SOME FIXED POINT RESULTS VIA $\mathcal{L}_{\mathfrak{D}}$-CONTRACTION AND RELATIONAL CONTRACTION IN JS-METRIC SPACES

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Our main objective in this paper is to establish two new fixed point results in generalized metric spaces, often referred as JS-metric spaces which was recently initiated by Jleli and Samet [Fixed Point Theory Appl., 2015:61 (2015), 14 pp]. Firstly, we obtain some fixed point (or, periodic point) results via $\mathcal{L}_{\mathfrak{D}}$-contractions in JS-metric spaces. Thereafter, we obtain a relation-theoretic analog of Banach contraction principle in JS-metric spaces employing an S-transitive binary relation. Our newly proved results unify, generalize, improve and extend several known fixed point results of the existing literature.

Keywords: Fixed point, $\mathcal{L}_{\mathfrak{D}}$-contraction, JS-metric space, binary relation, relational k-contraction, Banach contraction principle.

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1. Introduction

Historically, the earliest fixed point theorem was proved by L. Brouwer [5] but the most discussed and applicable fixed point theorem was essentially obtained by Polish mathematician S. Banach in 1922 which asserts that every contraction map defined on a complete metric space admits a unique fixed point. Due to its simplicity as well as extensive applications within and beyond mathematics, researchers took intense interest to extend and improve this result in various ways. In an attempt to improve this classical result some authors choose to weakened the involved contraction condition whereas few others attempted to enlarge the class of underlying spaces and such efforts are still on.

In 2014, Jleli and Samet [12] introduced the concept of θ-contraction and utilized the same to prove a notable generalization of Banach contraction principle in Branciari distance spaces [4]. Thereafter, Ahmad et al. [11] slightly modified the conditions on the auxiliary functions θ and proved a natural analogous result in metric spaces. On the other side, Khojasteh et al. [18] coined the idea of $\mathcal{Z}$-contraction using a family of control functions often known as simulation functions and unified several types of linear as well as nonlinear contractions of the existing literature. There already exists an extensive literature on simulation functions and related fixed point results. For the work of this kind, one can be referred to ([6, 16, 22]) and references cited therein. Inspired by these ideas, Cho [7] introduced an efficient and effective concept of $\mathcal{L}$-contraction by combining the idea of θ-contraction and $\mathcal{Z}$-contraction, and utilized the same to prove some fixed point results for $\mathcal{L}$-simulation functions in Branciari distance spaces. Recently, Hasanuzzaman et al. [9] slightly modified the definition of $\mathcal{L}$-contraction and proved some fixed point results in relational metric spaces.

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As mentioned earlier, in 2015, Jleli and Samet [13] introduced yet another type of generalized metric space, often referred as JS-metric space which contains various classes of distance spaces, namely: standard metric spaces, b-metric spaces, modular spaces with Fatou-property, dislocated metric space and some others. In recent past, this new concept attracted the attention of several researchers of this domain and by now there exists a considerable literature on JS-metric spaces (e.g., see [1, 3, 16, 17, 25, 26]).

On the other hand, in 1986, Turinici [27] initiated the order-theoretic idea in fixed point results. Later, in 2004, Ran and Reurings [23] established a natural order-theoretic analogue of Banach contraction principle and also utilized their result to establish the existence and uniqueness of solution of a suitable matrix equation. Soon, Nieto and Rodríguez-López [20] have modified Ran-Reurings theorem and established fixed point theorems. Later, Samet and Turinici [24] obtained fixed point results involving symmetric closure of an amorphous binary relation for nonlinear contractions. In 2015, Alam and Imdad [2] employed an arbitrary binary relation to obtain relation-theoretic variant of Banach contraction principle in metric spaces, wherein the contractive condition is required to merely hold on related elements (w.r.t. involved binary relation).

Motivated by forgoing observations and facts, our main intent is to prove a result on fixed point (also, periodic point) for $\mathcal{L}$-simulation functions via $\mathcal{L}_{\mathfrak{D}}$-contractions in JS-metric spaces. Also, we obtain a relation-theoretic analog of Banach contraction principle in JS-metric spaces employing an S-transitive binary relation. We have essentially adopted the notions and approach of Alam and Imdad [2] while proving a relation-theoretic analogue of Banach contraction principle in JS-metric spaces. Some illustrative examples are furnished to show the utility of our results. Additionally, we deduce some corollaries to substantiate the genuineness of our newly proved results.

2. **Preliminaries**

To make our exposition self-contained, we adopt the following notational and terminological conventions that will be used throughout the paper. In what follows, $\mathbb{N}$, $\mathbb{Q}$ and $\mathbb{R}$ respectively denote the sets of natural numbers, rational numbers and real numbers wherein $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Following Jleli and Samet [12], let Θ be the set of all function $\theta : (0,\infty) \to (1,\infty)$ satisfying the conditions:

(θ_1) θ is nondecreasing;

(θ_2) for each sequence $\{\beta_n\} \subset (0,\infty)$, $\lim_{n\to\infty} \theta(\beta_n) = 1 \iff \lim_{n\to\infty} \beta_n = 0$;

(θ_3) there exist $p \in (0,1)$ and $\gamma \in (0,\infty]$ such that $\lim_{\beta\to 0^+} \frac{\theta(\beta)-1}{\beta^p} = \gamma$.

In 2017, Ahmad et al. [11] replaced the condition (θ_3) by the following:

(θ_4) θ is continuous.

Let us denote Θ^* be the family of all such functions that satisfy the conditions: $(\theta_1), (\theta_2)$ and (θ_4). Here, for the sake of convenience we have enlisted some examples of such functions from the existing literature.

Example 2.1. [9] *Define* $\theta : (0,\infty) \to (1,\infty)$ *by* $\theta(\beta) = e^{e^{-\frac{1}{\sqrt{\beta}}}}$, *then* $\theta \in \Theta^*$.

Example 2.2. [12] *Define* $\theta : (0,\infty) \to (1,\infty)$ *by* $\theta(\beta) = e^{\sqrt{\beta}}$, *then* $\theta \in \Theta \cap \Theta^*$.

Example 2.3. [10] *Define* $\theta : (0,\infty) \to (1,\infty)$ *by*

$$\theta(\beta) = \begin{cases} e^{\sqrt{\beta}} & \beta \leq k; \\ e^{2(k+1)} & \beta > k, \end{cases}$$

where $k \geq 1$ (a fixed real number). Then $\theta \in \Theta$ but $\theta \notin \Theta^*$.

Example 2.4. [10] *Define $\theta : (0,\infty) \to (1,\infty)$ by $\theta(\beta) = e^{e^{-\frac{1}{\beta}}}$, then $\theta \in \Theta^*$ but $\theta \notin \Theta$.*

Thus, from the above examples, it is easy to observe that $\Theta \cap \Theta^* \neq \emptyset$, $\Theta \not\subseteq \Theta^*$, $\Theta^* \not\subseteq \Theta$. Using Θ^* (instead of Θ), authors in [11] have proved the following fixed point result:

Theorem 2.1. *On a complete metric space, every θ-contraction mapping (with $\theta \in \Theta^*$) possesses a unique fixed point.*

Recently, motivated by Khojasteh et al. [18], Cho [7] introduced the concept of $\mathcal{L}$-simulation functions as follows:

Definition 2.1. *A mapping $\zeta : [1,\infty) \times [1,\infty) \to \mathbb{R}$ is said to be a $\mathcal{L}$-simulation function if the following conditions are satisfied:*

(ζ_1) $\zeta(1,1) = 1$;
(ζ_2) $\zeta(x,y) < \frac{y}{x}$ for all $x, y > 1$;
(ζ_3) if $\{x_n\}, \{y_n\}$ are sequences in $(1,\infty)$ such that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n > 1$, then $\limsup_{n\to\infty} \zeta(x_n, y_n) < 1$.

Throughout the paper, the family of $\mathcal{L}$-simulation functions will be denoted by "$\mathcal{L}$". For the sake of convenience, we provide some basic examples of $\mathcal{L}$-simulation functions.

Example 2.5. [7] *Let us define the mappings $\zeta_i : [1,\infty) \times [1,\infty) \to \mathbb{R}$ for $i = 1,2,3$ as follows:*

- $\zeta_1(x,y) = \frac{y^k}{x}$ *for all $x, y \in [1,\infty)$, and $k \in (0,1)$;*
- $$\zeta_2(x,y) = \begin{cases} 1 & if\ (x,y) = (1,1); \\ \frac{x}{2y} & if\ x < y; \\ \frac{y^k}{x} & elsewhere, \end{cases}$$
  for all $x, y \in [1,\infty)$ and $k \in (0,1)$;
- $\zeta_3(x,y) = \frac{y}{x\varphi(y)}$ *for all $x, y \in [1,\infty)$, wherein $\varphi : [1,\infty) \to [1,\infty)$ is a non-decreasing lower semi continuous function such that $\varphi^{-1}(\{1\}) = \{1\}$.*

Then ζ_i are $\mathcal{L}$-simulation functions for $i = 1,2,3$.

By using $\mathcal{L}$-simulation functions, Cho [7] introduced $\mathcal{L}$-contraction in generalized metric spaces without using (θ_4). Recently, Hasanuzzaman et al. [9] redefined the definition of $\mathcal{L}$-contraction for $\theta \in \Theta^*$ in the setting of metric space which runs as follows:

Definition 2.2. [9] *Let (M,d) be a metric space and $S : M \to M$. Then S is said to be a $\mathcal{L}$-contraction w.r.t. ζ if there exist $\zeta \in \mathcal{L}$ and $\theta \in \Theta^*$ such that*

$$\zeta\big(\theta(d(Sr,Ss)), \theta(d(r,s))\big) \geq 1 \tag{1}$$

for all $r, s \in M$ with $d(Sr,Ss) > 0$.

Remark 2.1. *If we consider $\zeta(x,y) = \frac{y^k}{x}$ for all $x, y \in [1,\infty)$ with $k \in (0,1)$ then $\mathcal{L}$-contraction reduces to θ-contraction (in the sense of Ahmad et al. [11]) which was extensively utilized to prove several fixed point results in metric spaces.*

Using $\mathcal{L}$-contraction, Cho [7] obtained some fixed point results in Branciari distance spaces wherein author used the continuity of θ but fails to mention the same. Authors in [15] pointed out that there is a gap in the proof of Theorem 4 due to Cho [7], and modified the assumptions accordingly to obtain a valid proof. Here, we record the following metrical version of Theorem 4 due to Cho [7], whose modified version is available (in the form of a corollary) in [15].

Theorem 2.2. [7] *Let (M, d) be a complete metric space and $S : M \to M$ a $\mathcal{L}$-contraction w.r.t. some ζ. Then S has a unique fixed point.*

Now, we recall the definition of JS-metric spaces introduced by Jleli and Samet [13]. Let M be a non-empty set and $\mathcal{D} : M \times M \to [0, \infty]$ a given mappings. For every $r \in M$, we define

$$\mathcal{C}(\mathcal{D}, M, r) = \big\{\{r_n\} \subset M : \lim_{n\to\infty} \mathcal{D}(r_n, r) = 0\big\}$$

Definition 2.3. [13] *Let M be a non-empty set and $\mathcal{D} : M \times M \to [0, \infty]$ a function satisfies the following conditions:*
($\mathcal{D}_1$) $\mathcal{D}(r, s) = 0 \implies r = s$;
($\mathcal{D}_2$) $\mathcal{D}(r, s) = \mathcal{D}(s, r)$ for all $r, s \in M$;
($\mathcal{D}_3$) there exists $\mathcal{C} > 0$ such that if $(r, s) \in M \times M$ and $\{r_n\} \in \mathcal{C}(\mathcal{D}, M, r)$, then

$$\mathcal{D}(r, s) \leq \mathcal{C} \limsup_{n\to\infty} \mathcal{D}(r_n, s). \tag{2}$$

Then $\mathcal{D}$ is called JS-metric while the pair $(M, \mathcal{D})$ is referred as JS-metric space.

In JS-metric spaces, due to the absence of triangle inequality, the uniqueness of the limit of a sequence is seriously effected. To overcome this difficulty, Jleli and Samet [13] employed a suitable condition (namely ($\mathcal{D}_3$)) to ensure the uniqueness of the limit of a sequence. Thus far, we are not familiar with the topology of JS-metric spaces. However to prove our results in JS-metric spaces the much needed notions such as Cauchy sequence, convergence sequence and complete JS-metric spaces etc., are already available in Jleli and Samet [13].

Before proving our results, we need some basic relation-theoretic notions, definitions and related auxiliary results that will be utilized in Section 4.
Any subset $\mathcal{R}$ of $M \times M$ is said to be a binary relation on a non-empty set M. Incidentally, $\emptyset$ and $M \times M$ are known as the empty relation and the universal relation on M, respectively. Henceforth, a non-empty binary relation will be denoted by $\mathcal{R}$. If $(r, s) \in \mathcal{R}$ and $(s, t) \in \mathcal{R}$ imply $(r, t) \in \mathcal{R}$, for all $r, s, t \in M$ then $\mathcal{R}$ is said to be transitive relation on M. Furthermore, if S is a self mapping on M, then $\mathcal{R}$ is said to be a S-transitive if it is transitive on $S(M)$. The inverse of $\mathcal{R}$ is denoted by $\mathcal{R}^{-1}$ and is defined as $\mathcal{R}^{-1} := \{(r, s) \in M \times M : (s, r) \in \mathcal{R}\}$ and $\mathcal{R}^s = \mathcal{R} \cup \mathcal{R}^{-1}$. Two elements r and s of M are said to be $\mathcal{R}$-comparable if either $(r, s) \in \mathcal{R}$ or $(s, r) \in \mathcal{R}$ and is denoted by $[r, s] \in \mathcal{R}$.

Proposition 2.1. [2] *For a binary relation $\mathcal{R}$ defined on a non-empty set M,*

$$(r, s) \in \mathcal{R}^s \text{ if and only if } [r, s] \in \mathcal{R}.$$

Definition 2.4. [2] *Let $\mathcal{R}$ be a binary relation on a non-empty set M. A sequence $\{r_n\} \subset M$ is called $\mathcal{R}$-preserving if*

$$(r_n, r_{n+1}) \in \mathcal{R} \text{ for all } n \in \mathbb{N}_0.$$

Definition 2.5. [2] *Let $\mathcal{R}$ be a binary relation on a non-empty set M and $S : M \to M$. Then $\mathcal{R}$ is said to be S-closed if for any $r, s \in M$,*

$$(r, s) \in \mathcal{R} \text{ implies } (Sr, Ss) \in \mathcal{R}.$$

Definition 2.6. *Let $(M, \mathcal{D})$ be a JS-metric space equipped with a binary relation $\mathcal{R}$. Then, M is $\mathcal{R}$-complete if every $\mathcal{R}$-preserving $\mathcal{D}$-Cauchy sequence is $\mathcal{D}$-converges to some point in M.*

Remark 2.2. *Every complete JS-metric space is $\mathcal{R}$-complete for arbitrary binary relation $\mathcal{R}$. On the other hand, under the universal relation $\mathcal{R}$-completeness coincides with the usual completeness.*

Definition 2.7. *Let $(M, \mathcal{D})$ be a JS-metric space, $S : M \to M$ and $\mathcal{R}$ be a binary relation on M. Then S is said to be $\mathcal{R}$-$\mathcal{D}$-continuous at $r \in M$ if for any $\mathcal{R}$-preserving sequence $\{r_n\} \subset M$ with $r_n \xrightarrow{\mathcal{D}} r$, implies $Sr_n \xrightarrow{\mathcal{D}} Sr$. If S is $\mathcal{R}$-$\mathcal{D}$-continuous at each point of M then we say that S is $\mathcal{R}$-$\mathcal{D}$-continuous.*

Remark 2.3. *Every $\mathcal{D}$-continuous mapping can be treated as $\mathcal{R}$-$\mathcal{D}$-continuous (irrespective of a binary relation $\mathcal{R}$). On the other hand, $\mathcal{R}$-$\mathcal{D}$-continuity coincides with the usual $\mathcal{D}$-continuity under the universal relation.*

Definition 2.8. *Let $(M, \mathcal{D})$ be a JS-metric space and $\mathcal{R}$ a binary relation on M. We say $\mathcal{R}$ is $\mathcal{D}$-self-closed if whenever $\mathcal{R}$-preserving sequence $\{r_n\}$ $\mathcal{D}$-converges to r, then there exists a subsequence $\{r_{n(l)}\}$ of $\{r_n\}$ with $[r_{n(l)}, r] \in \mathcal{R}$, for all $l \in \mathbb{N}_0$.*

Definition 2.9. [19] *For $r, s \in M$, a path (of length n, $n \in \mathbb{N}$) in $\mathcal{R}$ from r to s is a sequence (finite) $\{r_0, r_1, r_2, \cdots, r_n\} \subseteq M$ such that $r_0 = r$, $r_n = s$ with $(r_i, r_{i+1}) \in \mathcal{R}$, for each $i \in \{0, 1, \cdots, n-1\}$.*

Notice that a path of length n involves $n+1$ elements of M (not necessarily distinct). The following notations will be crucial in our subsequent discussions.

- (•) $M(S; \mathcal{R}) := \{r \in M : (r, Sr) \in \mathcal{R}\}$, where $S : M \to M$ be any given mapping;
- (•) $M_\delta(S; \mathcal{R}) := \{r \in M : (r, Sr) \in \mathcal{R}$ such that $\delta(\mathcal{D}, S, r) < \infty\}$; (where $\delta(\mathcal{D}, S, r)$ is defined in Eq. (4))
- (•) $\Upsilon(r, s, \mathcal{R}) :=$ the family of all paths from r to s in $\mathcal{R}$, where $r, s \in M$;
- (•) $Diam_{\mathcal{D}}(Fix(S)) := \sup\{\mathcal{D}(r^*, s^*) : r^*, s^* \in Fix(S)\}$.

3. $\mathcal{L}_{\mathcal{D}}$-contraction in JS-metric spaces

In this section, we prove some results on fixed point (also, periodic point) for $\mathcal{L}_{\mathcal{D}}$-contractions in JS-metric spaces which in turn recover many results in the existing literature besides yielding some new ones.

Definition 3.1. *Let $(M, \mathcal{D})$ be a JS-metric space and $S : M \to M$ such that for all $(r, s) \in M \times M$:*

$$\mathcal{D}(r, s) = 0 \implies \mathcal{D}(Sr, Ss) = 0$$

and

$$\mathcal{D}(Sr, Ss) = \infty \implies \mathcal{D}(r, s) = \infty.$$

Then S is said to be a $\mathcal{L}_{\mathcal{D}}$-contraction w.r.t. ζ if there exist $\zeta \in \mathcal{L}$ and $\theta \in \Theta^*$ such that

$$\mathcal{D}(Sr, Ss) > 0, \mathcal{D}(r, s) < \infty \implies \zeta\big(\theta(\mathcal{D}(Sr, Ss)), \theta(\mathcal{D}(r, s))\big) \geq 1 \tag{3}$$

for any $r, s \in M$.

Proposition 3.1. *Suppose that S is a $\mathcal{L}_{\mathcal{D}}$-contraction w.r.t. some $\zeta \in \mathcal{L}$. Then, any fixed point $r^* \in M$ satisfying*

$$\mathcal{D}(Sr^*, Sr^*) > 0, \mathcal{D}(r^*, r^*) < \infty \implies \mathcal{D}(r^*, r^*) = 0.$$

Proof. Let on contrary $\mathcal{D}(r^*, r^*) > 0$. Now, as S is $\mathcal{L}_{\mathcal{D}}$-contraction and $\mathcal{D}(Sr^*, Sr^*) > 0, \mathcal{D}(r^*, r^*) < \infty$, we have

$$1 \leq \zeta\big(\theta(\mathcal{D}(Sr^*, Sr^*)), \theta(\mathcal{D}(r^*, r^*))\big) < \frac{\theta(\mathcal{D}(r^*, r^*))}{\theta(\mathcal{D}(Sr^*, Sr^*))} = 1$$

a contradiction. Thus, $\mathcal{D}(r^*, r^*) = 0$. □

For every $r_0 \in M$, we denote

$$\delta(\mathcal{D}, S, r_0) = \sup\{\mathcal{D}(S^{n+p}(r_0), S^{n+q}(r_0)) : p, q \in \mathbb{N}\} \tag{4}$$

Now, we prove our first result on $\mathcal{L}_{\mathcal{D}}$-contraction in JS-metric spaces involving fixed point as well as periodic point.

Theorem 3.1. *Let $(M, \mathcal{D})$ be a complete JS-metric space and $S : M \to M$ a $\mathcal{L}_{\mathcal{D}}$-contraction w.r.t. some ζ. Suppose that there exists $r_0 \in M$ with $\delta(\mathcal{D}, S, r_0) < \infty$, then S has either a periodic point or a fixed point.*

Proof. Choose $r_0 \in M$ such that $\delta(\mathcal{D}, S, r_0) < \infty$. Now, if $\delta(\mathcal{D}, S, r_0) = 0$, then r_0 is fixed point of S. Otherwise, let $\delta(\mathcal{D}, S, r_0) = a > 0$. Consider a Picard sequence $\{r_n\}$ based on the point r_0 defined as $r_n = S^n(r_0)$ for all $n \in \mathbb{N}_0$. Now, if it happens that $\mathcal{D}(S^{n+p}(r_0), S^{n+q}(r_0)) = 0$, for some $p \neq q$ and $n \in \mathbb{N}_0$, then r_0 turns to be a periodic point of S. Therefore, without loss of generality we may assume that $\mathcal{D}(S^{n+p}(r_0), S^{n+q}(r_0)) > 0$ for all $p \neq q,\ n \in \mathbb{N}_0$. Now, as S is a $\mathcal{L}_{\mathcal{D}}$-contraction and $0 < \mathcal{D}(S^{n+p}(r_0), S^{n+q}(r_0)) \leq a$ for all $p \neq q,\ n \in \mathbb{N}_0$, we have

$$\begin{aligned} 1 &\leq \zeta\big(\theta(\mathcal{D}(S^{n+p}(r_0), S^{n+q}(r_0))), \theta(\mathcal{D}(S^{n+p-1}(r_0), S^{n+q-1}(r_0)))\\ &< \frac{\theta(\mathcal{D}(S^{n+p-1}(r_0), S^{n+q-1}(r_0))}{\theta(\mathcal{D}(S^{n+p}(r_0), S^{n+q}(r_0)))}\\ \theta(\mathcal{D}(S^{n+p}(r_0), S^{n+q}(r_0))) &< \theta(\mathcal{D}(S^{n+p-1}(r_0), S^{n+q-1}(r_0)), \end{aligned} \tag{5}$$

then due to the property (θ_1), we can have $\mathcal{D}(S^{n+p}(r_0), S^{n+q}(r_0)) < \mathcal{D}(S^{n+p-1}(r_0), S^{n+q-1}(r_0))$ for all $p, q \in \mathbb{N}$. Apparently,

$$\mathcal{D}(S^{n+p}(r_0), S^{n+q}(r_0)) \leq \delta(\mathcal{D}, S, S^n(r_0)) \leq \delta(\mathcal{D}, S, r_0).$$

Therefore, $\{\mathcal{D}(S^{n+p}(r_0), S^{n+q}(r_0))\}$ is a bounded and monotonically decreasing sequence of real numbers and hence there must exists $l \geq 0$ such that $\lim_{n\to\infty} \mathcal{D}(S^{n+p}(r_0), S^{n+q}(r_0)) = l$, for all $p, q \in \mathbb{N}$.
Infact, $l = 0$, to accomplish this, let on contrary $l > 0$. Now by using (θ_4), we obtain

$$\lim_{n\to\infty} \theta(\mathcal{D}(S^{n+3}(r_0), S^{n+2}(r_0))) = \lim_{n\to\infty} \theta(\mathcal{D}(S^{n+2}(r_0), S^{n+1}(r_0))) = \theta(l).$$

Now, if we set $x_n = \theta(\mathcal{D}(S^{n+2}(r_0), S^{n+1}(r_0)))$, $y_n = \theta(\mathcal{D}(S^{n+3}(r_0), S^{n+2}(r_0)))$ then $y_n < x_n$, for all $n \in \mathbb{N}$ (by (5)) and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n > 1$. Then by (ζ_3), we obtain

$$1 \leq \limsup_{n\to\infty} \zeta\big(\theta(\mathcal{D}(S^{n+3}(r_0), S^{n+2}(r_0))), \theta(\mathcal{D}(S^{n+2}(r_0), S^{n+1}(r_0)))\big) < 1,$$

which is a contradiction yielding thereby $l = 0, i.e.,$

$$\lim_{n\to\infty} \mathcal{D}(S^{n+p}(r_0), S^{n+q}(r_0)) = 0, \tag{6}$$

for all $p, q \in \mathbb{N}$. This shows that $\{r_n\}$ is a $\mathcal{D}$-Cauchy sequence in $(M, \mathcal{D})$. Thus, owing to the completeness of the underlying space $(M, \mathcal{D})$, there exists a $r^* \in M$ such that $r_n \xrightarrow{\mathcal{D}} r^*$, *i.e.*,

$$\{r_n\} \in \mathcal{C}(\mathcal{D}, M, r^*) \implies \lim_{n\to\infty} \mathcal{D}(r_n, r^*) = 0. \tag{7}$$

Finally, we show that r^* is a fixed point of S. To prove this, on contrary let us assume that $S(r^*) \neq r^*$. Thus, by $(\mathcal{D}_1)$, we have $\mathcal{D}(S(r^*), r^*) > 0$. Now, from (7) we can choose $n_0 \in \mathbb{N}$ in such a way that for all $n \geq n_0$, we deduce

$$\mathcal{D}(S(r^*), r^*) > \mathcal{D}(S^n(r_0), r^*) = \mathcal{D}(r_n, r^*),$$

which amounts to say $S(r^*) \neq S^n(r_0), i.e., \mathcal{D}(S(r^*), S^n(r_0)) > 0$ for all $n \geq n_0$. Consequently, $\theta(\mathcal{D}(S(r^*), S^n(r_0))) > 1$ for all $n \geq n_0$. Also, from (7), we can contract a subsequence $\{r_{k_n}\}$ of $\{r_n\}$ such a way that $r_{k_n} \neq r^*, i.e., \mathcal{D}(r^*, r_{k_n}) > 0$ for all $n \in \mathbb{N}$. Hence, $\theta(\mathcal{D}(r^*, r_{k_n})) > 1$ for all $n \in \mathbb{N}$. Now, using (3.1) and (ζ_2) (of Definition 2.1), we get

$$1 \leq \zeta\big(\theta(\mathcal{D}(S(r^*), S^n(r_0))), \theta(\mathcal{D}(r^*, r_{k_n}))\big) < \frac{\theta(\mathcal{D}(r^*, r_{k_n}))}{\theta(\mathcal{D}(S(r^*), S^n(r_0)))}$$

so that

$$\theta(\mathcal{D}(S(r^*), S^n(r_0))) < \theta(\mathcal{D}(r^*, r_{k_n})) \tag{8}$$

for all $n \geq n_0$.
Therefore, in view of condition ($\mathcal{D}_3$) (of Definition 2.3), there exists $\mathcal{C} > 0$ such that

$$\begin{aligned} \mathcal{D}(S(r^*), r^*) &\leq \mathcal{C} \limsup_{n\to\infty} \mathcal{D}(S(r^*), S^n(r_0)) \\ &\leq \mathcal{C} \limsup_{n\to\infty} \mathcal{D}(r^*, r_{k_n}) \\ &= 0, \end{aligned}$$

which is a contradiction to our assumption. Therefore, $S(r^*) = r^*, i.e.,$ r^* is a fixed point of S in M. □

Next, we present a corresponding uniqueness fixed point result provided $Fix(S) \neq \emptyset$.

Theorem 3.2. *In Theorem 3.1, if s^* is another fixed point of S such that $\mathcal{D}(Sr^*, Ss^*) > 0$ and $\mathcal{D}(r^*, s^*) < \infty$, then $r^* = s^*$.*

Proof. Let on contrary $r^* \neq s^*$.Then, by property ($\mathcal{D}_1$), we get $\mathcal{D}(r^*, s^*) > 0$. Now, as $\mathcal{D}(Sr^*, Ss^*) > 0, \mathcal{D}(r^*, s^*) < \infty$ and S is $\mathcal{L}_{\mathcal{D}}$-contraction, we have

$$1 \leq \zeta\big(\theta(\mathcal{D}(Sr^*, Ss^*)), \theta(\mathcal{D}(r^*, s^*))\big) < \frac{\theta(\mathcal{D}(r^*, s^*))}{\theta(\mathcal{D}(Sr^*, Ss^*))} = 1$$

a contradiction. Therefore, we obtain $r^* = s^*$. This completes the proof. □

Remark 3.1. *By varying $\mathcal{L}$-simulation functions in Theorem 3.1 and Theorem 3.2, a list of fixed point results can be obtained in JS-metric spaces that too include a large number of topological spaces such as: standard metric spaces, b-metric spaces, modular spaces with Fatou-property, dislocated metric space and similar others. For the sake of brevity, we have avoided the details here.*

As every metric space is also a JS-metric space, therefore Theorem 2.2 is an obvious consequence of Theorem 3.1 and Theorem 3.1. Additionally, if we choose $\zeta(x, y) = \frac{y^k}{x}$ for all $x, y \in [1, \infty)$ with $k \in (0, 1)$ then in view of Remark 2.1, we also obtain Theorem 2.1 as an immediate corollary. Therefore, in lieu of Remark 3.1, it is evident that we not only cover several fixed point results but also proved the results that are valid for different abstract distance spaces.

Next, we furnish an example to demonstrate our results.

Example 3.1. *Let $M = \mathbb{N}_0$ (the set of non-negative integers) be endowed with the JS-metric:*

$$\mathcal{D}(r, s) = \begin{cases} 0 & \text{if } r = s, \\ r + s & \text{if } r \neq s. \end{cases}$$

Clearly, the pair $(M, \mathcal{D})$ is a complete JS-metric Space. Now, define a mapping by

$$S(r) = \begin{cases} 0 & \text{if } r \in \{0, 1\}, \\ r - 1 & \text{otherwise.} \end{cases}$$

Here, if we choose $\theta(\beta) = e^{\sqrt{\beta e^{\beta}}}$ *and* $\zeta_*(x, y) = \frac{y^k}{x}$ *for all* $x, y \in [1, \infty)$*, then we show that* S *is* $\mathcal{L}_{\mathcal{D}}$*-contraction w.r.t.* ζ_* *for* $k = e^{-\frac{1}{2}}$*.*
Firstly, observe that

$$\mathcal{D}(Sr, Ss) > 0 \iff \textbf{cardinality}(\{r, s\} \cap \{0, 1\}) = 0 \text{ or } 1, r \neq s.$$

Therefore, the following two cases arise:
(Without loss of generality, we assume $x > y$ *in both the cases as* $\mathcal{D}$ *is symmetric in both variables.)*
Case 1: *When* **cardinality**$(\{r, s\} \cap \{0, 1\}) = 0$*. Then,* $\mathcal{D}(Sr, Ss) = r + s - 2$ *and* $\mathcal{D}(r, s) = r + s$*. In this case, we obtain*

$$\begin{aligned}
\zeta\big(\theta(\mathcal{D}(Sr, Ss)), \theta(\mathcal{D}(r, s))\big) &= \frac{[\theta(\mathcal{D}(r, s))]^k}{\theta(\mathcal{D}(Sr, Ss))} \\
&= \frac{[e^{\sqrt{\mathcal{D}(r,s)e^{\mathcal{D}(r,s)}}}]^{e^{-\frac{1}{2}}}}{e^{\sqrt{\mathcal{D}(Sr,Ss)e^{\mathcal{D}(Sr,Ss)}}}} \\
&= \frac{[e^{\sqrt{(r+s)e^{(r+s)}}}]^{e^{-\frac{1}{2}}}}{e^{\sqrt{(r+s-2)e^{(r+s-2)}}}} \geq 1.
\end{aligned}$$

Case 2: *When* **cardinality**$(\{r, s\} \cap \{0, 1\}) = 1$*. In this case,* $\mathcal{D}(Sr, Ss) = r - 1$ *and* $\mathcal{D}(r, s) = r + s$*, so we deduce*

$$\begin{aligned}
\zeta\big(\theta(\mathcal{D}(Sr, Ss)), \theta(\mathcal{D}(r, s))\big) &= \frac{[\theta(\mathcal{D}(r, s))]^k}{\theta(\mathcal{D}(Sr, Ss))} \\
&= \frac{[e^{\sqrt{\mathcal{D}(r,s)e^{\mathcal{D}(r,s)}}}]^{e^{-\frac{1}{2}}}}{e^{\sqrt{\mathcal{D}(Sr,Ss)e^{\mathcal{D}(Sr,Ss)}}}} \\
&= \frac{[e^{\sqrt{(r+s)e^{(r+s)}}}]^{e^{-\frac{1}{2}}}}{e^{\sqrt{(r-1)e^{(r-1)}}}} \geq 1.
\end{aligned}$$

Thus, S *is a* $\mathcal{L}_{\mathcal{D}}$*-contraction. Also, for* $r_0 = 2$ *we get* $\delta(\mathcal{D}, S, 2) = 3 < \infty$*. Therefore, applying Theorems 3.1 and Theorem 3.2, we conclude that* S *has a unique fixed point (namely,* $r = 0$*).*

4. Contraction Principle in relational JS-metric spaces

Inspired by Alam and Imdad [2], Jleli and Samet [13]; we define the notion of relational k-contraction in JS-metric spaces utilizing a binary relation as follows:

Definition 4.1. *Let* $\mathcal{R}$ *be a binary relation on JS-metric space* $(M, \mathcal{D})$ *and* $S : M \to M$*. We call* S *to be a relational* k*-contraction, if the following condition holds:*

$$\mathcal{D}(Sr, Ss) \leq k\mathcal{D}(r, s), \quad \forall\, r, s \in M \text{ with } (r, s) \in \mathcal{R} \tag{9}$$

for some $k \in (0, 1)$*.*

By symmetricity ($i.e., \mathcal{D}(r, s) = \mathcal{D}(s, r)$, for all $r, s \in M$) of the JS-metric $\mathcal{D}$, the following proposition is obvious.

Proposition 4.1. *Let* $(M, \mathcal{D})$ *be a JS-metric space endowed with a binary relation* $\mathcal{R}$*, and* $S : M \to M$*. Then, the following are equivalent:*

(i) $\mathcal{D}(Sr, Ss) \leq k\mathcal{D}(r, s), \quad \forall\, r, s \in M$ *with* $(r, s) \in \mathcal{R}$;
(ii) $\mathcal{D}(Sr, Ss) \leq k\mathcal{D}(r, s), \quad \forall\, r, s \in M$ *with* $[r, s] \in \mathcal{R}$.

Now, we prove a relation-theoretic analog of Banach contraction principle in JS-metric spaces.

Theorem 4.1. *Let $(M, \mathcal{D})$ be a JS-metric space, $\mathcal{R}$ a binary relation on M and $S : M \to M$. Suppose that the following conditions hold:*

(i) $M_\delta(S; \mathcal{R})$ is non-empty;
(ii) $\mathcal{R}$ is S-closed and S-transitive;
(iii) S is relational k-contraction;
(iv) $(M, \mathcal{D})$ is $\mathcal{R}$-$\mathcal{D}$-complete; and
(v) either S is $\mathcal{R}$-$\mathcal{D}$-continuous or $\mathcal{R}$ is $\mathcal{D}$-self-closed.

Then $\{S^n(r_0)\}_{n\in\mathbb{N}}$, converges to a fixed point r^* of S in M. Moreover, if $\mathcal{R}$ is reflexive and $\mathcal{D}(r^*, r^*) < \infty$, then $\mathcal{D}(r^*, r^*) = 0$.

Proof. Since $M_\delta(S; \mathcal{R}) \neq \emptyset$, then there exists $r_0 \in M$ such that $(r_0, Sr_0) \in \mathcal{R}$ with $\delta(\mathcal{D}, S, r_0) < \infty$. Then due to the S-closedness of $\mathcal{R}$, we have

$$(S^n(r_0), S^{n+1}(r_0)) \in \mathcal{R}, \text{ for all } n \in \mathbb{N}_0. \tag{10}$$

Now, as $\mathcal{R}$ is S-transitive, we obtain

$$(S^n(r_0), S^m(r_0)) \in \mathcal{R}, \text{for any } n, m \in \mathbb{N}_0 \text{ such that } n \leq m.$$

Using the condition (iii) and symmetricity of the JS-metric $\mathcal{D}$, we deduce

$$\mathcal{D}(S^{n+p}(r_0), S^{n+q}(r_0)) \leq k\mathcal{D}(S^{n-1+p}(r_0), S^{n-1+q}(r_0)), \quad \text{for all } p, q \in \mathbb{N}.$$

Which yields that

$$\delta(\mathcal{D}, S, S^n(r_0)) \leq k\delta(\mathcal{D}, S, S^{n-1}(r_0)).$$

Then, inductively we have

$$\delta(\mathcal{D}, S, S^n(r_0)) \leq k^n\delta(\mathcal{D}, S, r_0), \text{ for all } n \in \mathbb{N}.$$

Therefore, it follows that

$$\mathcal{D}(S^n(r_0), S^{n+m}(r_0)) \leq \delta(\mathcal{D}, S, S^n(r_0)) \leq k^n\delta(\mathcal{D}, S, r_0), \text{ for all } n, m \in \mathbb{N}_0.$$

Then, as $k \in (0, 1)$ and $\delta(\mathcal{D}, S, r_0) < \infty$, we get

$$\lim_{n,m\to\infty} \mathcal{D}(S^n(r_0), S^{n+m}(r_0)) = 0,$$

which confirms that $\{S^n(r_0)\}_{n\in\mathbb{N}_0}$ is an $\mathcal{R}$-preserving $\mathcal{D}$-Cauchy sequence in M. Now, due to the $\mathcal{R}$-$\mathcal{D}$-completeness of $(M, \mathcal{D})$, there exists some $r^* \in M$ such that $\{S^n(r_0)\}_{n\in\mathbb{N}_0}$ is $\mathcal{D}$-convergent to r^*, which implies $\{S^n(r_0)\}_{n\in\mathbb{N}_0} \in \mathcal{C}(\mathcal{D}, M, r^*)$.
Now, if S is $\mathcal{R}$-$\mathcal{D}$-continuous, then we have

$$\{S(S^n(r_0))\}_{n\in\mathbb{N}_0} \in \mathcal{C}(\mathcal{D}, M, S(r^*)),$$

i.e., $\{S^{n+1}(r_0)\}$ is $\mathcal{D}$-convergent to $S(r^*)$(as $n \to \infty$). Then, owing the uniqueness of the limit, we obtain $S(r^*) = r^*$(*i.e.*, r^* is a fixed point of S).

Alternatively, suppose that $\mathcal{R}$ is $\mathcal{D}$-self-closed. Then, there exists a subsequence $\{S^{n_l}(r_0)\}$ of $\{S^n(r_n)\}$ with $[S^{n_l}(r_0), r^*] \in \mathcal{R}$, for all $l \in \mathbb{N}_0$. Since, S is relational k-contraction then from Eq. (9) and Proposition 4.1, we have

$$\mathcal{D}(S^{n_l+1}(r_0), S(r^*)) \leq k\mathcal{D}(S^{n_l}(r_0), r^*), \quad \text{for all } l \in \mathbb{N}_0. \tag{11}$$

Now, letting $l \to \infty$ in the above equation, we obtain

$$\lim_{l\to\infty} \mathcal{D}(S^{n_l+1}(r_0), S(r^*)) = 0,$$

i.e., $\{S^{n_l+1}(r_0)\}_{n\in\mathbb{N}_0} \in \mathcal{C}(\mathcal{D}, M, S(r^*))$. This shows that $\{S^{n_l+1}(r_0)\}_{n\in\mathbb{N}_0}$ is $\mathcal{D}$-convergent to $S(r^*)$. Then, again by the virtue of uniqueness of the limit in JS-metric spaces leads us to conclude that $S(r^*) = r^*$.
Now, suppose that $\mathcal{D}(r^*, r^*) < \infty$, and if $\mathcal{R}$ is reflexive then utilizing Eq. (9), we deduce

$$\mathcal{D}(r^*, r^*) = \mathcal{D}(S(r^*), S(r^*)) \leq k\mathcal{D}(r^*, r^*), \quad (\text{as } (r^*, r^*) \in \mathcal{R})$$

for some $k \in (0,1)$ and hence $\mathcal{D}(r^*, r^*) = 0$, which concludes the proof. □

Observation 4.1. *In Theorem 4.1, it is easy to observe that only a S-transitive binary relation is utilized to deduce the conclusion that the Picard-iterates of $\mathcal{R}$-preserving sequences converge to a fixed point. Here, it can be pointed out that the reflexivity of the underlying binary relation is merely utilized to show that $\mathcal{D}(r^*, r^*) = 0$.*

Next, we prove corresponding uniqueness fixed point result.

Theorem 4.2. *Besides the assumptions of Theorem 4.1, if $\Upsilon(r, s; \mathcal{R}|_{S(M)})$ is non-empty for all $r, s \in S(M)$, then the fixed point of S is unique provided the $Diam_{\mathcal{D}}(Fix(S)) < \infty$.*

Proof. Due to Theorem 4.1, we have $Fix(S)$ is non-empty. Now, if $Fix(S)$ is singleton then the result is obvious. Otherwise, there exists at least two distinct elements $r^*, s^* \in Fix(S)$ such that $\mathcal{D}(r^*, s^*) < \infty$. Now, since $\Upsilon(r, s; \mathcal{R}|_{S(M)})$ is non-empty for all $r, s \in S(M)$ and $Fix(S) \subseteq S(M)$, then there exists a path of some finite length n from r^* to s^* in $\mathcal{R}|_{S(M)}$ say $\{Sr_0, Sr_1, Sr_2, \cdots, Sr_n\}$ such that $r^* = Sr_0, s^* = Sr_n$ with $(Sr_i, Sr_{i+1}) \in \mathcal{R}|_{S(M)}$ for each $i = 0, 1, 2, \cdots, n-1$. As $\mathcal{R}$ is S-transitive, we obtain

$$(r^*, Sr_1) \in \mathcal{R}, (Sr_1, Sr_2) \in \mathcal{R}, \cdots, (Sr_{n-1}, s^*) \in \mathcal{R} \text{ implies } (r^*, s^*) \in \mathcal{R}.$$

Now, as S is relational k-contraction, we have

$$\mathcal{D}(r^*, s^*) = \mathcal{D}(S(r^*), S(s^*)) \leq k\mathcal{D}(r^*, s^*), \quad (\text{for } k \in (0,1))$$

which implies that $\mathcal{D}(r^*, s^*) = 0$. Then, employing $(\mathcal{D}_1)$, we get $r^* = s^*$. Thus, the fixed point of S is unique. □

Remark 4.1. *If the underlying space M is equipped with a transitive binary relation then the conclusions of Theorem 4.1 and Theorem 4.2 continue to hold good as every transitive binary relation remains S-transitive (where, S is self-map on M).*

We have the following example to illustrate the above results.

Example 4.1. *Consider the set $M = [0, 3)$ equipped with the JS-metric:*

$$\mathcal{D}(r, s) = \begin{cases} r + s & \text{if } r = s, \\ \frac{r+s}{2} & \text{otherwise.} \end{cases}$$

Clearly, the pair $(M, \mathcal{D})$ is a JS-metric Space. Now, define a mapping by

$$S(r) = \begin{cases} \frac{r}{3} & \text{if } 0 \leq r \leq 2, \\ 2r - \frac{7}{2} & \text{otherwise.} \end{cases}$$

Now, consider a binary relation M as follows:

$$\mathcal{R} := \{(0,0), (0, \frac{1}{3}), (0, \frac{2}{3}), (0,1), (0,2), (\frac{1}{3}, \frac{1}{3}), (1,1), (1,2)\}$$

Clearly, $\mathcal{R}$ is S-transitive (being transitive) and S is $\mathcal{R}$-$\mathcal{D}$-continuous. Also, $\mathcal{R}$ is S-closed and $(M, \mathcal{D})$ is $\mathcal{R}$-$\mathcal{D}$-complete. As, $\delta(\mathcal{D}, S, 0) = 0 < \infty$ and $(0,0) \in \mathcal{R}$, therefore $M_\delta(S; \mathcal{R}) \neq \emptyset$. By a routine calculation, it can be easily verified that S is a relational k-contraction with $k = \frac{1}{3}$. Therefore, all the assumptions of Theorems 4.1 and 4.2 are satisfied and hence S has a unique fixed point (namely, $r = 0$).

As we know that JS-metric spaces includes various variant of metric spaces such as: standard metric spaces, b-metric spaces, dislocated metric spaces, modular metric spaces with fatou-property etc. (for further details one is referred to [13]). Hence, Theorem 4.1 and Theorem 4.2 can also be utilized to drive corresponding corollaries in all earlier mentioned spaces accordingly.

If we consider $\mathcal{R} = \{(r, s) \in M \times M \mid r \preceq s\}$ (where, '$\preceq$' stands for natural partial ordering on the set M) in Theorem 4.1 and Theorem 4.2 then we deduce the following corollaries yielding sharpened versions of corresponding results due to Jleli and Samet (Theorem 5.5 and Theorem 5.7, [13]) substantiating the genuineness of our newly proved results which in turn generalize two classical and core results due to Ran-Reuring [23] and Nieto-Rodríguez-López (Theorem 4, [21]) in metric spaces. Here, it is interesting to note that another core theorem due to Alam and Imdad (Theorem 3.1, [2]) can not be driven from Theorem 4.1 and Theorem 4.2.

Corollary 4.1. *Let $(M, \mathcal{D}, \preceq)$ be an ordered JS-metric space and S is a self map on M. Suppose that the following conditions hold:*

(i) $M_\delta(S; \preceq)$ *is non-empty;*
(ii) $\preceq$ *is S-closed (i.e., S is $\preceq$-monotone (see Definition 5.2 of* [13]*));*
(iii) S *is $\preceq$-k-contraction (i.e., S is weak k-contraction(see Definition 5.4 of* [13]*));*
(iv) $(M, \mathcal{D})$ *is $\mathcal{D}$-$\preceq$-complete; and*
(v) either S is $\preceq$-continuous or $\preceq$ is $\mathcal{D}$-self-closed.

Then $\{S^n(r_0)\}_{n\in\mathbb{N}}$, converges to a fixed point r^* of S in M. Moreover, if $\mathcal{D}(r^*, r^*) < \infty$, then $\mathcal{D}(r^*, r^*) = 0$.

Corollary 4.2. *In addition to the assumptions of Corollary 4.1, if $\Upsilon(r, s; \preceq|_{S(M)})$ is non-empty for all $r, s \in S(M)$ then S has unique fixed point provided the $Diam_{\mathcal{D}}(Fix(S)) < \infty$.*

Now, if we choose $\mathcal{R}$ to be universal relation ($i.e., \mathcal{R} = M \times M$) in Theorem 4.1, then we directly deduce one of the main results due to Jleli and Samet (Theorem 3.3, [13]).

5. Conclusion

In this paper, we obtain a fixed point (or, periodic point) result for $\mathcal{L}$-simulation function via newly introduced $\mathcal{L}_{\mathcal{D}}$-contraction in complete JS-metric spaces which yields known as well as unknown results in various abstract distance spaces. Also, we prove a relation-theoretic variant of Banach contraction principle in JS-metric spaces employing an S-transitive binary relation without completeness of the underlying JS-metric space which in turn generalize, extend and unify several order-theoretic fixed point results of the existing literature. On the similar lines we can undertake the investigation of existence results on common fixed points for two or more maps under suitable conditions.

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