AN EXTRAGRADIENT ALGORITHM FOR FIXED POINT PROBLEMS AND PSEUDOMONOTONE EQUILIBRIUM PROBLEMS

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Iterative methods for solving fixed point problems and pseudomonotone equilibrium problems have been considered. An extragradient algorithm has been presnted for finding a common element of the fixed points of pseudocontractive operators and the solutions of pseudomonotone equilibrium problems in Hilbert spaces. Weak convergence result of the suggested algorithm is proved.

Keywords: Fixed point, pseudomonotone equilibrium problem, pseudocontractive operators, extragradient method.

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1. **Introduction**

Let H be a real Hilbert space with inner product and norm denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let C be a nonempty closed and convex subset of H. Let $f : C\times C \to \mathbb{R}$ be a bifunction. Recall that the equilibrium problem (EP) is to seek a point $\tilde{u}\in C$ such that

$$f(\tilde{u}, u) \geq 0,\ \forall u \in C, \tag{1}$$

The solution set of equilibrium problem (1) is denoted by $EP(f, C)$.

Equilibrium problems have been studied extensively in the literature (see e.g. [2, 7, 17]). Many problems, such as variational inequalities ([3, 9, 12, 14, 20, 21, 23, 26, 28, 29, 31, 33, 34, 36]), fixed point problems ([5, 6, 24, 27, 30, 32]), Nash equilibrium in noncooperative games theory ([4, 8, 15]), can be formulated in the form of (1). An important method for solving (1) is proximal point method which was originally introduced by Martinet [13] and further developed by Rockafellar [18] for seeking a zero of maximal monotone operators.

In particular, in [4, 7], the technique of resolvent of bi-function f was used to solve (1). For every $\lambda > 0$ and $x\in H$, there exists a point $z \in C$ such that

$$f(z, y) + \frac{1}{\lambda}\langle z - x, y - x\rangle \geq 0, \forall y \in C.$$

Thus, we can define a resolvent operator $J^f_\lambda : H \to 2^C$ by

$$J^f_\lambda = \{z \in C | f(z, y) + \frac{1}{\lambda}\langle z - x, y - x\rangle \geq 0, \forall y \in C\}.$$

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Consequently, by using the resolvent technique, Tada and Takahashi [19] presented the following iterative algorithm for solving equilibrium problem (1) and a fixed point problem:

$$\begin{cases} u_n \in C \text{ such that } \langle f(u_n, u) + \frac{1}{\lambda_n}\langle u - u_n, u_n - x_n\rangle \geq 0, \ \forall u \in C, \\ x_{n+1} = (1-\delta_n)x_n + \delta_n T u_n, n \geq 0, \end{cases} \tag{2}$$

where $\{\lambda_n\} \subset (0, \infty)$, $\{\delta_n\} \subset (0,1)$ and $T : C \to C$ is a nonexpansive mapping.

Recently, Nguyen, Strodiot and Nguyen [16] presented a hybrid method for solving equilibrium problem (1) and a fixed point problem: let $x_0 \in H, Q_1 = C, x_1 = P_{Q_1}[x_0]$. Let $\alpha \in (0,2)$, $\gamma \in (0,1)$. Set $n = 1$.

Step 1. Compute $y_n = \min_{y^\dagger \in C}\{\lambda_n f(x_n, y^\dagger) + \frac{1}{2}\|x_n - y^\dagger\|^2\}$ and $w_n = (1-\gamma^m)x_n + \gamma^m y_n$ where m is the smallest nonnegative integer such that $f(w_n, x_n) - f(w_n, y_n) \geq \frac{\alpha}{2\lambda_n}\|x_n - y_n\|^2$.

Step 2. Calculate $z_n = P_C[x_n - \sigma_n g_n]$, where $g_n \in \partial_2 f(z_n, x_n)$ and $\sigma_n = \frac{f(w_n,x_n)}{\|g_n\|^2}$ if $y_n \neq x_n$ and $\sigma_n = 0$ otherwise.

Step 3. Calculate $t_n = \alpha_n z_n + (1-\alpha_n)Tz_n$, where $T : C \to C$ is a nonexpansive mapping.

Step 4. Compute $x_{n+1} = P_{Q_{n+1}}[x_0]$, where $Q_{n+1} = \{z \in Q_n \| \|t_n - z\|^2 \leq \|x_n - z\|^2 - (1-\alpha_n)\alpha_n\|z_n - Tz_n\|^2\}$.

Step 5. Set $n := n + 1$ and return to Step 1.

Very recently, iterative algorithms for solving (1) and fixed point problems have been future studied in the literature, see, for instance [1, 10, 11, 22].

Motivated and inspired by the above work in the literature, the main purpose of this paper is to investigate the pseudomonotone equilibrium problem and fixed point problem of pseudocontractive operators. We construct an iterative algorithm for finding a common solution of the pseudomonotone equilibrium problem and fixed point of pseudocontractive operators. Weak convergence analysis of the proposed procedure is given.

2. Preliminaries

Throughout, let C be a nonempty closed and convex subset of a real Hilbert space H. Let $g : C \to (-\infty, +\infty]$ be a function.

- g is said to be convex if $g(\alpha u^\dagger + (1-\alpha)v^\dagger) \leq \alpha g(u^\dagger) + (1-\alpha)g(v^\dagger)$ for every $u^\dagger, v^\dagger \in C$ and $\alpha \in [0,1]$.
- g is said to be ρ-strongly convex ($\rho > 0$) if

$$g(\alpha u^\dagger + (1-\alpha)v^\dagger) + \frac{\rho}{2}\alpha(1-\alpha)\|u^\dagger - v^\dagger\|^2 \leq \alpha g(u^\dagger) + (1-\alpha)g(v^\dagger) \tag{3}$$

  for every $u^\dagger, v^\dagger \in C$ and $\alpha \in (0,1)$.

Let $g : C \to (-\infty, +\infty]$ be a convex function. Then, the subdifferential ∂g of g is defined by

$$\partial g(u) := \{v^\dagger \in H : g(u) + \langle v^\dagger, u^\dagger - u\rangle \leq g(u^\dagger), \forall u^\dagger \in C\} \tag{4}$$

for each $u \in C$.

Recall that an operator $T : C \to C$ is said to be pseudocontractive if

$$\|Tu - Tu^\dagger\|^2 \leq \|u - u^\dagger\|^2 + \|(I-T)u - (I-T)u^\dagger\|^2$$

for all $u, u^\dagger \in C$ and T is called L-Lipschitz if

$$\|Tu - Tu^\dagger\| \leq L\|u - u^\dagger\|$$

for all $u, u^\dagger \in C$.

The following symbols are needed in the paper.

- $x_n \rightharpoonup p^\dagger$ indicates the weak convergence of x_n to $p^\dagger$ as $n \to \infty$.
- $x_n \to p^\dagger$ implies the strong convergence of x_n to $p^\dagger$ as $n \to \infty$.

- $Fix(T)$ means the set of fixed points of T.
- $\omega_w(x_n) = \{p^\dagger : \exists\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \rightharpoonup p^\dagger(i \to \infty)\}$.

Lemma 2.1 ([2]). *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let a function $h : C \to \mathbb{R}$ be subdifferentiable. Then $u^\dagger$ is a solution to the following minimization problem*

$$\min_{x\in C}\{h(x)\}$$

if and only if $0 \in \partial h(u^\dagger) + N_C(u^\dagger)$, where $N_C(u^\dagger)$ means the normal cone of C at $u^\dagger$ defined by

$$N_C(u^\dagger) = \{\omega \in H : \langle \omega, u - u^\dagger\rangle \le 0, \forall u \in C\}. \tag{5}$$

Lemma 2.2 ([16]). *In a Hilbert space H, we have*
(i) *for all $x, y, u, v \in H$,*

$$2\langle x - y, u - v\rangle = \|x - v\|^2 + \|y - u\|^2 - \|x - u\|^2 - \|y - v\|^2.$$

(ii) *for all $u, u^\dagger \in H$ and $\forall \kappa \in [0, 1]$,*

$$\|\kappa u + (1 - \kappa)u^\dagger\|^2 = \kappa\|u\|^2 + (1 - \kappa)\|u^\dagger\|^2 - \kappa(1 - \kappa)\|u - u^\dagger\|^2.$$

Lemma 2.3 ([25]). *Assume that the operator $T : C \to C$ is L-Lipschitz pseudocontractive. Then, for all $\tilde u \in C$ and $u^\dagger \in Fix(T)$, we have*

$$\|u^\dagger - T[(1 - \sigma)\tilde u + \sigma T\tilde u]\|^2 \le \|\tilde u - u^\dagger\|^2 + (1 - \sigma)\|\tilde u - T[(1 - \sigma)\tilde u + \sigma T\tilde u]\|^2,$$

where $0 < \sigma < \frac{1}{\sqrt{1+L^2}+1}$.

Lemma 2.4 ([35]). *If the operator $T : C \to C$ is continuous pseudocontractive, then*
(i) *the fixed point set $Fix(T) \subset C$ is closed and convex;*
(ii) *T satisfies demi-closedness, i.e., $u_n \rightharpoonup \tilde z$ and $Tu_n \to z^\dagger$ as $n \to \infty$ imply that $T\tilde z = z^\dagger$.*

3. **Main results**

In this section, we introduce an iterative algorithm for solving the fixed point problems and pseudomonotone equilibrium problems. Consequently, we show the convergence analysis of the suggested algorithm.

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a Lipschitz pseudocontractive operator with Lipschitz constant $L > 0$. Let $f : C\times C \to \mathbb{R}$ be a bifunction which satisfies the following assumptions:

(A1): $f(z^\dagger, z^\dagger) = 0$ for all $z^\dagger \in C$;
(A2): f is pseudomonotone on C, i.e., $f(u^\dagger, u) \ge 0$ implies $f(u, u^\dagger) \le 0$ for all $u, u^\dagger \in C$;
(A3): f is jointly sequently weakly continuous on $C\times C$ (recall that f is called jointly sequently weakly continuous on $C \times C$, if $x_n \rightharpoonup x^\dagger$ and $y_n \rightharpoonup y^\dagger$, then $f(x_n, y_n) \to f(x^\dagger, y^\dagger)$);
(A4): $f(z^\dagger, \cdot)$ is convex and subdifferentiable for all $z^\dagger \in C$;
(A5): f satisfies the Lipschitz-type condition: $\exists \mu_1, \mu_2 > 0$ such that

$$f(x^\dagger, y^\dagger) + f(y^\dagger, z^\dagger) \ge f(x^\dagger, z^\dagger) - \mu_1\|x^\dagger - y^\dagger\|^2 - \mu_2\|y^\dagger - z^\dagger\|^2,\ \forall x^\dagger, y^\dagger, z^\dagger \in C.$$

Let $\{\lambda_n\} \subset (0,\infty)$, $\{\delta_n\} \subset (0, 1)$ and $\{\sigma_n\} \subset (0, 1)$ be three sequences satisfying the following restrictions:

(C1): $\lambda_n \in [\underline{\lambda}, \overline{\lambda}]$, where $0 < \underline{\lambda} \le \overline{\lambda} < \min\{\dfrac{1}{2\mu_1}, \dfrac{1}{2\mu_2}\}$;
(C2): $0 < \underline{\delta} < \delta_n < \overline{\delta} < \sigma_n < \overline{\sigma} < \dfrac{1}{\sqrt{1 + L^2} + 1}, \forall n \ge 0$.

In the sequel, assume that the intersection $Fix(T) \cap EP(f, C) \ne \emptyset$.

Algorithm 3.1. *Let $x_0 \in C$ be an initial value. Assume that the current sequence $\{x_n\}$ has been given and then compute the next iterative sequence $\{x_{n+1}\}$ by the following form*

$$\begin{cases} z_n = (1-\delta_n)x_n + \delta_n T[(1-\sigma_n)x_n + \sigma_n T x_n], \\ y_n = \arg\min_{y^\dagger \in C} \left\{ f(z_n, y^\dagger) + \dfrac{1}{2\lambda_n}\|z_n - y^\dagger\|^2 \right\}, \\ x_{n+1} = \arg\min_{y^\dagger \in C} \left\{ f(y_n, y^\dagger) + \dfrac{1}{2\lambda_n}\|z_n - y^\dagger\|^2 \right\}, n \geq 0. \end{cases} \tag{6}$$

Remark 3.1. *Since $f(x,\cdot) + \frac{1}{2\lambda}\|x - \cdot\|^2$ is strongly convex, for each $x \in C$, $\min_{y^\dagger \in C}\{f(x, y^\dagger) + \frac{1}{2\lambda}\|x - y^\dagger\|^2\}$ has a unique solution. Therefore, the sequence $\{x_n\}$ generated by (6) is well-defined.*

Proposition 3.1. *For all $z^\dagger \in C$, we have*

$$f(z_n, z^\dagger) \geq f(z_n, y_n) + \frac{1}{\lambda_n}\langle z_n - y_n, z^\dagger - y_n\rangle, \tag{7}$$

and

$$f(y_n, z^\dagger) \geq f(y_n, x_{n+1}) + \frac{1}{\lambda_n}\langle x_{n+1} - z_n, x_{n+1} - z^\dagger\rangle. \tag{8}$$

Proof. According to Lemma 2.1, from the definition of y_n, we have

$$0 \in \partial_2 \left\{ f(z_n, y^\dagger) + \frac{1}{2\lambda_n}\|z_n - y^\dagger\|^2 \right\}\Big|_{y^\dagger = y_n} + N_C(y_n). \tag{9}$$

It follows from (9) that there exists $p_n \in \partial_2 f(z_n, y_n)$ such that

$$-p_n + \frac{1}{\lambda_n}(z_n - y_n) \in N_C(y_n).$$

This together with the definition (5) of the normal cone N_C implies that

$$\left\langle -p_n + \frac{1}{\lambda_n}(z_n - y_n), z^\dagger - y_n \right\rangle \leq 0, \ \forall z^\dagger \in C.$$

It follows that

$$\langle p_n, z^\dagger - y_n\rangle \geq \frac{1}{\lambda_n}\langle z_n - y_n, z^\dagger - y_n\rangle, \ \forall z^\dagger \in C. \tag{10}$$

By the definition (4) of subgradient of $f(z_n, \cdot)$ at y_n, we obtain

$$f(z_n, z^\dagger) \geq f(z_n, y_n) + \langle p_n, z^\dagger - y_n\rangle, \ \forall z^\dagger \in C. \tag{11}$$

Combining (10) and (11), we deduce the desired result (7). Similarly, we can show that the conclusion (8) also holds. □

Proposition 3.2. *Let $p \in Fix(T) \cap EP(f, C)$. Then, we have*

$$\begin{aligned} \|x_{n+1} - p\|^2 &\leq \|z_n - p\|^2 - \|x_{n+1} - z_n\|^2 - 2\langle z_n - y_n, x_{n+1} - y_n\rangle \\ &\quad + 2\mu_1\lambda_n\|z_n - y_n\|^2 + 2\mu_2\lambda_n\|y_n - x_{n+1}\|^2 \\ &\leq \|x_n - p\|^2 - \delta_n(\sigma_n - \delta_n)\|x_n - T[(1-\sigma_n)x_n + \sigma_n T x_n]\|^2 \\ &\quad - (1 - 2\mu_2\lambda_n)\|x_{n+1} - y_n\|^2 - (1 - 2\mu_1\lambda_n)\|y_n - z_n\|^2. \end{aligned} \tag{12}$$

Proof. Since $p \in EP(f, C)$, then $f(p, y_n) \geq 0$. By the pseudomonotonicity (A2) of f, we have, $f(y_n, p) \leq 0$. This together with (8) implies that

$$f(y_n, x_{n+1}) + \frac{1}{\lambda_n}\langle x_{n+1} - z_n, x_{n+1} - p\rangle \leq 0. \tag{13}$$

Applying the Lipschitz property (A5) of f, we obtain

$$f(y_n, x_{n+1}) \geq f(z_n, x_{n+1}) - f(z_n, y_n) - \mu_1\|z_n - y_n\|^2 - \mu_2\|y_n - x_{n+1}\|^2. \tag{14}$$

By virtue of (13) and (14), we deduce

$$\begin{aligned} \frac{1}{\lambda_n}\langle x_{n+1} - z_n, p - x_{n+1}\rangle \geq f(z_n, x_{n+1}) - f(z_n, y_n) - \mu_1\|z_n - y_n\|^2 \\ - \mu_2\|y_n - x_{n+1}\|^2. \end{aligned} \tag{15}$$

Setting $z^\dagger = x_{n+1}$ in (7), we have

$$f(z_n, x_{n+1}) - f(z_n, y_n) \geq \frac{1}{\lambda_n}\langle z_n - y_n, x_{n+1} - y_n\rangle. \tag{16}$$

In terms of (15) and (16), we get

$$\begin{aligned} \langle x_{n+1} - z_n, p - x_{n+1}\rangle \geq \langle z_n - y_n, x_{n+1} - y_n\rangle - \mu_1\lambda_n\|z_n - y_n\|^2 \\ - \mu_2\lambda_n\|y_n - x_{n+1}\|^2. \end{aligned} \tag{17}$$

Using Lemma 2.2, we deduce

$$2\langle x_{n+1} - z_n, p - x_{n+1}\rangle = \|z_n - p\|^2 - \|x_{n+1} - z_n\|^2 - \|x_{n+1} - p\|^2. \tag{18}$$

Combining (17) and (18), we derive

$$\begin{aligned} \|z_n - p\|^2 - \|x_{n+1} - z_n\|^2 - \|x_{n+1} - p\|^2 \geq 2\langle z_n - y_n, x_{n+1} - y_n\rangle - 2\mu_1\lambda_n\|z_n - y_n\|^2 \\ - 2\mu_2\lambda_n\|y_n - x_{n+1}\|^2. \end{aligned}$$

It follows that

$$\begin{aligned} \|x_{n+1} - p\|^2 &\leq \|z_n - p\|^2 - \|x_{n+1} - z_n\|^2 - 2\langle z_n - y_n, x_{n+1} - y_n\rangle \\ &\quad + 2\mu_1\lambda_n\|z_n - y_n\|^2 + 2\mu_2\lambda_n\|y_n - x_{n+1}\|^2 \\ &= \|z_n - p\|^2 - (1 - 2\mu_2\lambda_n)\|x_{n+1} - y_n\|^2 - (1 - 2\mu_1\lambda_n)\|y_n - z_n\|^2. \end{aligned} \tag{19}$$

In the light of (6) and Lemmas 2.2 and 2.3, we obtain

$$\begin{aligned} \|z_n - p\|^2 &= \|(1 - \delta_n)(x_n - p) + \delta_n(T[(1 - \sigma_n)x_n + \sigma_n Tx_n] - p)\|^2 \\ &= (1 - \delta_n)\|x_n - p\|^2 - \delta_n(1 - \delta_n)\|T[(1 - \sigma_n)x_n + \sigma_n Tx_n] - x_n\|^2 \\ &\quad + \delta_n\|T[(1 - \sigma_n)ux_n + \sigma_n Tx_n] - p\|^2 \\ &\leq (1 - \delta_n)\|x_n - p\|^2 - \delta_n(1 - \delta_n)\|T[(1 - \sigma_n)x_n + \sigma_n Tx_n] - x_n\|^2 \\ &\quad + \delta_n(\|x_n - p\|^2 + (1 - \sigma_n)\|x_n - T[(1 - \sigma_n)x_n + \sigma_n Tx_n]\|^2) \\ &= \|x_n - p\|^2 - \delta_n(\sigma_n - \delta_n)\|x_n - T[(1 - \sigma_n)x_n + \sigma_n Tx_n]\|^2 \\ (by\ (C2)) &\leq \|x_n - p\|^2. \end{aligned} \tag{20}$$

Substituting (20) into (19), we have

$$\begin{aligned} \|x_{n+1} - p\|^2 &\leq \|z_n - p\|^2 - \|x_{n+1} - z_n\|^2 - 2\langle z_n - y_n, x_{n+1} - y_n\rangle \\ &\quad + 2\mu_1\lambda_n\|z_n - y_n\|^2 + 2\mu_2\lambda_n\|y_n - x_{n+1}\|^2 \\ &= \|x_n - p\|^2 - \delta_n(\sigma_n - \delta_n)\|x_n - T[(1 - \sigma_n)x_n + \sigma_n Tx_n]\|^2 \\ &\quad - (1 - 2\mu_2\lambda_n)\|x_{n+1} - y_n\|^2 - (1 - 2\mu_1\lambda_n)\|y_n - z_n\|^2. \end{aligned}$$

□

Theorem 3.1. *The sequence* $\{x_n\}$ *generated by* (6) *converges weakly to some point in* $Fix(T) \cap EP(f, C)$.

Proof. Let $p \in Fix(T) \cap EP(f, C)$. Thanks to assumptions (C1) and (C2), from (12), we deduce

$$\|x_{n+1} - p\| \leq \|x_n - p\|,$$

which implies that the limit $\lim_{n\to\infty} \|x_n - p\|$ exists. Thus, the sequence $\{x_n\}$ is bounded. Consequently, $\{z_n\}$ is also bounded due to (20).

By (12), we have

$$\begin{aligned} &\delta_n(\sigma_n - \delta_n)\|x_n - T[(1-\sigma_n)x_n + \sigma_n T x_n]\|^2 + (1 - 2\mu_2\lambda_n)\|x_{n+1} - y_n\|^2 \\ &\quad + (1 - 2\mu_1\lambda_n)\|y_n - z_n\|^2 \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2. \end{aligned} \tag{21}$$

Since $\lim_{n\to\infty} \|x_n - p\|$ exists, $\|x_n - p\|^2 - \|x_{n+1} - p\|^2 \to 0$. According to (21) and the restrictions (C1) and (C2), we get

$$\lim_{n\to\infty} \|x_n - T[(1-\sigma_n)x_n + \sigma_n T x_n]\| = 0 \tag{22}$$

and

$$\lim_{n\to\infty} \|x_{n+1} - y_n\| = 0 \text{ and } \lim_{n\to\infty} \|y_n - z_n\| = 0. \tag{23}$$

From (6), we derive

$$\|z_n - x_n\| = \delta_n \|x_n - T[(1-\sigma_n)x_n + \sigma_n T x_n]\|,$$

which together with (22) and $\liminf_{n\to\infty} \delta_n > 0$ (by (C2)) implies that

$$\lim_{n\to\infty} \|z_n - x_n\| = 0. \tag{24}$$

Take into account (23) and (24), we deduce

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{25}$$

On the other hand, using the Lipschitz property of T, we have

$$\begin{aligned} \|x_n - Tx_n\| &\leq \|x_n - T[(1-\sigma_n)x_n + \sigma_n T x_n]\| + \|T[(1-\sigma_n)x_n + \sigma_n T x_n] - Tx_n\| \\ &\leq \|x_n - T[(1-\sigma_n)x_n + \sigma_n T x_n]\| + L\sigma_n\|x_n - Tx_n\|. \end{aligned}$$

It follows that

$$\|x_n - Tx_n\| \leq \frac{1}{1 - L\sigma_n}\|x_n - T[(1-\sigma_n)x_n + \sigma_n T x_n]\|. \tag{26}$$

Since $\liminf_{n\to\infty} \sigma_n < \frac{1}{L}$, combining (22) and (26), we deduce

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{27}$$

Note that the sequence $\{z_n\}$ is bounded. Selecting any $x^\dagger \in \omega_w(z_n)$, there exists a subsequence $\{z_{n_i}\} \subset \{z_n\}$ such that

$$z_{n_i} \rightharpoonup x^\dagger \in C. \tag{28}$$

From (7), we obtain

$$f(z_{n_i}, z^\dagger) \geq f(z_{n_i}, y_{n_i}) + \frac{1}{\lambda_{n_i}}\langle z_{n_i} - y_{n_i}, z^\dagger - y_{n_i}\rangle, \ \forall z^\dagger \in C. \tag{29}$$

Thanks to (23), (A1) and (A3), we get

$$\lim_{i\to\infty} f(z_{n_i}, y_{n_i}) = 0.$$

This together with (29) implies that

$$f(x^\dagger, z^\dagger) \geq 0, \ \forall z^\dagger \in C.$$

Therefore, $x^\dagger \in EP(f, C)$.

By (24) and (28), we have $x_{n_i} \rightharpoonup x^\dagger \in C$. Combining with (27), we deduce

$$\lim_{i\to\infty} \|x_{n_i} - Tx_{n_i}\| = 0.$$

Applying Lemma 2.4, we conclude that $x^\dagger \in Fix(T)$.

Now, we have shown that $\omega_w(x_n) \subset EP(f,C) \cap Fix(T)$. At the same time, according to (25), $\omega_w(x_n)$ is singleton. Therefore, the whole sequence $\{x_n\}$ converges weakly to $x^\dagger$. □

Setting $T = I$, the identity operator, we obtain the following iterative algorithm for finding a solution in $EP(f,C)$.

Algorithm 3.2. *Let $x_0 \in C$ be an initial value. Assume that the current sequence $\{x_n\}$ has been given and then compute the next iterative sequence $\{x_{n+1}\}$ by the following form*

$$\begin{cases} y_n = \arg\min_{y^\dagger\in C} \left\{ f(x_n, y^\dagger) + \dfrac{1}{2\lambda_n}\|x_n - y^\dagger\|^2 \right\}, \\ x_{n+1} = \arg\min_{y^\dagger\in C} \left\{ f(y_n, y^\dagger) + \dfrac{1}{2\lambda_n}\|x_n - y^\dagger\|^2 \right\}, n \geq 0. \end{cases}$$

Corollary 3.1. *Suppose that $EP(f,C) \neq \emptyset$. Then, the sequence $\{x_n\}$ generated by Algorithm 3.2 converges weakly to some point in $EP(f,C)$.*

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REFERENCES

[1] P.K. Anh and D.V. Hieu, *Parallel hybrid methods for variational inequalities, equilibrium problems and common fixed point problems*, Vietnam J. Math., **44** (2016), 351–374.2

[2] H.H. Bauschke and P.L. Combettes, *Convex analysis and monotone operator theory in hilbert spaces*, Springer, New York (2011).

[3] J.Y. Bello-Cruz and A.N. Iusem, *Convergence of direct methods for paramonotone variational inequalities*, Comput. Optim. Appl., **46** (2010), 247–263.

[4] E. Blum and W. Oettli, *From optimization and variational inequalities to equilibrium problems*, Math. Student, **63** (1994), 123–145.

[5] L.C. Ceng, A. Petrusel, J.C. Yao and Y. Yao, *Hybrid viscosity extragradient method for systems of variational inequalities, fixed Points of nonexpansive mappings, zero points of accretive operators in Banach spaces*, Fixed Point Theory, **19** (2018), 487–502.

[6] L.C. Ceng, A. Petrusel, J.C. Yao and Y. Yao, *Systems of variational inequalities with hierarchical variational inequality constraints for Lipschitzian pseudocontractions*, Fixed Point Theory, **20** (2019), 113–134.

[7] P.L. Combettes and S.A. Hirstoaga, *Equilibrium programming in Hilbert spaces*, J. Nonlinear Convex Anal., **6** (2005), 117–136.

[8] R.W. Cottle and J.C. Yao, *Pseudo-monotone complementarity problems in Hilbert space*, J. Optim. Theory Appl., **75** (1992), 281–295.

[9] D.V. Hieu, P.K. Anh and L.D. Muu, *Modified extragradient-like algorithms with new stepsizes for variational inequalities*, Comput. Optim. Appl., **73** (2019), 913–932.

[10] D.V. Hieu, L.D. Muu and P.K. Anh, *Parallel hybrid extragradientmethods for pseudomonotone equilibrium problems and nonexpansive mappings*, Numer. Algorithms, **73** (2016), 197–217.

[11] H. Iiduka and I. Yamada, *A subgradient-type method for the equilibrium problem over the fixed point set and its applications*, Optim., **58** (2009), 251–261.

[12] Y. Malitsky, *Proximal extrapolated gradient methods for variational inequalities*, Optim. Meth. & Software, **33** (2018), 140–164.

[13] B. Martinet, *Régularisation dínéquations variationelles par approximations successives*, Rev. Fr. Autom. Inform. Rech. Opér., Anal. Numér., **4**, (1970), 154–159.

[14] C. Martinez-Yanes and H.K. Xu, *Strong convergence of the CQ method for fixed point iteration processes*, Nonlinear Anal., **64** (2006), 2400–2411.

[15] L.D. Muu and W. Oettli, *Convergence of an adaptive penalty scheme for finding constrained equilibria*, Nonlinear Anal., **18** (1992), 1159–1166.

[16] T.T.V. Nguyen, J.J. Strodiot and V.H. Nguyen, *Hybrid methods for solving simultaneously an equilibrium problem and countably many fixed point problems in a Hilbert space*, J. Optim. Theory Appl., **160** (2014), 809–831.

[17] T.D. Quoc, L.D. Muu and N.V. Hien, *Extragradient algorithms extended to equilibrium problems*, Optim., **57** (2008), 749–776.

[18] R.T. Rockafellar, *Monotone operators and the proximal point algorithm*, SIAM J. Control Optim., **14** (1976), 877–898.

[19] A. Tada and W. Takahashi, *Weak and strong convergence theorems for a nonexpansive mapping and an equilibrium problem*, J. Optim. Theory Appl., **133** (2007), 359–370.

[20] B.S. Thakur and M. Postolache, *Existence and approximation of solutions for generalized extended nonlinear variational inequalities*, J. Inequal. Appl., **2013** (2013), Art. No. 590.

[21] D.V. Thong and A. Gibali, *Extragradient methods for solving non-Lipschitzian pseudo-monotone variational inequalities*, J. Fixed Point Theory Appl., **21** (2019), Art. ID. UNSP 20.

[22] P.T. Vuong, J.J. Strodiot and V.H. Nguyen, *On extragradient-viscosity methods for solving equilibrium and fixed point problems in a Hilbert space*, Optim., **64** (2015), 429–451.

[23] J. Yang and H.W. Liu, *A modified projected gradient method for monotone variational inequalities*, J. Optim. Theory Appl., **179** (2018), 197–211.

[24] Y. Yao, L. Leng, M. Postolache and X. Zheng, *Mann-type iteration method for solving the split common fixed point problem*, J. Nonlinear Convex Anal., **18** (2017), 875–882.

[25] Y. Yao, Y.C. Liou and J.C. Yao, *Split common fixed point problem for two quasi-pseudocontractive operators and its algorithm construction*, Fixed Point Theory Appl., **2015** (2015), Art. No. 127.

[26] Y. Yao, M. Postolache, Y.C. Liou and Z.S. Yao, *Construction algorithms for a class of monotone variational inequalities*, Optim. Lett., **10** (2016), 1519–1528.

[27] Y. Yao, M. Postolache and J.C. Yao, *An iterative algorithm for solving the generalized variational inequalities and fixed points problems*, Mathematics, **7** (2019), Art. ID. 61.

[28] Y. Yao, M. Postolache and J.C. Yao, *Iterative algorithms for generalized variational inequalities*, U. Politeh. Buch. Ser. A, **81** (2019), 3–16.

[29] Y. Yao, M. Postolache and J.C. Yao, *Strong convergence of an extragradient algorithm for variational inequality and fixed point problems*, U.P.B. Sci. Bull., Series A, **83** (2020), 3–12.

[30] Y. Yao and N. Shahzad, *Strong convergence of a proximal point algorithm with general errors*, Optim. Lett., **6** (2012), 621–628.

[31] C. Zhang, Z. Zhu, Y. Yao and Q. Liu, *Homotopy method for solving mathematical programs with bounded box-constrained variational inequalities*, Optim., **68** (2019), 2293–2312.

[32] H. Zegeye, N. Shahzad and Y. Yao, *Minimum-norm solution of variational inequality and fixed point problem in Banach spaces*, Optim., **64** (2015), 453–471.

[33] X.P. Zhao, J.C. Yao and Y. Yao, *A proximal algorithm for solving split monotone variational inclusions*, U.P.B. Sci. Bull., Series A, in press.

[34] X. Zhao and Y. Yao, *Modified extragradient algorithms for solving monotone variational inequalities and fixed point problems*, Optim., DOI: 10.1080/02331934.2019.1711087.

[35] H. Zhou, *Strong convergence of an explicit iterative algorithm for continuous pseudocontractions in Banach spaces*, Nonlinear Anal., **70** (2009), 4039–4046.

[36] L.J. Zhu, Y. Yao and M. Postolache, *Projection methods with linesearch technique for pseudomonotone equilibrium problems and fixed point problems*, U.P.B. Sci. Bull., Series A, in press.