HEIGHT OF HYPERIDEALS IN NOETHERIAN KRASNER HYPERRINGS

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Inspired by the classical concept of height of a prime ideal in a ring, we proposed in a precedent paper the notion of height of a prime hyperideal in a Krasner hyperring. In this note we first generalize some results concerning the height of a prime hyperideal in a Noetherian Krasner hyperring, with the intent to extend this definition to the case of a general hyperideal in a such hyperring. The main results in this note show that, in a commutative Noetherian Krasner hyperring, the height of a minimal prime hyperideal over a proper hyperideal generated by n elements is less than or equal to n, the converse of this claim being also true. Based on this result, it can be proved that the height of such a prime hyperideal is limited by the height of a corresponding quotient hyperideal.

Keywords: Krasner hyperring, prime/maximal hyperideal, Noetherian hyperring, height of a prime hyperideal

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1. Introduction

Krasner hyperrings [7], introduced as a tool in the approximation of valued fields, are the most studied types of hyperrings with applications not just in hyperstructure theory (for example concerning homomorphisms [13], fundamental relations [11], (m, n)-hyperrings [10], composition hyperoperation [3], etc.), but also in number theory [2] or algebraic geometry [15].

Motivated by one of the central theorem in commutative algebra, namely Krull's principal ideal theorem, and its generalization, known as Krull's height theorem [5, 6], we have started the study of their application in the framework of Krasner hyperrings. We first defined [1] the notion of *height of a prime hyperideal* in a commutative Krasner hyperring, linking it to the notion of *dimension* of such a hyperring. In the same paper [1], we investigated the properties of these two notions in a Noetherian Krasner hyperring, concluding with the extension of Krull's principal ideal theorem to the case of hyperrings. To be more precise, we showed that if R is a commutative Noetherian Krasner hyperring and I is a proper principal hyperideal of R, then the height of a minimal prime hyperideal of R over I is at most one.

In this article, we focus on the generalization of this result, i.e. the similar result of Krull's height theorem, which says that: In a commutative Noetherian Krasner hyperring R, the height of a minimal prime hyperideal over a proper hyperideal of R generated by n elements is at most n (see Theorem 3.3). The other main objective of our note is expressed by Theorem 4.1, that is a converse result of Theorem 3.3, since it shows that, in a commutative Noetherian Krasner hyperring R having a prime hyperideal P of height n, there always exists a proper hyperideal I included in P and generated by n elements, with height equal to n. Some ideas to continue our study in future are mentioned in the concluding section of the paper.

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2. **Preliminaries**

In this section, we gather some results and definitions related to hyperideals of hyperrings, which will be used in the next sections. For more details regarding this topic, the readers are refereed to the book [4] and to the PhD thesis defended in 2013 by N. Ramaruban [14] at the University of Cincinnati. The expository papers of Nakassis [12] or Massouros [9], dedicated to a survey of hyperring and hyperfield theory till 1990's, are another sources for the basic theory of hyperrings.

Throughtout this paper, unless otherwise stated, R denotes a *Krasner hyperring*, first introduced in [7], called here, by short, *hyperring*.

Definition 2.1. *A (Krasner) hyperring is a hyperstructure* $(R, +, \cdot)$ *where*

(1) $(R, +)$ *is a canonical hypergroup;*
(2) $(R, \cdot)$ *is a semigroup endowed with a two-sided absorbing element* 0*;*
(3) *the product distributes from both sides over the sum.*

Definition 2.2. [4] *A subhyperring I of a hyperring R is a left (respectively right) hyperideal of R, if $r \cdot a \in I$ (respectively $a \cdot r \in I$), for all $r \in R$ and $a \in I$. It is called a hyperideal of R if it is both a left and a right hyperideal of R.*

A proper hyperideal M of R is called a maximal hyperideal of R if the only hyperideals of R that contain M are M itself and R.

A hyperideal P of a hyperring R is called a prime hyperideal of R if, for every pair of elements a and b of R, whenever $ab \in P$, either $a \in P$ or $b \in P$.

It is well known that, in a commutative unitary hyperring R, for any proper hyperideal I of R, there exists a maximal hyperideal containing I. Moreover, in such a hyperring, each maximal hyperideal is a prime hyperideal, so there exists at least one prime hyperideal in R.

A nonzero hyperring R having a unique maximal hyperideal is called a *local hyperring*.

Definition 2.3. [14] *A prime hyperideal P of R is called a minimal prime hyperideal over a hyperideal I of R if it is minimal (with respect to inclusion) among all prime hyperideals of R containing I. A prime hyperideal P is called a minimal prime hyperideal if it is a minimal prime hyperideal over the zero hyperideal of R.*

Definition 2.4. [14] *A hyperring R is called Noetherian if it satisfies the ascending chain condition on hyperideals of R: for every ascending chain of hyperideals $I_1 \subseteq I_2 \subseteq I_3 \subseteq \dots$ there exists $N \in \mathbb{N}$ such that $I_n = I_N$, for every natural number $n \geq N$ (this is equivalent with saying that, every ascending chain of hyperideals has a maximal element).*

A hyperring R is called Artinian if it satisfies the descending chain condition on hyperideals of R: for every descending chain of hyperideals $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$ there exists $N \in \mathbb{N}$ such that $I_n = I_N$, for every natural number $n \geq N$ (this is equivalent with saying that, every descending chain of hyperideals has a minimal element).

Remark 2.1. *It is obvious that any finite hyperring is both Noetherian and Artinian, since it contains just finite ascending (descending) chains of hyperideals.*

Corollary 2.1. [14] *Let R be a hyperring in which the zero hyperideal is a product $M_1M_2 \dots M_n$ of (not necessarily distinct) maximal hyperideals. Then R is Noetherian if and only if it is Artinian.*

A general method to obtain a Krasner hyperring was proposed by Krasner [8], starting from a ring S and a group G, where $G \subseteq S$. Here we use it in a particular case.

Example 2.1. *Suppose the set of all congruence classes of integers modulo* 12*, i.e.* $\mathbb{Z}_{12} = \{\overline{0}, \overline{1}, \overline{2}, \dots, \overline{11}\}$ *and its multiplicative subgroup of units* $G = \{\overline{1}, \overline{5}, \overline{7}, \overline{11}\}$ *and construct R as* $\mathbb{Z}_{12}/G$*, i.e.*

$$R = \{\overline{r}G \mid \overline{r} \in \mathbb{Z}_{12}\} = \{\overline{\overline{r}} \mid \overline{r} \in \mathbb{Z}_{12}\}.$$

There is: $\overline{\overline{0}} = \{\overline{0}\}$, $\overline{\overline{1}} = \{\overline{1},\overline{5},\overline{7},\overline{11}\}$, $\overline{\overline{2}} = \{\overline{2},\overline{10}\} = \overline{\overline{10}}$, $\overline{\overline{3}} = \{\overline{3},\overline{9}\} = \overline{\overline{9}}$, $\overline{\overline{4}} = \{\overline{4},\overline{8}\} = \overline{\overline{8}}$, $\overline{\overline{6}} = \{\overline{6}\}$. *Now on* R *define the hyperaddition* $\oplus$ *and multiplication* $\cdot$ *by*

$$\overline{\overline{r}} \oplus \overline{\overline{s}} = \{\overline{\overline{t}} \mid \overline{\overline{t}} \cap (\overline{r} + \overline{s}) \neq \emptyset\}$$
$$\overline{\overline{r}} \cdot \overline{\overline{s}} = \overline{\overline{rs}}.$$

When computing e.g. $\overline{\overline{2}} \oplus \overline{\overline{4}}$ *we have to consider that* $\overline{\overline{2}} + \overline{\overline{4}} = \{\overline{2},\overline{10}\} + \{\overline{4},\overline{8}\} = \{\overline{6},\overline{10},\overline{2}\}$. *Or when computing* $\overline{\overline{1}} \oplus \overline{\overline{3}}$ *we have to consider that* $\overline{\overline{1}} + \overline{\overline{3}} = \{\overline{1},\overline{5},\overline{7},\overline{11}\} + \{\overline{3},\overline{9}\} = \{\overline{4},\overline{8},\overline{10},\overline{2}\}$. *The hyperaddition* $\oplus$ *is shown in Table 1:*

$\oplus$	$\overline{\overline{0}}$	$\overline{\overline{1}}$	$\overline{\overline{2}}$	$\overline{\overline{3}}$	$\overline{\overline{4}}$	$\overline{\overline{6}}$
$\overline{\overline{0}}$	$\overline{\overline{0}}$	$\overline{\overline{1}}$	$\overline{\overline{2}}$	$\overline{\overline{3}}$	$\overline{\overline{4}}$	$\overline{\overline{6}}$
$\overline{\overline{1}}$		$\overline{\overline{0}},\overline{\overline{2}},\overline{\overline{4}},\overline{\overline{6}}$	$\overline{\overline{1}},\overline{\overline{3}}$	$\overline{\overline{2}},\overline{\overline{4}}$	$\overline{\overline{1}},\overline{\overline{3}}$	$\overline{\overline{1}}$
$\overline{\overline{2}}$			$\overline{\overline{0}},\overline{\overline{4}}$	$\overline{\overline{1}}$	$\overline{\overline{2}},\overline{\overline{6}}$	$\overline{\overline{4}}$
$\overline{\overline{3}}$				$\overline{\overline{0}},\overline{\overline{6}}$	$\overline{\overline{1}}$	$\overline{\overline{3}}$
$\overline{\overline{4}}$					$\overline{\overline{0}},\overline{\overline{4}}$	$\overline{\overline{2}}$
$\overline{\overline{6}}$						$\overline{\overline{0}}$

TABLE 1. The hyperaddition $\oplus$

Then $R = \{\overline{0}G, \overline{1}G, \overline{2}G, \ldots, \overline{11}G\} = \{\overline{\overline{0}},\overline{\overline{1}},\overline{\overline{2}},\overline{\overline{3}},\overline{\overline{4}},\overline{\overline{6}}\}$ *is a Krasner hyperring (by Krasner's construction) with zero divisors* $\overline{\overline{3}}$ *and* $\overline{\overline{4}}$ *since* $\overline{3}G \cdot \overline{4}G = \overline{0}G$. *Since* R *is a finite hyperring, it is both Noetherian and Artinian.*

Denote $I = \{\overline{\overline{0}},\overline{\overline{3}},\overline{\overline{6}}\}$, $J = \{\overline{\overline{0}},\overline{\overline{2}},\overline{\overline{4}},\overline{\overline{6}}\}$, $K = \{\overline{\overline{0}},\overline{\overline{6}}\}$, $L = \{\overline{\overline{0}},\overline{\overline{4}}\}$ *and* $Z = \{\overline{\overline{0}}\}$. *Then* I *and* J *are both prime and maximal hyperideals, the zero hyperideal* Z *is not prime while hyperideals* K *and* L *are neither prime nor maximal. Also,* I *and* J *are minimal prime hyperideals (over the zero hyperideal* Z*). The minimal prime hyperideal over* L *is* J.

The *radical* of a hyperideal I of a hyperring R, denoted by $r(I)$, is defined as

$$r(I) = \{x \mid x^n \in I, \quad \text{for some} \ \ n \in \mathbb{N}\}.$$

It can be proved that the radical of I is the intersection of all prime hyperideals of R containing I.

An element x of a hyperring R is called *nilpotent*, if $x^n = 0$, for some $n > 0$. The set of all nilpotent elements of R is called the *nilradical* of R and denoted by $\mathcal{N}(R)$. It is clear that $\mathcal{N}(R) = r(0)$, the radical of the zero hyperideal. Thus, the nilradical $\mathcal{N}(R)$ of a commutative hyperring R is the intersection of all prime hyperideals of R. Moreover, in any Artinian hyperring R, every prime hyperideal is maximal, while the nilradical $\mathcal{N}(R)$ is nilpotent. This means there exists $t \in \mathbb{N}$ such that $\mathcal{N}(R)^t = 0$; in other notation, $r(0)^t = 0$.

Example 2.2. *If we continue with Example 2.1 and use its notation, then e.g.* $r(L) = J$. *Also,* $\mathcal{N}(R) = I \cap J = K = \{\overline{\overline{0}},\overline{\overline{6}}\}$. *The nilpotent elements of* R *are* $\overline{\overline{0}}$ *and* $\overline{\overline{6}}$, *since there is* $\overline{\overline{6}}^2 = \overline{\overline{0}}$.

Proposition 2.1. [14] *Let* I *be a hyperideal of* R *and* $\frac{R}{I} = \{r + I \mid r \in R\}$. *Defining the hyperoperations* $\oplus$ *and* $\otimes$ *on* $\frac{R}{I}$ *as follows:*

$$(a + I) \oplus (b + I) = a + b + I \quad \textit{and} \quad (a + I) \otimes (b + I) = a \cdot b + I,$$

we get that $(\frac{R}{I}, \oplus, \otimes)$ *is a hyperring, too.*

We call the above hyperring $(\frac{R}{I}, \oplus, \otimes)$ the *quotient hyperring*. The following proposition is known in literature as *Lattice Isomorphism Theorem.*

Proposition 2.2. [14] *There is a one-to-one, order-preserving correspondence between the hyperideals I of the hyperring R that contain J and the hyperideals $\hat{I}$ of $\frac{R}{J}$, given by $I = \varphi^{-1}(\hat{I})$, where $\varphi : R \to \frac{R}{J}$ is defined by $\varphi(r) = r + J$.*

Proposition 2.3. [14] *The hyperideal P is a prime hyperideal of R if and only if $\frac{R}{P}$ is a hyperdomain, i.e. a hyperring with no zero divisors.*

Proposition 2.4. *Let I and J be hyperideals of a commutative hyperring R, such that $I \subseteq J$. Then the hyperideal $\frac{J}{I}$ of the quotient hyperring $\frac{R}{I}$ is prime if and only if J is a prime hyperideal of R.*

Proof. The statement follows from the third isomorphism theorem for hyperrings, saying that

$$\frac{\frac{R}{I}}{\frac{J}{I}} \cong \frac{R}{J},$$

and from Proposition 2.3. □

In the following we recall the concepts of extension and contraction of hyperideals and some of their properties.

Definition 2.5. [14] *Let $f : R \to S$ be a hyperring homomorphism, I be a hyperideal of R and J be a hyperideal of S.*

(i) *The hyperideal $\langle f(I)\rangle$ of S generated by the set $f(I)$ is called the extension of I and it is denoted by I^e. Explicitly, we have*

$$\langle f(I)\rangle = \{x \in S \mid x \in \sum_{i=1}^{n} f(a_i)b_i, \text{ where } a_i \in I;\ b_i \in S;\ n \in \mathbb{N}\}.$$

(ii) *The hyperideal $f^{-1}(J) = \{a \in R \mid f(a) \in J\}$ is called the contraction of J and it is denoted by J^c. It is known that, if J is a prime hyperideal in S, then J^c is a prime hyperideal in R.*

Further on we use the above notation in an intuitive way without brackets, i.e. we write I^{ec} instead of $(I^e)^c$.

Proposition 2.5. [14] *Let $f : R \to S$ be a hyperring homomorphism, I and J be hyperideals of R and S, respectively. Then it follows that:*

(i) *$I \subseteq I^{ec}$ and $J \supseteq J^{ce}$.*

(ii) *$J^c = J^{cec}$ and $I^e = I^{ece}$.*

(iii) *If I is a prime hyperideal of R, then it is the contraction of a prime hyperideal of S if and only if $I^{ec} = I$.*

In the following theorem notice that the *Jacobson radical* of a hyperring R is defined to be the intersection of all maximal hyperideals of R.

Theorem 2.1. [14] (Nakayama's Lemma) *Let M be a finitely generated R-hypermodule and I a hyperideal of R contained in the Jacobson radical of R. Then $M = IM$ implies that $M = \{0\}$.*

We continue with some results regarding the hyperrings/hypermodules of fractions. Let R be any hyperring and let S be any multiplicatively closed subset of R, with $1 \in S$. Define a relation $\equiv$ on $R \times S$ by $(a, s) \equiv (b, t)$ if and only if $0 \in (at - bs)u$, for some $u \in S$. Denote the equivalence class of (a, s) with $\frac{a}{s}$ and let $S^{-1}R$ denote the set of all equivalence classes. We endow the set $S^{-1}R$ with a hyperring structure, by defining the addition and the multiplication between fractions as follows:

$$\frac{a}{s} + \frac{b}{t} = \frac{at + bs}{st} \quad \text{and} \quad \frac{a}{s} \cdot \frac{b}{t} = \frac{ab}{st}.$$

We know that $S^{-1}R$ forms a hyperring under these operations [14].

Remark 2.2. *If P is a prime hyperideal of a hyperring R, then $S = R \backslash P$ is a multiplicatively closed subset of R. In this case, we denote $S^{-1}R = R_P$. As proved in* [14], *the elements $\frac{a}{s}$, with $a \in P$, form a hyperideal M in R_P, which is the only maximal hyperideal of R_P. Therefore, R_P is a local hyperring.*

Proposition 2.6. [14] *Let S be a multiplicatively closed subset of a hyperring R.*

i) Every hyperideal in $S^{-1}R$ is an extended hyperideal.
ii) If I is a hyperideal in R, then $I^e = S^{-1}R$ if and only if $I \cap S = \emptyset$.
iii) A hyperideal I is a contracted hyperideal of R if and only if no element of S is a zero divisor in R/I.
iv) The prime hyperideals of $S^{-1}R$ are in one-to-one correspondence with the prime hyperideals of R that do not meet S, with the correspondence given by $P \leftrightarrow S^{-1}P$.

Similarly, one constructs the *hypermodule of fractions*. Let M be an R-hypermodule and S be a multiplicatively closed subset of R. Define a relation $\equiv$ on $M \times S$ by $(m, s) \equiv (m_1, s_1)$ if and only if there exists $t \in S$ such that $0 \in t(ms_1 - m_1 s)$, that is $ms_1 t = m_1 st$. This is clearly an equivalence relation. Let $\frac{m}{s}$ denote the equivalence class of the pair (m, s), and let $S^{-1}M$ denote the set of all such fractions. Then $S^{-1}M$ is an $S^{-1}R$-hypermodule.

If P is a prime hyperideal of R and M is an R-hypermodule, then the R_P-hypermodule $(R \setminus P)^{-1}M$ is simply denoted by M_P.

Definition 2.6. [14] *A hyperideal Q in a hyperring R is called primary if $Q \neq R$ and if whenever $xy \in Q$ either $x \in Q$ or $y^n \in Q$, for some $n \in \mathbb{N}$. If $P = r(Q)$, then Q is called a P-primary hyperideal of R.*

It is obvious that every prime hyperideal is also primary.

Definition 2.7. [14] *A primary decomposition of a hyperideal I in the hyperring R is an expression of I as a finite intersection of primary hyperideals, say $I = \bigcap_{i=1}^{n} Q_i$, where each Q_i is primary. If, moreover*

i) the radicals $r(Q_i)$ are all distinct, and
ii) $Q_i \not\supseteq \bigcap_{j \neq i} Q_j$, $1 \leq i \leq n$,

then the primary decomposition is called minimal. We say that a hyperideal I of R is decomposable, if it has a primary decomposition.

Theorem 2.2. [14] *In a Noetherian hyperring R, every hyperideal has a primary decomposition.*

Denote by $(I : x)$ the quotient hyperideal $\{a \in R \mid ax \subset I\}$.

Theorem 2.3. [14] (First Uniqueness Theorem) *Let I be a decomposable hyperideal of R and let $I = \bigcap_{i=1}^{n} Q_i$ be a minimal primary decomposition of I. Let $P_i = r(Q_i)$, $1 \leq i \leq n$. Then P_i are precisely the prime hyperideals which occur in the set of hyperideals $r(I : x)$, $x \in R$, and hence are independent of the particular decomposition of I.*

Definition 2.8. [14] *The prime hyperideals P_i in Theorem 2.3 are said to belong to I. The minimal elements of the set $\{P_1, P_2, \ldots, P_n\}$ are called the minimal or isolated prime hyperideals of I. The others are called embedded prime hyperideals.*

Lemma 2.1. [1] *Let I be a decomposable hyperideal of a commutative hyperring R and P be a prime hyperideal of R. Then $I \subseteq P$ and P is a prime hyperideal, which is a minimal (with respect to inclusion) prime hyperideal containing I among all prime hyperideals of R if and only if P is a minimal prime hyperideal over I. In particular, the minimal prime hyperideal P, which contains I, belongs to I.*

Example 2.3. *If we continue with Example 2.1 and use its notation, then I and J are primary hyperideals of R (since they are prime), $L = \{\overline{\overline{0}}, \overline{\overline{4}}\}$ is a primary hyperideal of R since $\overline{\overline{2}} \cdot \overline{\overline{2}} = \overline{\overline{4}} \in L$ and $\overline{\overline{2}}^2 \in L$, i.e. $n = 2$. On contrary, neither the zero hyperideal Z nor the hyperideal $K = \{\overline{\overline{0}}, \overline{\overline{6}}\}$ are primary as $\overline{\overline{3}} \cdot \overline{\overline{4}} = \overline{\overline{0}} \in Z$ but $\overline{\overline{3}}^n \notin Z$ and $\overline{\overline{4}}^n \notin Z$ for every*

$n \in \mathbb{N}$ (for the case of Z) and (for the case of K) $\overline{\overline{2}} \cdot \overline{\overline{3}} = \overline{\overline{6}} \in K$ but $\overline{\overline{2}}^n \notin K$ and $\overline{\overline{3}}^n \notin K$ for every $n \in \mathbb{N}$.

Example 2.4. *If we continue with Example 2.1 and use its notation, then a primary decomposition of K is $K = I \cap J$. In this respect notice that $r(I) = I \neq r(J) = J$ and $I \not\supseteq J$, $J \not\supseteq I$, so the decomposition is minimal. Moreover, I and J are hyperideals that belong to K. Thus they are minimal prime hyperideals of K. Further, a primary decomposition of L is $L = J \cap L$. Since $r(L) = J = r(J)$, this decomposition is not minimal.*

Example 2.5. *If we continue with Example 2.1 and use its notation, then $(K : I) = \{\overline{\overline{a}} \in R \mid \overline{\overline{a}} I \subset K\} = \{\overline{\overline{0}}, \overline{\overline{2}}, \overline{\overline{4}}, \overline{\overline{6}}\} = J$.*

3. **Height of prime hyperideals**

In this section, we first recall the definition of the height of a prime hyperideal of a hyperring, introduced in [1], together with the main results obtained in a Noetherian hyperring. In [1], the first two authors have proved that, in such a hyperring, the height of a minimal prime hyperideal P over a principal hyperideal I generated by one element is always less than or equal to 1. Here we extend this result to the case of a minimal prime hyperideal P over a hyperideal I generated by n elements, showing that its height is less than or equal to n.

Definition 3.1. [14] *Let R be a non-trivial commutative hyperring.*

(*i*) *An expression of the type*

$$P_0 \subset P_1 \subset \ldots \subset P_n$$

(note the strict inclusions), where $P_0, \ldots, P_n$ are prime hyperideals of R, is called a chain of prime hyperideals of R; the length of such a chain is the number of the "links" between the terms of the chain, that is, 1 less than the number of prime hyperideals in the sequence. Thus the above displayed chain has length n. Note that, for a prime hyperideal P, we consider

$$P$$

to be a chain, with just one prime hyperideal of R, of length 0. Since R is non-trivial, it contains at least one prime hyperideal, so there certainly exists at least one chain of prime hyperideals of R of length 0.
The supremum of the lengths of all chains of prime hyperideals of R is called the dimension of R, denoted by $dim(R)$.

(*ii*) *A chain of the type*

$$P_0 \subset P_1 \subset \ldots \subset P_n$$

of prime hyperideals of R is called saturated when, for every $i \in \mathbb{N}$, with $1 \leq i \leq n$, there is no prime hyperideal P such that $P_{i-1} \subset P \subset P_i$, that is, if and only if we cannot make a chain of length $n+1$ by inserting an additional prime hyperideal of R between two terms in the chain.

(*iii*) *A chain of the type*

$$P_0 \subset P_1 \subset \ldots \subset P_n$$

of prime hyperideals of R is called maximal, when it is saturated, P_n is a maximal prime hyperideal of R and P_0 is a minimal prime hyperideal of R.

Example 3.1. [1] *The dimension of a hyperfield, or of a non-trivial Artinian hyperring, is 0, while the dimension of the hyperring $\frac{\mathbb{Z}}{G}$, where $G = \{-1, 1\}$ is the multiplicative subgroup of $\mathbb{Z}$, is 1. This is another example of a hyperring obtained with Krasner's construction* [8].

Definition 3.2. [1] *Let P be a prime hyperideal of a non-trivial commutative hyperring R. The height of P, denoted by $ht_R P$, is defined to be the supremum of the lengths of all chains*

$$P_0 \subset P_1 \subset \ldots \subset P_n$$

of prime hyperideals of R, for which $P_n = P$ (if this supremum exists). If the supremum does not exist, we define $ht_R P$ to be ∞.

In the following, for the sake of completeness of this paper, we recall some fundamental results obtained in [1].

Lemma 3.1. [1] *Let P be a prime hyperideal of the commutative hyperring R and I be a hyperideal of R such that $I \subseteq P$. Then the set*

$$\Theta = \{P' \mid P' \textit{is a prime hyperideal and } I \subseteq P' \subseteq P\}$$

has a minimal element with respect to the inclusion.

Lemma 3.2. [1] *Let R be a commutative Noetherian hyperring and let P be a minimal prime hyperideal over a proper hyperideal I of R. Let S be a multiplicatively closed subset of R such that $P \cap S = \emptyset$. Then $S^{-1}P$ is a minimal prime hyperideal over the hyperideal $S^{-1}I$ of $S^{-1}R$.*

Theorem 3.1. [1] *Let R be a commutative Noetherian hyperring in which every prime hyperideal is maximal. Then*

i) R contains finitely many maximal hyperideals.
ii) R is an Artinian hyperring.

Theorem 3.2. [1] *Let R be a commutative Noetherian hyperring and let $a \in R$ be a non-unit element. Let P be a minimal prime hyperideal over the principal hyperideal $\langle a \rangle$ of R. Then $ht_R P \leq 1$.*

Now we can extend Theorem 3.2 to the case when P is a minimal prime hyperideal over a hyperideal I generated not by one element but by n elements.

Theorem 3.3. *Let R be a commutative Noetherian hyperring. Suppose that I is a proper hyperideal of R generated by n elements and P is a minimal prime hyperideal over I. Then $ht_R P \leq n$.*

Proof. We prove the result by induction on n. If $n = 0$, then obviously $I = 0$ and P is a minimal prime hyperideal of R, hence $ht_R P = 0$. If $n = 1$, then $I = \langle a \rangle$ is a principal hyperideal of R generated by one element and the claim follows immediately from Theorem 3.2.

Assume now that $n > 1$ and that the result is true for smaller values of n. Since P is a prime hyperideal of R, it follows that $S = R \setminus P$ is a multiplicatively closed subset of R. By Lemma 3.2, we know that $S^{-1}I = I_P$ is a proper hyperideal of $S^{-1}R = R_P$ and $S^{-1}P = PR_P$ is a minimal prime hyperideal over $S^{-1}I$. Moreover, by Remark 2.2, R_P is a local hyperring having $S^{-1}P$ as a unique maximal hyperideal and $ht_R P = ht_{R_P} PR_P$. Thus we can assume the additional hypothesis that R is a local hyperring with $M = P$ its unique maximal ideal.

Suppose that $I = \langle a_1, a_2, \ldots, a_n \rangle$. For every non maximal prime hyperideal P' of R, there exists a non maximal prime hyperideal P'' of R, such that $P' \subseteq P''$, and the chain $P'' \subset M$ of prime hyperideals is saturated. Indeed, defining

$$\Lambda = \{P_1 \mid P_1 \text{ is a non maximal prime hyperideal of } R, P' \subseteq P_1\},$$

then $\Lambda \neq \emptyset$ and Λ has a maximal element P'', because R is a Noetherian hyperring and it is clear now that $P'' \subset M$ is a saturated chain. Therefore, there exists a non maximal prime hyperideal Q, such that the chain $Q \subset M$ is saturated, and since M is a minimal prime hyperideal over I, it results that $I \nsubseteq Q$. Thus there exists $i \in \{1, 2, \ldots, n\}$, such that $a_i \notin Q$. Let suppose that $a_n \notin Q$. Then, it is enough to show that $ht_R Q \leq n - 1$, implying that $ht_R M \leq n$ and this completes the proof.

Since M is the only prime hyperideal of R containing $Q + \langle a_n \rangle$, it results that the hyperring $\frac{R}{Q+\langle a_n \rangle}$ is a local and Noetherian hyperring with the unique maximal hyperideal $\frac{M}{Q+\langle a_n \rangle}$. Besides this, every prime hyperideal of $\frac{R}{Q+\langle a_n \rangle}$ is clearly maximal. Therefore, by using Theorem 3.1, $\frac{R}{Q+\langle a_n \rangle}$ is also an Artinian local hyperring. According to the properties of the Artinian hyperrings grouped in Section 2, in any Artinian local hyperring, the maximal hyperideal is nilpotent. Hence $\frac{M}{Q+\langle a_n \rangle}$ is nilpotent. Therefore, there exists $t \in \mathbb{N}$ such that

$(\frac{M}{Q+\langle a_n\rangle})^t = \frac{M^t}{Q+\langle a_n\rangle} = 0$ and so $M^t \subseteq Q + \langle a_n\rangle$. On the other hand, since $a_1, a_2, \ldots, a_{n-1} \in M$, we have also $a_1^t \in M^t, \ldots, a_{n-1}^t \in M^t$. Thus, there exist $d_1, d_2, \ldots, d_{n-1}$ in Q and $r_1, r_2, \ldots, r_{n-1}$ in R such that $a_i^t = d_i + a_n r_i$ and $\sum_{i=1}^{n-1}\langle d_i\rangle \subseteq Q$.

If we show that Q is a minimal prime hyperideal over $\sum_{i=1}^{n-1}\langle d_i\rangle$, then by using the inductive hypothesis, we have that $ht_R Q \leq n-1$.

Let $\overline{R} = \frac{R}{\sum_{i=1}^{n-1}\langle d_i\rangle}$ and $nat : R \to \overline{R}$ be the natural hyperring homomorphism such that $nat(r) = r + \sum_{i=1}^{n-1}\langle d_i\rangle$. Since $a_i^t = d_i + a_n r_i$, each prime hyperideal P' of R which contains $d_1, d_2, \ldots, d_{n-1}, a_n$, must contain also $a_1, a_2, \ldots, a_n$. Indeed, if $d_i \in P'$, then $d_i + a_n r_i \in P'$, thus $a_i^t \in P'$, where P' is a prime hyperideal. So $a_i \in P'$. Now, since M is a minimal prime hyperideal containing $a_1, a_2, \ldots, a_n$, it results that M is the only one prime hyperideal of R which contains $d_1, d_2, \ldots, d_{n-1}, a_n$. By using Proposition 2.4, it follows that $\frac{M}{\sum_{i=1}^{n-1}\langle d_i\rangle}$ is a prime hyperideal of $\overline{R}$ and, moreover, it is a minimal prime hyperideal over the principal hyperideal $< \overline{a_n} >= \frac{a_n}{\sum_{i=1}^{n-1}\langle d_i\rangle}$, because otherwise there would exist a prime hyperideal $\frac{Q}{\sum_{i=1}^{n-1}\langle d_i\rangle}$ which would be minimal and would contain $< \overline{a_n} >$. Thus $Q \subseteq M$ and Q contains $d_1, d_2, \ldots, d_{n-1}, a_n$, which is a contradiction. Now, according to Theorem 3.2, it follows that

$$ht_{\overline{R}} \ \frac{M}{\sum_{i=1}^{n-1}\langle d_i\rangle} \leq 1,$$

because otherwise, the chain

$$\frac{Q}{\sum_{i=1}^{n-1}\langle d_i\rangle} \subset \frac{M}{\sum_{i=1}^{n-1}\langle d_i\rangle}$$

of prime hyperideals of $\overline{R}$ could be extended from the left side and this would again be a contradiction with the fact that $ht_{\overline{R}} \ \frac{M}{\sum_{i=1}^{n-1}\langle d_i\rangle} \leq 1$. Using the inductive hypothesis, we conclude that $ht_R Q \leq n-1$, implying that $ht_R M \leq n$, which completes the proof. □

Lemma 3.3. *A hyperring R with the multiplicative identity 1 is an R-hypermodule.*

Proof. Since R is a hyperring, its additive part is a canonical hypergroup. Thus, it is enough to define a map $\varphi : R \times R \longrightarrow R$, given by $\varphi(r, m) = r \cdot m$, to endow R with a multiplicative operation, that confers to the hyperring R also the role of R-hypermodule. □

Remark 3.1. *Suppose that R is a commutative hyperring; then, by Lemma 3.3, R is an R-hypermodule and therefore every hyperideal of R is a subhypermodule of the R-hypermodule R and viceversa.*

Proposition 3.1. *A commutative hyperring R is Noetherian if and only if every hyperideal of R is finitely generated.*

Proof. By using Remark 3.1 and Proposition 9.2 [14], stating that an R-hypermodule is Noetherian if and only if each of its subhypermodules is finitely generated, the proof is straightforward. □

Theorem 3.4. *Any prime hyperideal of a commutative Noetherian hyperring has a finite height.*

Proof. Let R be a Noetherian hyperring and P a prime hyperideal of R. According to Proposition 3.1, P is finitely generated. Suppose that $P = \langle p_1, p_2, \ldots, p_n\rangle$. Besides, P is a minimal prime hyperideal which contains P. Therefore, by Theorem 3.3, it follows that $ht_R P \leq n$. □

Proposition 3.2. *Let R be a commutative Noetherian hyperring and P_1, P_2 two prime hyperideals of R such that $P_1 \subseteq P_2$. Then $ht_R P_1 \leq ht_R P_2$.*

Proof. By using Theorem 3.4, ht_RP_1 and ht_RP_2 are finite. Now suppose that $ht_RP_1 = n$ and $Q_0 \subseteq Q_1 \subseteq \ldots \subseteq Q_n = P_1$ is a chain of prime hyperideals of R. Then obviously $Q_0 \subseteq Q_1 \subseteq \ldots \subseteq Q_n \subseteq P_2$ is also a chain of prime hyperideals of R, meaning that $ht_RP_2 \geq n+1$. Therefore $ht_RP_1 \leq ht_RP_2$. □

Example 3.2. *If we continue with Example 2.1 and use its notation, then we see that* $I = \langle\bar{\bar{3}}\rangle$, $J = \langle\bar{\bar{2}}\rangle$, $K = \langle\bar{\bar{6}}\rangle$, $L = \langle\bar{\bar{4}}\rangle$, $Z = \langle\bar{\bar{0}}\rangle$ *and there are no more hyperideals in* R, *which is obviously commutative. Moreover, e.g.* $ht_RI = ht_RJ = 0$.

4. **Height of hyperideals in Noetherian hyperrings**

The aim of this section is to introduce and characterize the notion of height of a hyperideal (so not necessarily a prime one) in Noetherian hyperrings. Based on it, we prove that the converse of Theorem 3.3 is also true.

Definition 4.1. *Let R be a commutative Noetherian hyperring and I a proper hyperideal of R. Then define the height of I as follows:*

$$ht_RI = \min\{ht_RP \mid P \text{ is a prime hyperideal of } R \text{ and } I \subseteq P\}.$$

For a proper hyperideal I of R, there clearly exists at least one prime hyperideal P of R such that $I \subseteq P$.

Remark 4.1. *According to Lemma 3.1, in a commutative hyperring R, every prime hyperideal which contains I, contains also a minimal prime hyperideal of R. Moreover, by Definition 2.7 and Definition 2.8, it follows that we can redefine the height of a hyperideal I in a Noetherian hyperring R in the following ways:*

$$ht_RI = \min\{ht_RP \mid P \text{ is a minimal prime hyperideal of } R \text{ and } I \subseteq P\}$$

and

$$ht_RI = \min\{ht_RP \mid P \text{ is an isolated prime hyperideal of } I\}.$$

Based on Proposition 3.2, the following lemma is obvious.

Lemma 4.1. *Let I and J be two hyperideals in a commutative Noetherian hyperring R, such that $I \subseteq J$. Then $ht_RI \leq ht_RJ$.*

Proposition 4.1. *Let R be a commutative Noetherian hyperring and I a proper hyperideal of R generated by n elements. Then $ht_RI \leq n$.*

Proof. Since I is a proper hyperideal of R, there exists a minimal prime hyperideal P of R, such that $I \subseteq P$. Thus, by using Lemma 4.1 and Theorem 3.3, $ht_RI \leq ht_RP$, while $ht_RP \leq n$. Therefore $ht_RI \leq n$. □

Lemma 4.2. *In a commutative Noetherian hyperring R, let I be a hyperideal and P a prime hyperideal of R, such that $I \subseteq P$ and $ht_RI = ht_RP$. Then P is a minimal prime hyperideal of I.*

Proof. Suppose that P is not a minimal prime hyperideal of I. Then, by Lemma 3.1, there exists a prime hyperideal Q of R which is minimal and $I \subseteq Q \subset P$. Using Lemma 4.1, we find that $ht_RI \leq ht_RQ < ht_RP$, obtaining a contradiction. So P is a minimal prime hyperideal of I. □

Remark 4.2. *According to Theorem 2.2, every proper hyperideal of a commutative Noetherian hyperring R has a primary decomposition. Suppose that $\{P_1, P_2, \ldots, P_n\}$ is a set of prime hyperideals of R which belong to I (see Definition 2.8). This set is finite and its minimal elements are precisely the minimal prime hyperideals of I (see Lemma 2.1). Therefore, I has finitely many minimal prime hyperideals.*

Proposition 4.2. *Let R be a commutative hyperring and $P_1, P_2, \dots, P_n$, where $n \geq 2$, prime hyperideals of R. Assume that I is a hyperideal of R such that*

$$I \subseteq \bigcup_{i=1}^{n} P_i.$$

Then $I \subseteq P_j$, for some j, with $1 \leq j \leq n$.

Proof. We prove the theorem by induction on n. For $n = 2$, we have $I \subseteq P_1 \cup P_2$, with P_1 and P_2 prime hyperideals of R. Assume that $I \nsubseteq P_1$ and $I \nsubseteq P_2$. Thus there exist $a_1 \in I \setminus P_1$ and $a_2 \in I \setminus P_2$, implying that $a_1 \in P_2$ and $a_2 \in P_1$. Therefore $a_1 + a_2 \subseteq I \subseteq P_1 \cup P_2$. Hence $a_1 + a_2$ is a subset of either P_1 or P_2. If $a_1 + a_2 \subseteq P_1$, then $a_1 \in (a_1 + a_2) - a_2 \subseteq P_1$, which is a contradiction. Similarly, if $a_1 + a_2 \subseteq P_2$, then we get again a contradiction. Thereby we must have $I \subseteq P_1$ or $I \subseteq P_2$.

Suppose that the result has been proved for any $n \leq k$, where $k \geq 2$, and assume now that $n = k + 1$. So we have $I \subseteq \bigcup_{i=1}^{k+1} P_i$. Suppose that, for any $j = 1, 2, \dots, k + 1$, it happens that $I \nsubseteq \bigcup_{i=1, i\neq j}^{k+1} P_i$. Thus, for each $j = 1, 2, \dots, k + 1$, there exists

$$a_j \in I \setminus \bigcup_{i=1, i\neq j}^{k+1} P_i.$$

The inductive hypothesis implies that $a_j \in P_j$, for any $j = 1, 2, \dots, k+1$ and moreover, since P_{k+1} is a prime hyperideal, it follows that $a_1 \cdots a_k \notin P_{k+1}$. Hence $a_1 \cdots a_k \in \bigcap_{i=1}^{k} P_i \setminus P_{k+1}$ and $a_{k+1} \in P_{k+1} \setminus \bigcup_{i=1}^{k} P_i$. Now consider an element $b \in a_1 \cdots a_k + a_{k+1}$. Then $b \notin P_{k+1}$, because otherwise we would get that

$$a_1 \dots a_k \in b - a_{k+1} \subseteq P_{k+1},$$

which is a contradiction. Besides, $b \notin P_j$, for some $j = 1, 2, \dots, k$, because otherwise it would result that

$$a_{k+1} \in b - a_1 \dots a_k \subseteq P_j,$$

that is again a contradiction. But clearly $b \in I$, obtaining a contradiction to the assumption that $I \subseteq \cup_{i=1}^{k+1} P_i$. Thereby there exists at least one $j \in \mathbb{N}$, with $1 \leq j \leq k + 1$, for which $I \subseteq \cup_{i=1, i\neq j}^{k+1} P_i$. Applying the inductive hypothesis for $n = k$, it results that $I \subseteq P_i$, for some $i \in \mathbb{N}$, with $1 \leq i \leq k + 1$, and the proof is now complete. □

Now we have all elements to prove that the converse of Theorem 3.3 is also true.

Theorem 4.1. *Let R be a commutative Noetherian hyperring and P a prime hyperideal of R, with $ht_R P = n$. Then there exists a proper hyperideal I of R having the following properties:*

(*i*) *$I \subseteq P$,*
(*ii*) *I is generated by n elements,*
(*iii*) *$ht_R I = n$.*

Proof. We prove the theorem by induction on n. If $n = 0$, then $ht_R P = 0$ and in this case it is sufficient to take $I = 0$ the zero hyperideal. Thus, $I = 0 \subseteq P$, I is generated by 0 elements and $ht_R I = 0$. Moreover, P is a minimal prime hyperideal over 0.

Now suppose that $n > 0$ and the theorem has been proved for smaller value of n. Assume that $ht_R P = n$, thus there exists a chain of prime hyperideals $P_1 \subset P_2 \subset \dots \subset P_n \subset P_{n+1} = P$. It results that $ht_R P_n = n - 1$. Indeed, since $P_n \subset P_{n+1} = P$, by using Lemma 4.1, it follows that $ht_R P_n \leq ht_R P_{n+1} = n$, while having the chain of prime hyperideals $P_1 \subset P_2 \subset \dots \subset P_n$ it means that $ht_R P_n \geq n - 1$. Therefore, $ht_R P_n = n - 1$. Based on the inductive hypothesis for the prime hyperideal P_n, there exists a proper hyperideal J of R generated by $n - 1$ elements, such that $J \subseteq P_n$ and $ht_R J = n - 1$.

Consider $J = \langle a_1, a_2, \dots, a_n \rangle$. Since $ht_R J = ht_R P_n$ and $J \subseteq P_n$, by Lemma 4.2 it follows that P_n is a minimal prime hyperideal of J. According to Remark 4.2, J has finitely many minimal prime hyperideals, for example the set $\{P_n, Q_1, Q_2, \dots, Q_k\}$. We claim that there exists $i \in \mathbb{N}$, $1 \leq i \leq k$, such that $ht_R Q_i = n-1$. Indeed, because Q_i is a minimal prime hyperideal of J and J is generated by $n-1$ elements, based on Theorem 3.3, it results that $ht_R Q_i \leq n-1$. On the other hand, $J \subseteq Q_i$ leads to the fact that $ht_R J = n-1 \leq ht_R Q_i$. Thus $ht_R Q_i = n-1$.

Besides, we have $P = P_{n+1} \nsubseteq P_n \cup Q_1 \cup Q_2 \cup \ldots \cup Q_k$, because otherwise, by Proposition 4.2, P would have to be a subset of P_n or Q_i, where $1 \leq i \leq k$, and $ht_R P = m$, while $ht_R Q_i = ht_R P_n = n-1$, which would be a contradiction. So

$$P = P_{n+1} \nsubseteq P_n \cup Q_1 \cup Q_2 \cup \ldots \cup Q_k.$$

Let $a_{n+1} \in P_{n+1} \backslash (P_n \cup Q_1 \cup Q_2 \cup \ldots \cup Q_k)$ and define $I = J \cup < a_{n+1} > = < a_1, a_2, \ldots, a_{n+1} >$. Then clearly, $I \subseteq P$ and I is generated by n elements. It remains to show that $ht_R I = n$.

We know that $J \subseteq I \subseteq P$. Thus $n-1 = ht_R J \leq ht_R I \leq ht_R P = n$. Therefore $ht_R I = n$ or $ht_R I = n-1$. Suppose that $ht_R I = n-1$ and P' is a minimal prime hyperideal of R, such that $I \subseteq P'$. Then, by Theorem 3.3, it results that $ht_R P' = n-1$. Hence $ht_R P' = ht_R J$ and, by Lemma 4.2, P' must be a minimal prime hyperideal of J. Thus P' must be one of the hyperideals $Q_1, Q_2, \ldots, Q_k$ or P_n, which is a contradiction, because $a_{n+1} \in P'$ and a_{n+1} does not belong to any Q_i or P_n. So the only possibility is to have $ht_R I = n$. Now the proof is complete. □

Based on Theorem 3.3 and its converse, Theorem 4.1, we establish a relation between the height of a minimal prime hyperideal P over a proper hyperideal I of R and the height of the quotient hyperideal $\frac{P}{I}$ in the quotient hyperring $\frac{R}{I}$.

Theorem 4.2. *Let R be a commutative Noetherian hyperring, P a prime hyperideal of R and I a proper hyperideal of R generated by n elements, such that $I \subseteq P$. Then:*

$$ht_{\frac{R}{I}} \frac{P}{I} \leq ht_R P \leq ht_{\frac{R}{I}} \frac{P}{I} + n.$$

Proof. Suppose that $ht_{\frac{R}{I}} \frac{P}{I} = m$. Then there exists a chain of prime hyperideals of $\frac{R}{I}$, denoted $\frac{P_0}{I} \subset \frac{P_1}{I} \subset \ldots \subset \frac{P_m}{I} = \frac{P}{I}$, such that P_0 is a minimal prime hyperideal over I and $P_m = P$, leading to the following chain of prime hyperideals of R

$$P_0 \subset P_1 \subset \ldots \subset P_m = P.$$

Hence $m \leq ht_R P$ and thus $ht_{\frac{R}{I}} \frac{P}{I} \leq ht_R P$.

Consider $I = \langle a_1, a_2, \ldots, a_n \rangle$. Since $ht_{\frac{R}{I}} \frac{P}{I} = m$, by using Theorem 4.1, there exists a proper hyperideal $\frac{J}{I}$ of $\frac{R}{I}$ such that $\frac{J}{I} \subseteq \frac{P}{I}$, $\frac{J}{I}$ is generated by m elements, for example $\alpha_1 + I, \alpha_2 + I, \ldots, \alpha_m + I$ and $ht_{\frac{R}{I}} \frac{J}{I} = m$. Since $I = < a_1, a_2, \ldots, a_n > \subseteq P$ and $J = \langle \alpha_1, \alpha_2, \ldots, \alpha_m \rangle \subseteq P$, we have the inclusion $\langle a_1, a_2, \ldots, a_n, \alpha_1, \alpha_2, \ldots, \alpha_m \rangle \subseteq P$. It follows that P is a minimal prime hyperideal over the principal hyperideal $\langle a_1, a_2, \ldots, a_n, \alpha_1, \alpha_2, \ldots, \alpha_m \rangle$, because otherwise, there would exist a hyperideal Q of R such that

$$\langle a_1, a_2, \ldots, a_n, \alpha_1, \alpha_2, \ldots, \alpha_m \rangle \subseteq Q \subsetneqq P.$$

So we have

$$\langle a_1 + I, a_2 + I, \ldots, a_n + I, \alpha_1 + I, \alpha_2 + I, \ldots, \alpha_m + I \rangle \subseteq \frac{Q}{I} \subsetneqq \frac{P}{I}.$$

Since $a_i \in I$, it results that $a_i + I = I$. So we must have

$$\langle \alpha_1 + I, \alpha_2 + I, \ldots, \alpha_m + I \rangle = \frac{J}{I} \subseteq \frac{Q}{I} \subsetneqq \frac{P}{I}.$$

Since $\frac{P}{I}$ is a minimal prime hyperideal over $\frac{J}{I}$, we clearly have $P = Q$, which is a contradiction. Therefore, P is a minimal prime hyperideal over $\langle a_1, a_2, \ldots, a_n, \alpha_1, \alpha_2, \ldots, \alpha_m \rangle$.

Finally, according to Theorem 3.3, we conclude that

$$ht_R P \leq m + n = ht_{\frac{R}{I}} \frac{P}{I} + n.$$

□

5. **Conclusions and future work**

In this note we have investigated the possibility to extend the Krull's principal ideal theorem and the Krull's height theorem to the hyperstructure theory, in particular to Noetherian Krasner hyperrings. After defining and studying the main properties of the central notion of this theory, namely the height of a prime hyperideal in a commutative Noetherian Krasner hyperring, we have proved that the above mentioned theorems are valid also in these kind of hyperstructures.

In future we intend to extend these results also in other classes of hyperrings, investigating their similarities/differences with the classical results obtained for commutative rings.

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