PROJECTION SUBGRADIENT ALGORITHMS FOR SOLVING PSEUDOMONOTONE VARIATIONAL INEQUALITIES AND PSEUDOMONOTONE EQUILIBRIUM PROBLEMS

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In this paper, we investigate pseudomonotone variational inequalities and pseudomonotone equilibrium problems in Hilbert spaces. We present a projection subgradient algorithm with self-adaptive technique for finding a common solution of pseudomonotone variational inequalities and pseudomonotone equilibrium problems. The proof of the strong convergence theorem is additionally established under some mild conditions on the operators and parameters.

Keywords: Pseudomonotone variational inequality, pseudomonotone equilibrium problem, pseudomonotone operators, projection, subgradient.

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1. **Introduction**

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H. Let $\phi : C \to C$ be a nonlinear operator. Recall that the variational inequality problem (VI) is to seek a point $\tilde{u} \in C$ such that

$$\langle\phi(\tilde{u}), u - \tilde{u}\rangle \geq 0,\ \forall u \in H. \tag{1}$$

The solution set of VI (1) is denoted by $VI(C, \phi)$.
VI (1) is said to be pseudomonotone variational inequality if ϕ is pseudomonotone on C, i.e.,

$$\langle\phi(\tilde{u}), u - \tilde{u}\rangle \geq 0 \Rightarrow \langle\phi(u), u - \tilde{u}\rangle \geq 0,\ \forall u, \tilde{u} \in C.$$

Theories and numerical iterative methods have been proposed, adopted and extended broadly as algorithmic solutions to the notion of variational inequalities. This concept, that mainly relates to many important operators, plays a critical role in sciences and engineering, such as fixed point problems ([7, 11, 18, 23, 26, 33]), optimization problems ([4, 28]), obstacle problems, as a unified framework for the study of a large number of significant real-word problems. For more information, please refer to [2, 27, 29, 31, 34, 35, 40]. Among them, the basic methods for solving (1) are projection method ([1]), proximal point method ([5, 14, 17]), extragradient method ([3, 13, 21]), Tikhonov regularization method, hybrid method ([6]) and subgradient method ([32]). Especially, in order to relax the constraints added on operator ϕ, self-adaptive technique is used without knowing the Lipschitz constant of the operator ϕ in advance, see [1].

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Let $\varphi : C \times C \to \mathbb{R}$ be a bifunction. Recall that the equilibrium problem (EP) is to seek a point $u^\dagger \in C$ such that

$$\varphi(u^\dagger, u) \geq 0, \ \forall u \in C. \tag{2}$$

The solution set of VI (1) is denoted by $EP(C, \varphi)$.
EP (2) is said to be pseudomonotone equilibrium problem if φ is pseudomonotone on C, i.e.,

$$\varphi(u^\dagger, u) \geq 0 \Rightarrow \varphi(u, u^\dagger) \leq 0, \ \forall u^\dagger, u \in C.$$

Theories and methods of equilibrium problems offer a powerful tool for studying a large number of nonlinear problems, such as optimization problems ([8, 44]), variational inequality problems ([30, 39, 41]), complementarity problems, minimax inequality problems ([12]), and fixed point problems ([19, 20, 25, 36]). The most approaches to the equilibrium problem are relied on the resolvent of equilibrium bifunction ([16, 22]) in which a strongly monotone regularization problem is solved at each iterative step. Iterative algorithms for solving (2) have been presented and developed in the literature, see, for instance ([9, 24, 37, 38, 42]).

Motivated and inspired by the work in this field, the main purpose of this paper is to investigate pseudomonotone variational inequality (1) and pseudomonotone equilibrium problem (2) in Hilbert spaces. We suggest a projection subgradient algorithm with self-adaptive technique for finding a common solution of pseudomonotone variational inequality (1) and pseudomonotone equilibrium problem (2). Under some mild conditions on the operators and parameters, strong convergence result of the proposed algorithm is shown.

2. **Notations and Lemmas**

Let C be a nonempty convex and closed subset of a real Hilbert space H. $\rightharpoonup$ means the weak convergence and $\to$ means the strong convergence. Use $\omega_w(p^k)$ to denote the set of all weak cluster points of the sequence $\{u^k\}$, i.e., $\omega_w(u^k) = \{u^\dagger : \exists \{u^{k_i}\} \subset \{u^k\}$ such that $u^{k_i} \rightharpoonup u^\dagger$ as $i \to \infty\}$.

For any $x^\dagger \in H$, there exists a unique point $proj_C[x^\dagger] \in C$ such that

$$\|x^\dagger - proj_C[x^\dagger]\| \leq \|x - x^\dagger\|, \ \forall x \in C.$$

It is known that $proj_C$ satisfies the following inequality

$$\|proj_C[v^*] - proj_C[v^\dagger]\|^2 \leq \langle proj_C[v^*] - proj_C[v^\dagger], q^* - v^\dagger \rangle, \ \forall v^*, v^\dagger \in H. \tag{3}$$

Furthermore,

$$\langle v^* - proj_C[v^*], v^\dagger - proj_C[v^*] \rangle \leq 0, \ \forall v^* \in H, v^\dagger \in C. \tag{4}$$

Let $\phi : C \to C$ be an operator. ϕ is said to be L-Lipschitz if

$$\|\phi(u) - \phi(u^\dagger)\| \leq L\|u - u^\dagger\|, \ \forall u, u^\dagger \in C,$$

where $L > 0$ is a constant.

Let $\varphi : C \times C \to \mathbb{R}$ be a bi-function. φ is said to be jointly sequently weakly continuous, if there exist two sequence $\{u^k\} \subset C$ and $\{v^k\} \subset C$ satisfying $u^k \rightharpoonup u^\dagger$ and $v^k \rightharpoonup v^\dagger$, then $\varphi(u^k, v^k) \to \varphi(u^\dagger, v^\dagger)$.

Let $f : C \to (-\infty, +\infty]$ be a proper, lower semicontinuous and convex function. Then, the subdifferential ∂f of f is defined by

$$\partial f(u) := \{v^\dagger \in H : f(u) + \langle v^\dagger, u^\dagger - u \rangle \leq f(u^\dagger), \forall u^\dagger \in C\} \tag{5}$$

for each $u \in C$.

It is well known that $u^\dagger$ is a solution of the optimization problem $\min_{u \in C} f(u)$ if and only if $0 \in \partial f(u^\dagger) + N_C(u^\dagger)$, where $N_C(u^\dagger)$ stands for the normal cone of C at $u^\dagger$ defined by $N_C(u^\dagger) = \{\omega \in H : \langle \omega, u - u^\dagger \rangle \leq 0, \forall u \in C\}$.

Let $\varphi : C\times C \to \mathbb{R}$ be a bi-function. In what follows, assume that φ satisfies conditions (BF1)-(BF4) below
(BF1): $\varphi(u^\dagger, u^\dagger) = 0$ for all $u^\dagger \in C$;
(BF2): φ is pseudomonotone on $EP(C, \varphi)$;
(BF3): φ is jointly sequently weakly continuous on $C \times C$;
(BF4): $\varphi(u^\dagger, \cdot)$ is convex and subdifferentiable for all $u^\dagger \in C$.

Lemma 2.1 ([16]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\varphi\colon C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions (BF1)-(BF4). Let $\{\varpi_k\}$ be a sequence satisfying $\varpi_k \in [\underline{\varpi}, \overline{\varpi}] \subset (0, 1], \forall k \geq 0$. For given $v^k \in C$, set*

$$y^k = \arg\min_{u^\dagger\in C} \left\{\varphi(v^k, u^\dagger) + \frac{1}{2\varpi_k}\|v^k - u^\dagger\|^2\right\}.$$

If v^k is bounded, then y^k is bounded.

Lemma 2.2 ([43]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\varphi\colon C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions (BF1)-(BF4). Let $\{u^k\}$ and $\{v^k\}$ be two sequences in C. Assume that $u^k \rightharpoonup \bar{u} \in C$ and $v^k \rightharpoonup \bar{v} \in C$. Then, for any $\epsilon > 0$, there exist $\eta > 0$ and a positive integer N_0 such that*

$$\partial_2\varphi(v^k, u^k) \subset \partial_2\varphi(\bar{v}, \bar{u}) + \frac{\epsilon}{\eta}B, \forall k \geq N_0,$$

where $B := \{x \in H | \|x\| \leq 1\}$.

Lemma 2.3 ([10]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\phi : C \to C$ be a continuous and pseudomonotone operator. Then $x^\dagger \in VI(C, \phi)$ iff $x^\dagger$ solves the following variational inequality*

$$\langle\phi(u^\dagger), u^\dagger - x^\dagger\rangle \geq 0, \ \forall u^\dagger \in H.$$

Lemma 2.4 ([15]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{x^k\}$ be a sequence in H and $p^\dagger$ be a fixed point in H. Assume that $\omega_w(x^k) \subset C$ and $\|x^k - p^\dagger\| \leq \|proj_C[p^\dagger] - p^\dagger\|, \forall k \geq 0$. Then $x^k \to proj_C[p^\dagger]$.*

3. **Main results**

In this section, we present our iterative algorithm for finding a common solution of pseudomonotone variational inequality (1) and pseudomonotone equilibrium problem (2). Consequently, we show the convergence of the proposed algorithm.

Let C be a nonempty closed convex subset of a real Hilbert space H. Let the operator ϕ be pseudomonotone on H, weakly sequentially continuous and L-Lipschitz continuous on C. Let $\varphi : C \times C \to \mathbb{R}$ be a function satisfying the conditions (BF1)-(BF4). Assume that $\Omega := VI(C, \psi) \cap EP(C, \varphi) \neq \emptyset$. Let $\sigma \in (0, 1)$, $\tau \in (0, 1)$, $\delta \in (0, 2)$, $\theta \in (0, 1)$ and $\eta \in (0, 1)$ be five constants. Let $\{\varpi_k\} \subset (0, 1)$, $\{\varsigma_k\} \subset (0, 1)$ and $\{\beta_k\} \subset (0, 2)$ be three real number sequences satisfying the following conditions:
(C1): $\varpi_k \in [\varpi_0, 1]$ with $0 < \varpi_0 \leq 1, \forall k \geq 0$;
(C2): $0 < \liminf_{n\to\infty} \beta_k \leq \limsup_{n\to\infty} \beta_k < 2$;
(C3): $0 < \liminf_{k\to\infty} \varsigma_k \leq \limsup_{k\to\infty} \varsigma_k < 1$.

Next, we present our iterative algorithm.

Algorithm 3.1. *Let x^0 be a fixed point in H. Set $C_1 = C$ and compute $x^1 = proj_{C_1}[x^0]$. Set $k = 0$.*
Step 1. Calculate

$$w^k = proj_C[x^k - \sigma\tau^n \phi(x^k)], \tag{6}$$

where $n = \min\{0, 1, 2, 3, \cdots\}$ and satisfies

$$\sigma\tau^n \|\phi(x^k) - \phi(w^k)\| \le \theta \|x^k - w^k\|, \tag{7}$$

Set $\tau^n = \tau_k$. If $w^k = x^k$, then set $v^k = x^k$ and go to Step 3. Otherwise, compute

$$v^k = proj_C\Big[x^k - \delta(1-\theta)\|x^k - w^k\|^2 \frac{t^k}{\|t^k\|^2}\Big], \tag{8}$$

where $t^k = x^k - w^k + \sigma\tau_k \varphi(w^k)$.
Step 2. Compute

$$y^k = \arg\min_{y^\dagger \in C} \Big\{\varphi(v^k, y^\dagger) + \frac{1}{2\varpi_k}\|v^k - y^\dagger\|^2\Big\}. \tag{9}$$

Find the smallest positive integer m such that

$$\varphi(z^{k,m}, y^k) + \frac{\delta}{2\varpi_k}\|v^k - y^k\|^2 \le 0, \tag{10}$$

where

$$z^{k,m} = (1 - \eta^m)v^k + \eta^m y^k. \tag{11}$$

Set $\eta^m = \eta_k$ and $z^{k,m} = z^k$ and calculate

$$u^k = \begin{cases} proj_C\Big[v^k + \dfrac{\varsigma_k \beta_k \varphi(z^k, y^k)}{(1-\varsigma_k)\|\psi^k\|^2}\psi^k\Big], & 0 \notin \partial_2\varphi(z^k, z^k), \\ v^k, & 0 \in \partial_2\varphi(z^k, z^k), \end{cases} \tag{12}$$

where $\psi^k \in \partial_2\varphi(z^k, z^k)$.
Step 3. Calculate

$$C_{k+1} = \{u^\dagger \in C_k : \|u^k - u^\dagger\| \le \|x^k - u^\dagger\|\}, \tag{13}$$

and

$$x^{k+1} = proj_{C_{k+1}}[x^0]. \tag{14}$$

Step 4. Set $k := k + 1$ and return to Step 1.

Proposition 3.1. *We have the following statements:*
(i) *There exists n such that (7) holds and $0 < \frac{\tau\theta}{\sigma L} < \tau_k \le 1, k \ge 0$.*
(ii) *If $w^k = x^k$, then $w^k \in VI(C, \phi)$. If $w^k \ne x^k$, then $t^k \ne 0$.*
(iii) *For each $u^\dagger \in C$, we have (see [16, 43])*

$$\varphi(v^k, u^\dagger) \ge \varphi(v^k, y^k) + \frac{1}{\varpi_k}\langle v^k - y^k, u^\dagger - y^k\rangle. \tag{15}$$

(iv) *There exists m such that (10) holds. In this case, $\varphi(z^k, y^k) < 0$ when $z^k \ne y^k$.*

Proof. (i) By the L-Lipschitz continuity of ϕ, $\sigma\tau^n\|\phi(x^k) - \phi(w^k)\| \le \sigma\tau^n L\|x^k - w^k\|$. Then, we can choose n such that $\sigma\tau^n L \le \theta$, i.e., $\tau^n \le \frac{\theta}{\sigma L}$. If $n = 0$, then $\tau_k = 1$. If $n > 0$, then $\frac{\tau\theta}{\sigma L} < \tau_k < 1$.
(ii) If $proj_C[x^k - \sigma\tau_k\phi(x^k)] = x^k$, then $w^k \in VI(C, \phi)$ due to the property (4) of $proj_C$.

Take $z^\dagger \in \Omega$. Since $w^k \in C$ and $x^k \in C$, $\langle \phi(z^\dagger), w^k - z^\dagger \rangle \geq 0$ and $\langle \phi(z^\dagger), x^k - z^\dagger \rangle \geq 0$. With the help of the pseudomonotonicity of ϕ, we deduce

$$\langle \phi(w^k), w^k - z^\dagger \rangle \geq 0, \tag{16}$$

and

$$\langle \phi(x^k), x^k - z^\dagger \rangle \geq 0. \tag{17}$$

According to (4) and (6), we obtain

$$\langle x^k - \sigma\tau^n \phi(x^k) - w^k, w^k - z^\dagger \rangle \geq 0. \tag{18}$$

Owing to (16)-(18), we obtain

$$\begin{aligned}
\langle t^k, x^k - z^\dagger \rangle &= \langle x^k - w^k - \sigma\tau_k \phi(x^k), x^k - z^\dagger \rangle + \sigma\tau_k \langle \phi(x^k), x^k - z^\dagger \rangle \\
&\quad + \sigma\tau_k \langle \phi(w^k), x^k - w^k \rangle + \sigma\tau_k \langle \phi(w^k), w^k - z^\dagger \rangle \\
&\geq \langle x^k - w^k - \sigma\tau_k \phi(x^k), x^k - z^\dagger \rangle + \sigma\tau_k \langle \phi(w^k), x^k - w^k \rangle \\
&= \langle x^k - w^k - \sigma\tau_k (\phi(x^k) - \phi(w^k)), x^k - w^k \rangle \\
&\quad + \langle x^k - w^k - \sigma\tau_k \phi(x^k), w^k - z^\dagger \rangle \\
&\geq \langle x^k - w^k - \sigma\tau_k (\phi(x^k) - \phi(w^k)), x^k - w^k \rangle \\
&\geq \|x^k - w^k\|^2 - \sigma\tau_k \|\phi(x^k) - \phi(w^k)\| \|x^k - w^k\| \\
&\geq (1-\theta)\|x^k - w^k\|^2 \\
&> 0.
\end{aligned} \tag{19}$$

So, $t^k \neq 0$.
(iv) If $v^k = y^k$, then $z^k = y^k$ and $\varphi(z^k, y^k) = 0$. Thus, (10) holds and set $m = 1$. Suppose that (10) is not satisfied when $v^k \neq y^k$. In this case, for any $m \geq 1$, we deduce

$$\varphi(z^{k,m}, y^k) + \frac{\delta}{2\varpi_k}\|v^k - y^k\|^2 > 0. \tag{20}$$

Letting $m \to \infty$ in (11), we obtain that $z^{k,m} \to v^k$. Hence, $\varphi(z^{k,m}, v^k) \to 0$ and $\varphi(z^{k,m}, y^k) \to \varphi(v^k, y^k)$. This together with (20) implies that

$$\varphi(v^k, y^k) \geq -\frac{\delta}{2\varpi_k}\|v^k - y^k\|^2. \tag{21}$$

Setting $z^\dagger = v^k$ in (15), we deduce

$$\varphi(v^k, y^k) \leq -\frac{1}{\varpi_k}\|v^k - y^k\|^2. \tag{22}$$

It follows from (21) and (22) that $0 \leq (\frac{1}{\varpi_k} - \frac{\delta}{2\varpi_k})\|v^k - y^k\|^2 \leq 0$ which implies that $v^k = y^k$. It is impossible. Therefore, the search rule (10) is well-defined. It is obvious that $\varphi(z^k, y^k) \leq -\frac{\delta}{2\varpi_k}\|v^k - y^k\|^2 < 0$ when $v^k \neq y^k$. □

Theorem 3.1. *The sequence $\{x^k\}$ generated by* (14) *converges strongly to $proj_\Omega[x^0]$.*

Proof. Step 1. For any k, C_k is closed convex and the sequence $\{x^k\}$ is valid.

We use induction to prove $\Omega \subset C_k, \forall k \geq 1$. (i) $\Omega \subset C_1$ is obvious. (ii) Suppose that $\Omega \subset C_k$ for some $k \in \mathbb{N}$. Pick up $z^\dagger \in \Omega \subset C_k$. By (8), we obtain

$$\begin{aligned}\|v^k - z^\dagger\|^2 &\leq \|x^k - z^\dagger - \delta(1-\theta)\|x^k - w^k\|^2 \frac{t^k}{\|t^k\|^2}\|^2 \\ &= \|x^k - z^\dagger\|^2 - 2\delta(1-\theta)\frac{\|x^k - w^k\|^2}{\|t^k\|^2}\langle t^k, x^k - z^\dagger\rangle + \delta^2(1-\theta)^2\frac{\|x^k - w^k\|^4}{\|t^k\|^2}\end{aligned}$$

which together with (19) implies that

$$\begin{aligned}\|v^k - z^\dagger\|^2 &\leq \|x^k - z^\dagger\|^2 - (2-\delta)\delta(1-\theta)^2\frac{\|x^k - w^k\|^4}{\|t^k\|^2} \\ &\leq \|x^k - z^\dagger\|^2.\end{aligned} \tag{23}$$

Since $z^\dagger \in EP(C, \varphi)$, $\varphi(z^\dagger, z^k) \geq 0$. It follows from the pseudomonotonicity of φ that $\varphi(z^k, z^\dagger) \leq 0$. Owing to $\psi^k \in \partial_2\varphi(z^k, z^k)$ and $\varphi(z^k, z^k) = 0$, by the subdifferential inequality, we have $\varphi(z^k, z^\dagger) \geq \langle \psi^k, z^\dagger - z^k\rangle$. It yields that $\langle \psi^k, z^\dagger - z^k\rangle \geq -\varphi(z^k, z^\dagger) \geq 0$. Then,

$$\langle \psi^k, v^k - z^\dagger\rangle = \langle \psi^k, v^k - z^k\rangle + \langle \psi^k, z^k - z^\dagger\rangle \geq \langle \psi^k, v^k - z^k\rangle.$$

Observe that $v^k - z^k = \frac{\eta_k}{1-\eta_k}(z^k - y^k)$ and $\varphi(z^k, y^k) \geq \langle \psi^k, y^k - z^k\rangle$. Therefore,

$$\langle \psi^k, v^k - z^\dagger\rangle \geq \frac{\eta_k}{1-\eta_k}\langle \psi^k, z^k - y^k\rangle \geq \frac{-\eta_k}{1-\eta_k}\varphi(z^k, y^k). \tag{24}$$

By virtue of (12) and (24), we obtain

$$\begin{aligned}\|u^k - z^\dagger\|^2 &\leq \left\|v^k + \frac{\varsigma_k\beta_k\varphi(z^k, y^k)}{(1-\varsigma_k)\|\psi^k\|^2}\psi^k - z^\dagger\right\|^2 \\ &= \|v^k - z^\dagger\|^2 + \frac{2\varsigma_k\beta_k\varphi(z^k, y^k)}{(1-\varsigma_k)\|\psi^k\|^2}\langle \psi^k, v^k - z^\dagger\rangle + \frac{\varsigma_k^2\beta_k^2\varphi^2(z^k, y^k)}{(1-\varsigma_k)^2\|\psi^k\|^2} \\ &\leq \|v^k - z^\dagger\|^2 - \frac{2\varsigma_k^2\beta_k\varphi^2(z^k, y^k)}{(1-\varsigma_k)^2\|\psi^k\|^2} + \frac{\varsigma_k^2\beta_k^2\varphi^2(z^k, y^k)}{(1-\varsigma_k)^2\|\psi^k\|^2} \\ &= \|v^k - z^\dagger\|^2 - \beta_k(2-\beta_k)\frac{\varsigma_k^2\varphi^2(z^k, y^k)}{(1-\varsigma_k)^2\|\psi^k\|^2}.\end{aligned} \tag{25}$$

Based on (23) and (25), we have

$$\begin{aligned}\|u^k - z^\dagger\|^2 &\leq \|x^k - z^\dagger\|^2 - (2-\delta)\delta(1-\theta)^2\frac{\|x^k - w^k\|^4}{\|t^k\|^2} \\ &\quad - \beta_k(2-\beta_k)\frac{\varsigma_k^2\varphi^2(z^k, y^k)}{(1-\varsigma_k)^2\|\psi^k\|^2} \\ &\leq \|x^k - z^\dagger\|^2.\end{aligned} \tag{26}$$

This implies that $z^\dagger \in C_{k+1}$.

In the case of $0 \in \partial_2\varphi(z^k, z^k)$, we have $u^k = v^k$ and $\|u^k - p\| \leq \|x^k - p\|$. Thus, $\Omega \subset C_k$ for all $k \geq 1$.

It is obviously that $C_k(\forall k \geq 1)$ is closed convex. Therefore, the sequence $\{x^k\}$ is valid.

Step 2. $\|x^{k+1} - x^k\| \to 0$, $\|w^k - x^k\| \to 0$, $\|v^k - x^k\| \to 0$ and $\|u^k - v^k\| \to 0$.

Thanks to (14), we have

$$\|x^k - x^0\| \leq \|x^0 - u\|, \ \forall u \in C_k, \tag{27}$$

which implies that $\{x^k\}$ is bounded. Then $\{v^k\}$ and $\{u^k\}$ are bounded. By Lemma 2.1, $\{y^k\}$ is bounded. So, $\{z^k\}$ is bounded. Applying Lemma 2.2, $\{\psi^k\}$ is bounded.

Noting that $x^k = proj_{C_k}[x^0]$ and $x^{k+1} \in C_k$, from (4), we have

$$\langle x^k - x^0, x^k - x^{k+1}\rangle \leq 0.$$

It follows that

$$\begin{aligned} \|x^k - x^{k+1}\|^2 &= 2\langle x^k - x^0, x^k - x^{k+1}\rangle + \|x^0 - x^{k+1}\|^2 - \|x^0 - x^k\|^2 \\ &\leq \|x^0 - x^{k+1}\|^2 - \|x^0 - x^k\|^2. \end{aligned} \tag{28}$$

In (27), setting $u = x^{k+1}$, we conclude that $\|x^k - x^0\| \leq \|x^{k+1} - x^0\|$. Therefore, $\lim_{k\to\infty} \|x^k - x^0\|$ exists. This together with (28) implies that

$$\lim_{k\to\infty} \|x^{k+1} - x^k\| = 0. \tag{29}$$

According to (13) and noting that $x^{k+1} \in C_k$, we get $\|u^k - x^{k+1}\| \leq \|x^k - x^{k+1}\|$. So,

$$\|u^k - x^k\| \leq \|u^k - x^{k+1}\| + \|x^{k+1} - x^k\| \leq 2\|x^{k+1} - x^k\|.$$

It follows from (29) that

$$\lim_{k\to\infty} \|u^k - x^k\| = 0. \tag{30}$$

In the light of (26), we obtain

$$\begin{aligned} &(2-\delta)\delta(1-\theta)^2 \frac{\|x^k - w^k\|^4}{\|t^k\|^2} + \beta_k(2-\beta_k)\frac{\varsigma_k^2 \varphi^2(z^k, y^k)}{(1-\varsigma_k)^2\|\psi^k\|^2} \\ &\leq \|x^k - z^\dagger\|^2 - \|u^k - z^\dagger\|^2 \\ &\leq \|u^k - x^k\|(\|x^k - z^\dagger\| + \|u^k - z^\dagger\|), \end{aligned}$$

which implies that

$$\lim_{k\to\infty} \frac{\|x^k - w^k\|^2}{\|t^k\|} = 0, \tag{31}$$

and

$$\lim_{k\to\infty} \frac{\varsigma_k \varphi(z^k, y^k)}{(1-\varsigma_k)\|\psi^k\|} = 0. \tag{32}$$

According to the boundedness of $\{t^k\}$, we deduce from (31) that

$$\lim_{k\to\infty} \|x^k - w^k\| = 0. \tag{33}$$

From (8), we have

$$\begin{aligned} \|v^k - x^k\| &= \|proj_C\Big[x^k - \delta(1-\theta)\|x^k - w^k\|^2 \frac{t^k}{\|t^k\|^2}\Big] - x^k\| \\ &\leq \delta(1-\theta)\frac{\|x^k - w^k\|^2}{\|t^k\|}. \end{aligned}$$

It results in that

$$\lim_{k\to\infty} \|v^k - x^k\| = 0. \tag{34}$$

By (12), we get

$$\begin{aligned}\|u^k - v^k\| &= \|proj_C\left[v^k + \frac{\varsigma_k \beta_k \varphi(z^k, y^k)}{(1-\varsigma_k)\|\psi^k\|^2}\psi^k\right] - v^k\| \\ &\leq \frac{\varsigma_k \beta_k \varphi(z^k, y^k)}{(1-\varsigma_k)\|\psi^k\|}.\end{aligned}$$

This together with (32) implies that

$$\lim_{k\to\infty} \|u^k - v^k\| = 0. \tag{35}$$

Step 3. $\omega_w(x^k) \subset VI(C,\phi) \cap EP(C,\varphi)$.

By the boundedness of the sequence $\{x^k\}$, there exists a subsequence $\{x^{k_i}\} \subset \{x^k\}$ satisfying $x^{k_i} \rightharpoonup p^\dagger \in \omega_w(x^k)$. First, we show $p^\dagger \in VI(C,\phi)$. From (18), we have

$$\langle x^{k_i} - \sigma\tau_{k_i}\phi(x^{k_i}) - w^{k_i}, w^{k_i} - x^\dagger\rangle \geq 0, \forall x^\dagger \in C.$$

It leads to that

$$\langle \phi(x^{k_i}), x^\dagger - x^{k_i}\rangle \geq \langle \phi(x^{k_i}), w^{k_i} - x^{k_i}\rangle + \frac{1}{\sigma\tau_{k_i}}\langle w^{k_i} - x^\dagger, w^{k_i} - x^{k_i}\rangle,\ \forall x^\dagger \in C. \tag{36}$$

As a result of (33) and (36), we obtain

$$\liminf_{i\to\infty}\langle \phi(x^{k_i}), x^\dagger - x^{k_i}\rangle \geq 0,\ \forall x^\dagger \in C. \tag{37}$$

Let $\{\zeta_j\}$ a positive real numbers sequence satisfying $\lim_{j\to\infty}\zeta_j = 0$. For each ζ_j, there exists the smallest positive integer k_i such that

$$\langle \phi(x^{k_{i_j}}), x^\dagger - x^{k_{i_j}}\rangle + \zeta_j \geq 0,\ \forall j \geq k_i. \tag{38}$$

It is obvious that $\phi(x^{k_{i_j}}) \neq 0$, otherwise $x^{k_{i_j}} \in VI(C,\phi)$ and hence $p^\dagger \in VI(C,\phi)$. Putting $f(x^{k_{i_j}}) = \frac{\phi(x^{k_{i_j}})}{\|\phi(x^{k_{i_j}})\|^2}$, we obtain $\langle f(x^{k_{i_j}}), \phi(x^{k_{i_j}})\rangle = 1$. Based on (38), we have

$$\langle \phi(x^{k_{i_j}}), x^\dagger + \zeta_j f(x^{k_{i_j}}) - x^{k_{i_j}}\rangle \geq 0.$$

Because of the pseudomonotonicity of ϕ, we obtain

$$\langle \phi(x^\dagger + \zeta_j f(x^{k_{i_j}})), x^\dagger + \zeta_j f(x^{k_{i_j}}) - x^{k_{i_j}}\rangle \geq 0.$$

It follows that

$$\begin{aligned}\langle \phi(x^\dagger), x^\dagger - x^{k_{i_j}}\rangle &\geq \langle \phi(x^\dagger) - \phi(x^\dagger + \zeta_j f(x^{k_{i_j}})), x^\dagger + \zeta_j f(x^{k_{i_j}}) - x^{k_{i_j}}\rangle \\ &\quad - \langle \phi(x^\dagger), \zeta_j f(x^{k_{i_j}})\rangle.\end{aligned} \tag{39}$$

Owing to $\phi(x^{k_{i_j}}) \rightharpoonup \phi(p^\dagger)$, we have

$$\liminf_{j\to\infty}\|\phi(x^{k_{i_j}})\| \geq \|\phi(p^\dagger)\| > 0.$$

Thus,

$$\lim_{j\to\infty}\|\zeta_j f(x^{k_{i_j}})\| = \lim_{j\to\infty}\frac{\zeta_j}{\|\phi(x^{k_{i_j}})\|} = 0.$$

This together with (39) implies that

$$\langle \phi(x^\dagger), x^\dagger - p^\dagger\rangle \geq 0. \tag{40}$$

Consequently, by Lemma 2.3, we conclude that $p^\dagger \in VI(C,\phi)$.

Next, we show $p^\dagger \in EP(C,\varphi)$. By (10), we have

$$\varphi(z^k, y^k) + \frac{\delta}{2\varpi_k}\|v^k - y^k\|^2 \leq 0. \tag{41}$$

Case 1. $\limsup_{k\to\infty} \eta_k > 0$. Without loss of generality, we assume that $\eta_{k_i} \geq \eta_0$ for some $\eta_0 > 0$ when $i \geq N_0$. By (32) and (41), we deduce

$$\lim_{i\to\infty} \|v^{k_i} - y^{k_i}\| = 0. \tag{42}$$

Noting that $v^{k_i} \rightharpoonup p^\dagger$, by (42), we have $y^{k_i} \rightharpoonup p^\dagger \in C$. By condition (C1), $\varpi_k \geq \varpi_0, k \geq 0$. Since y^{k_i} solves (9), for any $y^\dagger \in C$, we get

$$\begin{aligned}\varphi(v^{k_i}, y^{k_i}) + \frac{1}{2\varpi_{k_i}}\|v^{k_i} - y^{k_i}\|^2 &\leq \varphi(v^{k_i}, y^\dagger) + \frac{1}{2\varpi_{k_i}}\|v^{k_i} - y^\dagger\|^2 \\ &\leq \varphi(v^{k_i}, y^\dagger) + \frac{1}{2\varpi_0}\|v^{k_i} - y^\dagger\|^2.\end{aligned} \tag{43}$$

Letting $i \to \infty$ in (43), we obtain

$$0 \leq \varphi(p^\dagger, y^\dagger) + \frac{1}{2\varpi_0}\|p^\dagger - y^\dagger\|^2, \ \forall y^\dagger \in C. \tag{44}$$

Therefore, $p^\dagger \in EP(C, \varphi)$.

Case 2. $\lim_{k\to\infty} \eta_k = 0$, i.e., $\lim_{k\to\infty} \eta^m = 0$. Without loss of generality, we assume that $y^{k_i} \rightharpoonup q^\dagger \in C$ and $\varpi_{k_i} \to \rho^\dagger > 0$. By the definition of y^{k_i}, we have

$$\varphi(v^{k_i}, y^{k_i}) + \frac{1}{2\varpi_{k_i}}\|v^{k_i} - y^{k_i}\|^2 \leq \varphi(v^{k_i}, y^\dagger) + \frac{1}{2\varpi_{k_i}}\|v^{k_i} - y^\dagger\|^2, \ \forall y^\dagger \in C. \tag{45}$$

Letting $i \to \infty$ in (45), we derive

$$\varphi(p^\dagger, q^\dagger) + \frac{1}{2\rho^\dagger}\|p^\dagger - q^\dagger\|^2 \leq \varphi(p^\dagger, y^\dagger) + \frac{1}{2\rho^\dagger}\|p^\dagger - y^\dagger\|^2, \ \forall y^\dagger \in C. \tag{46}$$

Setting $y^\dagger = p^\dagger$ in (46) to deduce

$$\varphi(p^\dagger, q^\dagger) + \frac{1}{2\rho^\dagger}\|p^\dagger - q^\dagger\|^2 \leq 0. \tag{47}$$

By the search rule (10), we have

$$\varphi(z^{k_i, m-1}, y^{k_i}) + \frac{\delta}{2\varpi_{k_i}}\|v^{k_i} - y^{k_i}\|^2 > 0, \tag{48}$$

where

$$z^{k_i, m-1} = (1 - \eta^{m-1})v^{k_i} + \eta^{m-1} y^{k_i} \rightharpoonup p^\dagger \ (i \to \infty).$$

This together with (48) implies that

$$\varphi(p^\dagger, q^\dagger) + \frac{\delta}{2\rho^\dagger}\|p^\dagger - q^\dagger\|^2 \geq 0. \tag{49}$$

Taking into account (47) and (49), we deduce

$$0 \leq \frac{1-\delta}{2\rho^\dagger}\|p^\dagger - q^\dagger\|^2 \leq 0,$$

which implies that $p^\dagger = q^\dagger$. Therefore,

$$\varphi(p^\dagger, y^\dagger) + \frac{1}{2\rho^\dagger}\|p^\dagger - y^\dagger\|^2 \geq 0, \ \forall y^\dagger \in C,$$

which implies that $p^\dagger \in EP(C, \varphi)$. So, $\omega_w(x^k) \subset VI(C, \phi) \cap EP(C, \varphi)$.

Step 4. $x^k \to proj_\Omega[x^0]$.

(i) By (27), we have

$$\|x^k - x^0\| \leq \|x^0 - proj_\Omega[x^0]\|.$$

(ii) By Step 3, we have $\omega_w(x^k) \subset \Omega$.
All assumptions of Lemma 2.4 are satisfied. It follows that $x^k \to proj_\Omega[x^0]$. □

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