CONVERGENCE OF AN IMPLICIT NET FOR SOLVING EQUILIBRIUM PROBLEMS AND QUASI-VARIATIONAL INCLUSIONS

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In this article, we discuss iterative methods for finding a common solution of equilibrium problems and quasi-variational inclusion problems in Hilbert spaces. We introduce an implicit method which defines a net consisting of projection method and resolvent method. Convergence result of the proposed net is proved provided some additional conditions are fulfilled.

Keywords: equilibrium problem, quasi-variational inclusion, net, firmly nonexpansive, resolvent.
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1. **Introduction**

Let $\mathcal{H}$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$. Let $\mathcal{C}$ a nonempty closed convex subset of $\mathcal{H}$. Let $\varphi\colon \mathcal{C} \times \mathcal{C} \to \mathcal{R}$ be a bifunction. In this article, we concern the following equilibrium problem which aims to find a point $u^\dagger \in \mathcal{C}$ such that

$$\varphi(u^\dagger, u) \geq 0, \ \forall u \in \mathcal{C}. \tag{1}$$

Let the solution set of the equilibrium problem (1) be denoted by $\mathrm{Sol}(\mathcal{C}, \varphi)$.

As a powerful tool, the equilibrium problem has been continuously concerned and studied by many scholars, see e.g. [3, 5, 14, 15, 24]). Now, it is well-known ([2, 17]) that the formulation (1) includes variational inequality problems ([32, 34, 37]), optimization problems ([12, 21, 22]), split problems ([8]), as well as fixed point problems ([1, 9, 11, 23, 26–30, 33, 38]).

Note that solving equilibrium problem (1) can be translated into a fixed point problem by using the resolvent technique. In fact, the resolvent of a bifunction $\varphi\colon \mathcal{C} \times \mathcal{C} \to \mathcal{R}$ is the set-valued operator ([2])

$$F(u^\dagger) := \{w^\dagger \in \mathcal{C} : \varphi(w^\dagger, v^\dagger) + \langle v^\dagger - w^\dagger, w^\dagger - u^\dagger \rangle \geq 0, \ \forall v^\dagger \in \mathcal{C}\}.$$

Under some conditions, we have $\mathrm{Sol}(\mathcal{C}, \varphi) = \mathrm{Fix}(F)$, where $\mathrm{Fix}(F)$ stands for the set of fixed points of F. By utilizing the resolvent method, Combettes and Hirstoaga [6] proposed an iterative algorithm of finding a point in $\mathrm{Sol}(\mathcal{C}, \varphi)$.

Now, we consider the following generalized equilibrium problem of finding a point $u^\dagger \in \mathcal{C}$ such that

$$\varphi(u^\dagger, u) + \langle f(u^\dagger), u - u^\dagger \rangle \geq 0, \ \forall u \in \mathcal{C}. \tag{2}$$

Let the solution set of the generalized equilibrium problem (2) be denoted by $\mathrm{Sol}(\mathcal{C}, \varphi, f)$.

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With the help of the resolvent technique, Takahashi and Takahashi [25] introduced the following iterative algorithm for finding a common point in $\mathrm{Sol}(\mathcal{C}, \varphi, f) \cap \mathrm{Fix}(T)$

$$\begin{cases} \varphi(u_n, u) + \frac{1}{\varsigma_n}\langle u - u_n, u_n - (x_n - \varsigma_n f(x_n))\rangle \geq 0, \ \forall u \in \mathcal{C}, \\ x_{n+1} = \tau_n x_n + (1 - \tau_n)T[\mu_n \hat{u} + (1 - \mu_n)u_n], \ \forall n \geq 0, \end{cases}$$

where $T : \mathcal{C} \to \mathcal{C}$ is a nonexpansive operator.

Consequently, various methods and techniques are proposed for solving a common problem associated with equilibrium problems, please see [4, 13, 18, 19] and the references therein. Let $\psi \colon \mathcal{C} \to \mathcal{H}$ and $\Psi \colon \mathcal{H} \rightrightarrows 2^{\mathcal{H}}$ be two nonlinear operators. In this article, we investigate the following quasi-variational inclusion problem of finding a point $u^\dagger \in \mathcal{H}$ such that

$$0 \in \Psi(u^\dagger) + \psi(u^\dagger). \tag{3}$$

The solution set of (3) is denoted by $\mathrm{Sol}(\mathcal{C}, \Psi, \psi)$.

The quasi-variational inclusion and the relevant iterative algorithms have been investigated and proposed in the literature, see [7, 20, 31, 35, 36]. A basic algorithm for finding a point in $\mathrm{Sol}(\mathcal{C}, \Psi, \psi)$ is the following resolvent algorithm which generates a sequence $\{x_n\}$ iteratively by

$$x_0 \in \mathcal{C}, \ x_{n+1} = (I + \tau\Psi)_\alpha^{-1}(x_n - \tau\psi(x_n)), \ \forall n \geq 0.$$

In this paper, our main purpose is to investigate the common problem of the generalized equilibrium problem (2) and the quasi-variational inclusion (3). We construct an implicit algorithm which defines a net for finding a common solution of the generalized equilibrium problem (2) and the quasi-variational inclusion (3). Under some conditions, we show that the proposed net converges weakly to a point in $\mathrm{Sol}(\mathcal{C}, \varphi, f) \cap \mathrm{Sol}(\mathcal{C}, \Psi, \psi)$.

2. **Preliminaries**

Let $\mathcal{C}$ be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. Recall that an operator $g : \mathcal{C} \to \mathcal{H}$ is said to be κ-Lipschitz continuous if there is a positive constant κ such that

$$\|g(u^\dagger) - g(u)\| \leq \kappa\|x - y\|, \ \forall u^\dagger, u \in \mathcal{C}.$$

(i) g is said to be nonexpansive if $\kappa = 1$.
(ii) g is said to be contractive if $\kappa < 1$.

An operator $g : \mathcal{C} \to \mathcal{H}$ is said to be firmly nonexpansive if

$$\|g(u) - g(u^\dagger)\|^2 \leq \langle u - u^\dagger, g(u) - g(u^\dagger)\rangle$$

for all $u, u^\dagger \in \mathcal{C}$.

An operator $f : \mathcal{C} \to \mathcal{H}$ is said to be α-inverse strongly monotone if for some $\alpha > 0$, the following inequality holds

$$\langle f(u^\dagger) - f(u), u^\dagger - u\rangle \geq \alpha\|f(u^\dagger) - f(u)\|^2, \ \forall u^\dagger, u \in \mathcal{C}.$$

In this case, we call f α-inverse strongly monotone. It is easy to show that α-inverse-strongly monotone operator f is $\frac{1}{\alpha}$-Lipschitz continuous.

Let $\varphi : \mathcal{C} \times \mathcal{C} \to \mathcal{R}$ be a bifunction. Suppose that the following four conditions are fulfilled

$(\varphi 1):$ $\varphi(u^\dagger, u^\dagger) = 0, \ \forall u^\dagger \in \mathcal{C}$;
$(\varphi 2):$ $\varphi(u^\dagger, v^\dagger) + \varphi(v^\dagger, u^\dagger) \leq 0, \ \forall u^\dagger, v^\dagger \in \mathcal{C}$;
$(\varphi 3):$ $\lim_{t\downarrow 0} \varphi(tw^\dagger + (1 - t)u^\dagger, v^\dagger) \leq \varphi(u^\dagger, v^\dagger), \ \forall u^\dagger, v^\dagger, w^\dagger \in \mathcal{C}$;
$(\varphi 4):$ for each $u^\dagger \in \mathcal{C}$, $v^\dagger \mapsto \varphi(u^\dagger, v^\dagger)$ is convex and lower semicontinuous.

Recall that a linear bounded operator $\phi : \mathcal{H} \to \mathcal{H}$ is said to be σ strongly positive if there exists a constant $\sigma > 0$ such that

$$\langle \phi(u^\dagger), u^\dagger \rangle \geq \sigma \|x\|^2, \ \forall u^\dagger \in \mathcal{H}.$$

Let $\mathcal{C}$ be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. Recall that the well-known metric projection $proj_{\mathcal{C}} : \mathcal{H} \to \mathcal{C}$ is defined by

$$proj_{\mathcal{C}}(u^\dagger) := \arg\min_{u \in \mathcal{C}} \|u - u^\dagger\|, \ u^\dagger \in \mathcal{H}.$$

$proj_{\mathcal{C}}$ is firmly nonexpansive and satisfies

$$u^\dagger \in \mathcal{H}, \ \langle u^\dagger - proj_{\mathcal{C}}(u^\dagger), u - proj_{\mathcal{C}}(u^\dagger) \rangle \leq 0, \ \forall u \in \mathcal{C}. \tag{4}$$

Assume that $\Psi \colon \mathcal{H} \rightrightarrows 2^{\mathcal{H}}$ is a multi-valued operator. Write $\mathrm{dom}(\Psi) = \{u^\dagger \in \mathcal{H} : \Psi(u^\dagger) \neq \emptyset\}$ and $\Psi^{-1}(0) := \{u^\dagger \in \mathcal{H} : 0 \in \Psi(u^\dagger)\}$.

Recall that an operator $\Psi \colon \mathcal{H} \rightrightarrows 2^{\mathcal{H}}$ is said to be monotone if and only if

$$\langle u - u^\dagger, p - q \rangle \geq 0, \ \forall u, u^\dagger \in \mathrm{dom}(\Psi)$$

where $p \in \Psi(u)$ and $q \in \Psi(u^\dagger)$.

A monotone operator $\Psi \colon \mathcal{H} \rightrightarrows 2^{\mathcal{H}}$ is maximal monotone if and only if the graph of Ψ is not strictly contained in the graph of any other monotone operator.

Assume that $\Psi \colon \mathcal{H} \rightrightarrows 2^{\mathcal{H}}$ is a maximal monotone operator. Define an operator $J_\alpha^\Psi : \mathcal{H} \to \mathrm{dom}(\Psi)$ by the following way

$$J_\tau^\Psi := (I + \tau\Psi)^{-1}$$

where $\tau > 0$ is a constant.

J_α^Ψ is said to be the resolvent of ψ, which has the following properties

(i) J_α^Ψ is single-valued and firmly nonexpansive.
(ii) For any $\tau > 0$, $\Psi^{-1}(0) = \mathrm{Fix}(J_\tau^\Psi) := \{u^\dagger \in \mathcal{H} : J_\tau^\Psi(u^\dagger) = u^\dagger\}$.

Lemma 2.1 ([6]). *Let $\mathcal{C}$ be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. Let $\varphi : \mathcal{C} \times \mathcal{C} \to \mathcal{R}$ be a bifunction fulfilling conditions (φ1)-(φ4) above. Then, for $\varsigma > 0$ and $u^\dagger \in \mathcal{C}$, there exists a point $w^\dagger \in \mathcal{C}$ satisfying*

$$\varphi(w^\dagger, v^\dagger) + \frac{1}{\varsigma}\langle v^\dagger - w^\dagger, w^\dagger - u^\dagger \rangle \geq 0, \forall v^\dagger \in \mathcal{C}.$$

Write

$$F_\varsigma(u^\dagger) := \{w^\dagger \in \mathcal{C} : \varphi(w^\dagger, v^\dagger) + \frac{1}{\varsigma}\langle v^\dagger - w^\dagger, w^\dagger - u^\dagger \rangle \geq 0, \ \forall v^\dagger \in \mathcal{C}\}.$$

Then, we have

(i) *F_ς is single-valued and firmly nonexpansive;*
(ii) *$\mathrm{Sol}(\mathcal{C}, \varphi)$ is closed convex and $\mathrm{Sol}(\mathcal{C}, \varphi) = \mathrm{Fix}(F_\varsigma)$.*

Lemma 2.2 ([16]). *Let $\mathcal{C}$ be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. Let $f : \mathcal{C} \to \mathcal{H}$ be an α-inverse strongly monotone operator. Then, the following result holds*

$$\|(I - \varsigma f)u^\dagger - (I - \varsigma f)u\|^2 \leq \|u^\dagger - u\|^2 + \varsigma(\varsigma - 2\alpha)\|f(u^\dagger) - f(u)\|^2, \ \forall u^\dagger, u \in \mathcal{C},$$

where ς is a positive constant.

It is obviously that $I - \varsigma f$ is nonexpansive if $0 \leq \varsigma \leq 2\alpha$.

Lemma 2.3 ([10]). *Let $\mathcal{C}$ be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. Let $T : \mathcal{C} \to \mathcal{H}$ be a nonexpansive operator. Let $\{u_n\} \subset \mathcal{C}$ be a sequence. If $u_n \rightharpoonup u^\dagger \in \mathcal{C}$ and $u_n - Tu_n \to \hat{u}$, then we have $(I - T)u^\dagger = \hat{u}$.*

3. **Main results**

In this section, we propose an implicit net and show that it converges weakly to a common solution of the generalized equilibrium problem (2) and the quasi-variational inclusion (3).

Let $\mathcal{C}$ be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. Let $\varphi : \mathcal{C}\times\mathcal{C} \to \mathcal{R}$ be a bifunction satisfying conditions (φ1)-(φ4). Let $g : \mathcal{C} \to \mathcal{H}$ be a κ contractive operator. Let $\Psi\colon \mathcal{H} \rightrightarrows 2^{\mathcal{H}}$ be a maximal monotone operator fulfilling $\mathrm{dom}(\Psi) \subset \mathcal{C}$. Let $f : \mathcal{C} \to \mathcal{H}$ be an α-inverse strongly monotone operator and $\psi\colon \mathcal{C} \to \mathcal{H}$ be a β-inverse strongly monotone operator. Let $\phi : \mathcal{H} \to \mathcal{H}$ be a σ strongly positive bounded linear operator. Let τ, γ, ς and μ be four constants such that $\varsigma \in (0, 2\alpha)$, $\tau \in (0, 2\beta)$, $\gamma \in (0, \frac{\sigma}{\kappa})$ and $\mu \in (0, 1)$.

Algorithm 3.1. *For each $t \in (0, \frac{1}{(1-\mu)(\sigma-\gamma\kappa)})$, define a net $\{x_t\}$ by the following implicit manner*

$$\begin{cases}\varphi(u_t, x) + \frac{1}{\varsigma}\langle x - u_t, u_t - proj_{\mathcal{C}}[t\gamma g(x_t) + (I - t\phi)(x_t - \varsigma f(x_t))]\rangle \geq 0,\ \forall x \in \mathcal{C},\\ x_t = \mu J^{\Psi}_{\tau}(I - \tau\psi)x_t + (1 - \mu)u_t,\ \forall t \in (0, \frac{1}{(1-\mu)(\sigma-\gamma\kappa)}).\end{cases} \tag{5}$$

Theorem 3.1. *Suppose that $\Gamma := \mathrm{Sol}(\mathcal{C}, \varphi, f) \cap \mathrm{Sol}(\mathcal{C}, \Psi, \psi) \neq \emptyset$. Then, as $t \to 0+$, the net $\{x_t\}$ defined by Algorithm 3.1 converges weakly to a point in Γ.*

Proof. We divide our proof into several steps.

Step 1. The net $\{x_t\}$ defined by Algorithm 3.1 is well-defined.
For each $t > 0$, set

$$\Phi_t := F_{\varsigma} proj_{\mathcal{C}}[t\gamma g + (I - t\phi)(I - \varsigma f)]$$

and

$$G_t := \mu J^{\Psi}_{\tau}(I - \tau\psi) + (1 - \mu)\Phi_t.$$

Based on Lemma 2.1, we have $u_t = \Phi_t(x_t)$ and F_{ς} is firmly nonexpansive. By Lemma 2.2, $I - \varsigma f$ is nonexpansive for all $\varsigma \in (0, 2\alpha)$. Then, for any $x, y \in \mathcal{C}$, we have

$$\begin{aligned}\|\Phi_t(x) - \Phi_t(y)\| &= \|F_{\varsigma} proj_{\mathcal{C}}[t\gamma g + (I - t\phi)(I - \varsigma f)]x - F_{\varsigma} proj_{\mathcal{C}}[t\gamma g\\ &\quad + (I - t\phi)(I - \varsigma f)]y\|\\ &\leq t\gamma\|g(x) - g(y)\| + |I - t\phi|\|(I - \varsigma f)x - (I - \varsigma f)y\|\\ &\leq t\gamma\kappa\|x - y\| + (1 - \sigma t)\|x - y\|\\ &= [1 - (\sigma - \gamma\kappa)t]\|x - y\|.\end{aligned} \tag{6}$$

Since $J^{\Psi}_{\tau}(I - \tau\psi)$ is nonexpansive, from (6), we obtain

$$\begin{aligned}\|G_t x - G_t y\| &= \|\mu J^{\Psi}_{\tau}(I - \tau\psi)x + (1 - \mu)\Phi_t(x) - \mu J^{\Psi}_{\tau}(I - \tau\psi)y - (1 - \mu)\Phi_t(y)\|\\ &\leq \mu\|J^{\Psi}_{\tau}(I - \tau\psi)x - J^{\Psi}_{\tau}(I - \tau\psi)y\| + (1 - \mu)\|\Phi_t(x) - \Phi_t(y)\|\\ &\leq \mu\|x - y\| + (1 - \mu)[1 - (\sigma - \gamma\kappa)t]\|x - y\|\\ &= [1 - (1 - \mu)(\sigma - \gamma\kappa)t]\|x - y\|.\end{aligned}$$

If $t \in (0, \frac{1}{(1-\mu)(\sigma-\gamma\kappa)})$, then G_t is a contractive operator. Hence, for each $t \in (0, \frac{1}{(1-\mu)(\sigma-\gamma\kappa)})$, G_t has a unique fixed point in $\mathcal{C}$, denoted by x_t. Namely, $x_t = G_t(x_t)$. Therefore, (5) is well-defined.

Step 2. The net $\{x_t\}$ generated by (5) is bounded.

Let $p^\dagger \in \Gamma$. Then, we have $J_\tau^\Psi(p^\dagger - \tau\psi(p^\dagger)) = F_\varsigma proj_{\mathbb{C}}[p^\dagger - \varsigma f(p^\dagger)] = p^\dagger$. Hence,

$$\begin{aligned}\|u_t - p^\dagger\| &= \|F_\varsigma proj_{\mathbb{C}}[t\gamma g(x_t) + (I - t\phi)(x_t - \varsigma f(x_t))] - F_\varsigma proj_{\mathbb{C}}[(p^\dagger - \varsigma f(p^\dagger))]\| \\ &\le \|t\gamma g(x_t) + (I - t\phi)(x_t - \varsigma f(x_t)) - (p^\dagger - \varsigma f(p^\dagger))\| \\ &= \|t\gamma(g(x_t) - g(p^\dagger)) + (I - t\phi)[x_t - \varsigma f(x_t) - (p^\dagger - \varsigma f(p^\dagger))] \\ &\quad + t[\gamma g(p^\dagger) - \phi(p^\dagger - \varsigma f(p^\dagger))]\| \\ &\le t\gamma\|g(x_t) - g(p^\dagger)\| + |I - t\phi|\|x_t - \varsigma f(x_t) - (p^\dagger - \varsigma f(p^\dagger))\| \\ &\quad + t\|\gamma g(p^\dagger) - \phi(p^\dagger - \varsigma f(p^\dagger))\| \\ &\le [1 - (\sigma - \gamma\kappa)t]\|x_t - p^\dagger\| + t\|\gamma g(p^\dagger) - \phi(p^\dagger - \varsigma f(p^\dagger))\|.\end{aligned} \tag{7}$$

According to (5) and (7), we have

$$\begin{aligned}\|x_t - p^\dagger\| &= \|\mu J_\tau^\Psi(I - \tau\psi)x_t + (1 - \mu)u_t - p^\dagger\| \\ &\le \mu\|J_\tau^\Psi(I - \tau\psi)x_t - J_\tau^\Psi(p^\dagger - \tau\psi(p^\dagger))\| + (1 - \mu)\|u_t - p^\dagger\| \\ &\le \mu\|x_t - p^\dagger\| + (1 - \mu)\|u_t - p^\dagger\|.\end{aligned}$$

It follows that

$$\begin{aligned}\|x_t - p^\dagger\| &\le \|u_t - p^\dagger\| \\ &\le [1 - (\sigma - \gamma\kappa)t]\|x_t - p^\dagger\| + t\|\gamma g(p^\dagger) - \phi(p^\dagger - \varsigma f(p^\dagger))\|,\end{aligned} \tag{8}$$

which implies that

$$\|x_t - p^\dagger\| \le \frac{\|\gamma g(p^\dagger) - \phi(p^\dagger - \varsigma f(p^\dagger))\|}{\sigma - \gamma\kappa}.$$

So, $\{x_t\}$ is bounded.

Step 3. $\{x_t\}$ is relatively norm compact as $t \to 0+$.

Taking into account (7) and Lemma 2.2, we obtain

$$\begin{aligned}\|u_t - p^\dagger\|^2 &\le \|t(\gamma g(x_t) - \phi(p^\dagger - \varsigma f(p^\dagger))) + (I - t\phi)[x_t - \varsigma f(x_t) - (p^\dagger - \varsigma f(p^\dagger))]\|^2 \\ &\le [t\|\gamma g(x_t) - \phi(p^\dagger - \varsigma f(p^\dagger))\| + |I - t\phi|\|x_t - \varsigma f(x_t) - (p^\dagger - \varsigma f(p^\dagger))\|]^2 \\ &\le [t\sigma\|\gamma g(x_t) - \phi(p^\dagger - \varsigma f(p^\dagger))\|/\sigma + (1 - \sigma t)\|x_t - \varsigma f(x_t) - (p^\dagger - \varsigma f(p^\dagger))\|]^2 \\ &\le t\|\gamma g(x_t) - \phi(p^\dagger - \varsigma f(p^\dagger))\|^2/\sigma + (1 - \sigma t)\|x_t - \varsigma f(x_t) - (p^\dagger - \varsigma f(p^\dagger))\|^2 \\ &\le t\|\gamma g(x_t) - \phi(p^\dagger - \varsigma f(p^\dagger))\|^2/\sigma + (1 - \sigma t)[\|x_t - p^\dagger\|^2 \\ &\quad - \varsigma(2\alpha - \varsigma)\|f(x_t) - f(p^\dagger)\|^2].\end{aligned} \tag{9}$$

Combining (8) and (9), we receive

$$\begin{aligned}\|x_t - p^\dagger\|^2 &\le \|u_t - p^\dagger\|^2 \\ &\le t\|\gamma g(x_t) - \phi(p^\dagger - \varsigma f(p^\dagger))\|^2/\sigma + (1 - \sigma t)[\|x_t - p^\dagger\|^2 \\ &\quad - \varsigma(2\alpha - \varsigma)\|f(x_t) - f(p^\dagger)\|^2].\end{aligned}$$

It follows that

$$(1 - \sigma t)\varsigma(2\alpha - \varsigma)\|f(x_t) - f(p^\dagger)\|^2 \le t\|\gamma g(x_t) - \phi(p^\dagger - \varsigma f(p^\dagger))\|^2/\sigma \to 0 \ (t \to 0+).$$

Therefore,

$$\lim_{t \to 0+} \|f(x_t) - f(p^\dagger)\| = 0. \tag{10}$$

Owing to (5) and Lemma 2.1, we achieve

$$\begin{aligned}\|u_t - p^\dagger\|^2 &= \|F_\varsigma proj_{\mathbb{C}}[t\gamma g(x_t) + (I - t\phi)(x_t - \varsigma f(x_t))] - F_\varsigma proj_{\mathbb{C}}[(p^\dagger - \varsigma f(p^\dagger))]\|^2 \\ &\le \langle [t\gamma g(x_t) + (I - t\phi)(x_t - \varsigma f(x_t))] - [(p^\dagger - \varsigma f(p^\dagger))], u_t - p^\dagger\rangle \\ &= \frac{1}{2}\big(\big\|[t\gamma g(x_t) + (I - t\phi)(x_t - \varsigma f(x_t))] - [(p^\dagger - \varsigma f(p^\dagger))]\big\|^2 \\ &\quad + \|u_t - p^\dagger\|^2 - \big\|[t\gamma g(x_t) + (I - t\phi)(x_t - \varsigma f(x_t))] \\ &\quad - [(p^\dagger - \varsigma f(p^\dagger))] - u_t + p^\dagger\big\|^2\big).\end{aligned} \tag{11}$$

So,

$$\begin{aligned}\|u_t - p^\dagger\|^2 &\le \big\|[t\gamma g(x_t) + (I - t\phi)(x_t - \varsigma f(x_t))] - [(p^\dagger - \varsigma f(p^\dagger))]\big\|^2 \\ &\quad - \big\|[t\gamma g(x_t) + (I - t\phi)(x_t - \varsigma f(x_t))] - [(p^\dagger - \varsigma f(p^\dagger))] - u_t + p^\dagger\big\|^2 \\ &\le t\|\gamma g(x_t) - \phi(p^\dagger - \varsigma f(p^\dagger))\|^2/\sigma + (1 - \sigma t)\|x_t - p^\dagger\|^2 \\ &\quad - \big\|[t\gamma g(x_t) + (I - t\phi)(x_t - \varsigma f(x_t))] - u_t + \varsigma f(p^\dagger)\big\|^2.\end{aligned}$$

This together with (8) implies that

$$\begin{aligned}\|x_t - p^\dagger\|^2 &\le \|u_t - p^\dagger\|^2 \\ &\le t\|\gamma g(x_t) - \phi(p^\dagger - \varsigma f(p^\dagger))\|^2/\sigma + (1 - \sigma t)\|x_t - p^\dagger\|^2 \\ &\quad - \big\|[t\gamma g(x_t) + (I - t\phi)(x_t - \varsigma f(x_t))] - u_t + \varsigma f(p^\dagger)\big\|^2.\end{aligned}$$

Hence,

$$\big\|[t\gamma g(x_t) + (I - t\phi)(x_t - \varsigma f(x_t))] - u_t + \varsigma f(p^\dagger)\big\|^2 \le t\|\gamma g(x_t) - \phi(p^\dagger - \varsigma f(p^\dagger))\|^2/\sigma \to 0.$$

With the help of (10), we deduce

$$\lim_{t\to 0+} \big\|x_t - u_t\big\| = 0. \tag{12}$$

Then,

$$\lim_{t\to 0+} \|x_t - J_\tau^\Psi(I - \tau\psi)x_t\| = \lim_{t\to 0+}(1 - \mu)\|x_t - u_t\| = 0. \tag{13}$$

Thanks to (12), we attain

$$\begin{aligned}\|u_t - p^\dagger\|^2 &\le \langle [t\gamma g(x_t) + (I - t\phi)(x_t - \varsigma f(x_t))] - [(p^\dagger - \varsigma f(p^\dagger))], u_t - p^\dagger\rangle \\ &\le t\gamma\langle g(x_t) - g(p^\dagger), u_t - p^\dagger\rangle + (I - t\phi)\langle (x_t - \varsigma f(x_t)) - (p^\dagger - \varsigma f(p^\dagger)), u_t - p^\dagger\rangle \\ &\quad + t\langle \gamma g(p^\dagger) - \phi(p^\dagger - \varsigma f(p^\dagger)), u_t - p^\dagger\rangle \\ &\le t\gamma\|g(x_t) - g(p^\dagger)\|\|u_t - p^\dagger\| + |I - t\phi|\|(x_t - \varsigma f(x_t)) - (p^\dagger - \varsigma f(p^\dagger))\| \\ &\quad \times \|u_t - p^\dagger\| + t\langle \gamma g(p^\dagger) - \phi(p^\dagger - \varsigma f(p^\dagger)), u_t - p^\dagger\rangle \\ &\le [1 - (\sigma - \gamma\kappa)t]\|x_t - p^\dagger\|\|u_t - p^\dagger\| + t\langle \gamma g(p^\dagger) - \phi(p^\dagger - \varsigma f(p^\dagger)), u_t - p^\dagger\rangle \\ &\le \frac{1 - (\sigma - \gamma\kappa)t}{2}\|x_t - p^\dagger\|^2 + \frac{1}{2}\|u_t - p^\dagger\|^2 \\ &\quad + t\langle \gamma g(p^\dagger) - \phi(p^\dagger - \varsigma f(p^\dagger)), u_t - p^\dagger\rangle.\end{aligned}$$

Hence,

$$\begin{aligned}\|x_t - p^\dagger\|^2 \le \|u_t - p^\dagger\|^2 &\le [1 - (\sigma - \gamma\kappa)t]\|x_t - p^\dagger\|^2 \\ &\quad + 2t\langle \gamma g(p^\dagger) - \phi(p^\dagger - \varsigma f(p^\dagger)), u_t - p^\dagger\rangle.\end{aligned}$$

It follows that

$$\|x_t - p^\dagger\|^2 \le \frac{2}{\sigma - \gamma\kappa}\langle \gamma g(p^\dagger) - \phi(p^\dagger - \varsigma f(p^\dagger)), u_t - p^\dagger\rangle. \tag{14}$$

Next we show that $\{x_t\}$ is relatively norm compact as $t \to 0+$. Let $\{t_n\} \subset (0,1)$ be a sequence such that $t_n \to 0$ as $n \to \infty$. Put $x_n^{(1)} := x_{t_n}$ and $u_n^{(1)} := u_{t_n}$. From (13), we get

$$\lim_{n\to\infty} \|x_n^{(1)} - J_\tau^\Psi(I - \tau\psi)x_n^{(1)}\| = 0. \tag{15}$$

By (14), we have

$$\|x_n^{(1)} - p^\dagger\|^2 \le \frac{2}{\sigma - \gamma\kappa}\langle \gamma g(p^\dagger) - \phi(p^\dagger - \varsigma f(p^\dagger)), u_n^{(1)} - p^\dagger\rangle. \tag{16}$$

Since $\{x_n^{(1)}\}$ is bounded, there exists a subsequence $\{x_{n_i}^{(1)}\} \subset \{x_n^{(1)}\}$ such that $x_{n_i}^{(1)} \rightharpoonup x^\dagger \in \mathcal{C}$ as $i \to \infty$. Applying Lemma 2.3, we deduce that $x^\dagger \in \mathrm{Fix}(J_\tau^\Psi(I - \tau\psi)) = \mathrm{Sol}(\mathcal{C}, \Psi, \psi)$.

On the other hand, utilizing (12), we have $u_{n_i}^{(1)} \rightharpoonup x^\dagger$. Note that

$$u_{n_i}^{(1)} = \Phi_{t_{n_i}}(x_{n_i}^{(1)}) = F_\varsigma proj_{\mathcal{C}}[t_{n_i}\gamma g(x_{n_i}^{(1)}) + (I - t_{n_i}\phi)(x_{n_i}^{(1)} - \varsigma f(x_{n_i}^{(1)}))]$$

and the operator $F_\varsigma proj_{\mathcal{C}}(I - \varsigma f)$ is nonexpansive. By Lemma 2.3, we get $x^\dagger \in \mathrm{Sol}(\mathcal{C}, \varphi, f)$. Therefore, $x^\dagger \in \Gamma$. Substituting $p^\dagger$ with $x^\dagger$ in (16), we obtain

$$\|x_{n_i}^{(1)} - x^\dagger\|^2 \le \frac{2}{\sigma - \gamma\kappa}\langle \gamma g(x^\dagger) - \phi(x^\dagger - \varsigma f(x^\dagger)), u_{n_i}^{(1)} - x^\dagger\rangle. \tag{17}$$

Since $u_{n_i}^{(1)} \rightharpoonup x^\dagger \in \mathcal{C}$, it follows from (17) that $x_{n_i}^{(1)} \to x^\dagger$. This has proved the relative norm-compactness of the net $\{x_t\}$ as $t \to 0+$.

Step 4. The whole net $x_t \to x^\dagger$ as $t \to 0+$.

Since $\{x_t\}$ is relatively norm compact as $t \to 0+$. Let $\{s_n\} \subset (0,1)$ be another sequence such that $s_n \to 0$ as $n \to \infty$. Set $x_n^{(2)} := x_{s_n}$ and $u_n^{(2)} := u_{s_n}$. Since $\{x_n^{(2)}\}$ is bounded, there is another subsequence $\{x_{n_j}^{(2)}\} \subset \{x_n^{(2)}\}$ satisfying $x_{n_j}^{(2)} \rightharpoonup y^\dagger$ as $j \to \infty$. Consequently, we deduce $y^\dagger \in \Gamma$ and

$$\|x_{n_j}^{(2)} - y^\dagger\|^2 \le \frac{2}{\sigma - \gamma\kappa}\langle \gamma g(y^\dagger) - \phi(y^\dagger - \varsigma f(y^\dagger)), u_{n_j}^{(2)} - y^\dagger\rangle. \tag{18}$$

Since $u_{n_j}^{(2)} \rightharpoonup y^\dagger \in \mathcal{C}$, it follows from (18) that $x_{n_j}^{(2)} \to y^\dagger$. Take into account of (14), we acquire

$$\|x_{n_i}^{(1)} - y^\dagger\|^2 \le \frac{2}{\sigma - \gamma\kappa}\langle \gamma g(y^\dagger) - \phi(y^\dagger - \varsigma f(y^\dagger)), u_{n_i}^{(1)} - y^\dagger\rangle. \tag{19}$$

and

$$\|x_{n_j}^{(2)} - x^\dagger\|^2 \le \frac{2}{\sigma - \gamma\kappa}\langle \gamma g(x^\dagger) - \phi(x^\dagger - \varsigma f(x^\dagger)), u_{n_j}^{(2)} - x^\dagger\rangle. \tag{20}$$

Letting $i \to \infty$ in (19) and noting that $u_{n_i}^{(1)} \rightharpoonup x^\dagger (i \to \infty)$, we have

$$\langle \gamma g(y^\dagger) - \phi(y^\dagger - \varsigma f(y^\dagger)), x^\dagger - y^\dagger\rangle \ge 0. \tag{21}$$

Letting $j \to \infty$ in (20) and noting that $u_{n_j}^{(2)} \rightharpoonup y^\dagger (j \to \infty)$, we have

$$\langle \gamma g(x^\dagger) - \phi(x^\dagger - \varsigma f(x^\dagger)), y^\dagger - x^\dagger\rangle \ge 0. \tag{22}$$

Combining (21) and (22), we conclude that $x^\dagger = y^\dagger$. Therefore, the whole net $x_t \to x^\dagger$ as $t \to 0+$. The proof is completed. □

4. **Conclusions**

Equilibrium problems and quasi-variational inclusion problems provide a unified frame for solving many problems arising from science and engineering. In this paper, we investigate a common problem of the generalized equilibrium problem (2) and the quasi-variational inclusion (3) in Hilbert spaces. With the help of resolvent method, we propose an implicit algorithm [Algorithm 3.1] for finding a common solution of the generalized equilibrium problem (2) and the quasi-variational inclusion (3). We show that the net $\{x_t\}$ defined by Algorithm 3.1 weakly converges to a common solution of the generalized equilibrium problem (2) provided some mild conditions are satisfied.

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