilde{F^j}_i^1$ such that $\langle \tilde{F^j}_i^1, U_{W_i}\rangle = X_i^j$. Therefore, we have $\sum_{i\in I}\omega_i P_{W_i} f_i^j = 0$. Since $\{W_i\}_{i\in I}$ is a Riesz fusion basis, we have $P_{W_i} f_i^j = 0$. Therefore, $\langle \tilde{F^j}_i^1, U_{W_i}\rangle\omega_i U_{W_i} = X_i^j\omega_i U_{W_i} = 0$ and then $X_i\omega_i U_{W_i} = 0$.

□

## 4. Modular Fusion Frames

In this section we focus on structured fusion frames in the separable Hilbert space $\mathcal{H}$ using modular frames in the Hilbert $C^*$-module $\mathcal{H}^m$. First, we introduce some notations. We remind that a unitary system $\mathcal{U}$ on $\mathcal{H}^m$ is defined as the set of unitary operators acting on $\mathcal{H}^m$ which contains the identity operator. We call $G = (g_1, \cdots, g_m)$ in $\mathcal{H}^m$ as a complete frame element for a unitary system $\mathcal{U}$ on $\mathcal{H}^m$ if $\mathcal{U}G = \{UG : U \in \mathcal{U}\}$ is a frame. If $\mathcal{U}G$ is an orthonormal basis for $\mathcal{H}^m$, then $G$ is called wandering element for $\mathcal{U}$.

Therefore, we define $W$ as a complete fusion frame element for a unitary system $\mathcal{U}$ on $\mathcal{H}^m$ if $\mathcal{U}\omega_{\mathcal{U}}U_W = \{U\omega_U U_W : U \in \mathcal{U}\}$ is a frame in $\mathcal{H}^m$ and as a result is a fusion frame in $\mathcal{H}$. Moreover, $\mathcal{U}\omega_{\mathcal{U}}U_W$ is an orthonormal basis for $\mathcal{H}^m$, then $U_W$ is called wandering element for $\mathcal{U}$.

The following proposition shows that a unitary system $\mathcal{U}$ generates fusion frames.

**Proposition 4.1.** *Let $\mathcal{U}$ be a unitary system on the Hilbert $\mathcal{B}(\mathbb{C}^m)$-module $\mathcal{H}^m$. Suppose that W is a subspaace of $\mathcal{H}$ with dimension m. If $\mathcal{H}^m$ has an orthonormal basis and G is a wandering element for $\mathcal{U}$, we have the following statements*

- *$U_W$ is a complete Riesz basis element for $\mathcal{U}$ if and only if there exists an invertible and adjointable operator $T \in \{A \in End(\mathcal{H}^m) : AU\omega_U U_W = UAU_W, U \in \mathcal{U}\}$ such that $\omega_I U_W = TG$.*
- *$U_W$ is a complete Parseval frame element for $\mathcal{U}$ if and only if there exists a co-isometry $T \in \{A \in End(\mathcal{H}^m) : AU\omega_U U_W = UAU_W, U \in \mathcal{U}\}$ such that $\omega_I U_W = TG$.*
- *$U_W$ is a complete frame element for $\mathcal{U}$ if and only if there exists an adjointable operator $T \in \{A \in End(\mathcal{H}^m) : AU\omega_U U_W = UA\omega_U U_W, U \in \mathcal{U}\}$ with $C\langle F, F\rangle \leq \langle T^*F, T^*F\rangle$ for some $C > 0$ and any $F \in \mathcal{H}^m$ such that $\omega_I U_W = TG$.*
- *$U_W$ is a complete Bessel element for $\mathcal{U}$ if and only if there is an adjointable operator $T \in \{A \in End(\mathcal{H}^m) : AU\omega_U U_W = UA\omega_U U_W, U \in \mathcal{U}\}$ such that $\omega_I U_W = TG$.*

*Proof.* By the Proposition 5.1 in [22], $\mathcal{U}\omega_{\mathcal{U}}U_W$ is a frame ( Riesz basis, Parseval frame, and Bessel sequence) for the Hilbert $\mathcal{B}(\mathbb{C}^m)$-module $\mathcal{H}^m$. Therefore, by Theorems 3.1, 3.2, and 3.3 the fusion frame which is correspondent to $\mathcal{U}\omega_{\mathcal{U}}U_W$ is a fusion frame (Riesz fusion basis, Parseval fusion frame, and Bessel fusion sequence). □

**Remark 4.1.** *By this approach we can define Gabor fusion frames, Wavelet fusion frames and any fusion frames which is construted by a group of unitary operators.*

**Example 4.1.** *The translation operator on $L^2(\mathbb{R})^m$ is defined as*

$$\mathcal{T}_k : \mathcal{H}^m \to \mathcal{H}^m$$
$$\mathcal{T}_k(F) = \mathcal{T}_k((f_1, \cdots, f_m)) = (T_k(f_1), \cdots, T_k(f_m)),$$

*where $T_k$ is the usual translation operator on $L^2(\mathbb{R})$. It is clear that the matrix $\mathcal{T}_k$ is a unitary operator for $L^2(\mathbb{R})^m$ and $\mathcal{T}_k^* = \mathcal{T}_k^{-1} = \mathcal{T}_{-k}$.*

*Now the modulation operator is defined on $L^2(\mathbb{R})^m$ as*

$$\mathcal{M}_l : \mathcal{H}^m \to \mathcal{H}^m$$
$$\mathcal{M}_l(F) = \mathcal{M}_l((f_1, \cdots, f_m)) = (M_l(f_1), \cdots, M_l(f_m)),$$

*where $M_l$ is the usual modulation operator on $L^2(\mathbb{R})$. Like the translation operator, the modulation operator is also a unitary operator and $\mathcal{M}_l^* = \mathcal{M}_l^{-1} = \mathcal{M}_{-l}$.*

*It is obvious that the combination of two unitary operators is a unitary operator. Therefore, the set $\{\mathcal{M}_l\mathcal{T}_k\}_{k,l\in\mathbb{Z}}$ constitutes a unitary system which includes identity operator when $k = l = 0$. The Gabor transform is then defined on $\mathcal{H}^m$ for the window multifunction $G \in L^2(\mathbb{R})^m$ as*

$$\mathcal{V}_G F(k, l) = \langle F, \mathcal{M}_l\mathcal{T}_k G\rangle.$$

*Now we consider W as a subspace of $L^2(\mathbb{R})$ and $U_W \in L^2(\mathbb{R})^m$ as the matrix associated to W. Moreover, we consider $\tilde{F}^1 = (f, 0, \cdots, 0) \in L^2(\mathbb{R})^m$ for any $f \in L^2(\mathbb{R})$. Then, the Gabor transform on $L^2(\mathbb{R})^m$ can easily transfered to the Gabor fusion transform on $L^2(\mathbb{R})$ which is defined by*

$$\mathcal{V}_{U_W}\tilde{F}^1(k, l) = \langle \tilde{F}^1, \mathcal{M}_l\mathcal{T}_k\omega_{l,k}U_W\rangle.$$

*As $\{\mathcal{M}_l\mathcal{T}_k\omega_{l,k}U_W\}_{k,l\in\mathbb{Z}}$ is a Gabor frame in the Hilbert $\mathcal{B}(\mathbb{C}^m)$- module $L^2(\mathbb{R})^m$, the collection of corresponding subspaces is a Gabor fusion frame.*

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