INSTABILITY OF A FIFTH ORDER NON-LINEAR VECTOR DELAY DIFFERENTIAL EQUATION WITH MULTIPLE DEVIATING ARGUMENTS

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In this work, we study a fifth order non-linear vector delay differential equation with multiple deviating arguments. Some criteria for guaranteeing the instability of zero solution of the equation are given by using the Lyapunov-Krasovskii functional approach. Comparing with the previous literature, our result is new and complements some known results.

Keywords: Vector differential equation, fifth order, instability, multiple deviating arguments.
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1. Introduction

In applied sciences, some practical problems concerning mechanics, the engineering technique fields, economy, control theory, physics, chemistry, biology, medicine, atomic energy, information theory, etc. are associated with certain differential equations of the higher order with and without delay. Perhaps, the most effective basic tool in the literature to investigate the qualitative behaviors of certain differential equations of the higher order with and without delay is the Lyapunov function and Lyapunov-Krasovskii functional approach. Lyapunov functions and functionals have been successfully used and are still being used to obtain stability, instability, boundedness and the existence of periodic solutions of differential equations, differential equations with functional delays and functional differential equations (Ezeilo [2], Li and Duan [4], Li and Yu [5], Sadek [6], Sun and Hou [7], Tiryaki [8], Tunç [9-17], Tunç and Erdogan [18], Tunç and Karta [19], Tunç and Şevli [20]).

It should be noted that in 2000, using the Lyapunov function approach, Li and Duan [4] discussed the instability of the solutions of the fifth order nonlinear scalar differential equation without delay

$$x^{(5)}(t) + f_5(x(t), x'(t), ..., x^{(4)}(t))x^{(4)}(t) + f_4(x(t), x'(t), ..., x^{(4)}(t))x^{(3)}(t) + f_3(x''(t)) + f_2(x'(t)) + a_1 x(t) = 0.$$

Later, in 2011, using the same technique, Tunç [14] studied the instability of the solutions for the fifth order nonlinear scalar delay differential equation

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$$x^{(5)}(t)+f_5(x(t-r),x'(t-r),...,x^{(4)}(t-r))x^{(4)}(t)$$
$$+f_4(x(t-r),x'(t-r),...,x^{(4)}(t-r))x^{(3)}(t)$$
$$+f_3(x''(t-r))+f_2(x'(t))+a_1x(t)=0.$$

Therefore, it is worthwhile to continue the discussion of the instability to differential equations of the higher order.

In this paper, we consider the fifth order nonlinear vector delay differential equation with multiple deviating arguments:

$$X^{(5)}(t)+F(X(t),X(t-\tau_1),...,X(t-\tau_n),...,X^{(4)}(t),...,X^{(4)}(t-\tau_n))X^{(4)}(t)$$
$$+G(X(t),X(t-\tau_1),...,X(t-\tau_n),...,X^{(4)}(t),...,X^{(4)}(t-\tau_n))X^{(3)}(t)$$
$$+\sum_{i=1}^{n}H_i(X''(t-\tau_i))+\Psi(X'(t))+AX(t)=0, \tag{1}$$

where the primes in Eq. (1) denote differentiation with respect to t, $t\in\Re_+$, $\Re_+=[0,\infty)$;
$X\in\Re^n$, τ_i are certain positive constants, the fixed delays, $t-\tau_i>0$, A is an $n\times n-$ symmetric matrix, F and G are continuous $n\times n-$symmetric matrix functions for the respective arguments, $H_i:\Re^n\to\Re^n$, $\Psi:\Re^n\to\Re^n$ with $H_i(0)=\Psi(0)=0$ are continuous for the respective arguments. Let $J_{H_i}(X'')$ and $J_\Psi(X')$ denote the linear operators from $H_i(X'')$ and $\Psi(X')$ to

$$J_{H_1}(X'')=\left(\frac{\partial h_{1i}}{\partial x''_j}\right),...,J_{H_n}(X'')=\left(\frac{\partial h_{ni}}{\partial x''_j}\right)$$

and

$$J_\Psi(X')=\left(\frac{\partial\psi_i}{\partial x'_j}\right),\ (i,j=1,2,...,n),$$

where $(x'_1,...,x'_n)$, $(x''_1,...,x''_n)$, $(h_{1i}),...,(h_{ni})$ and $(\psi_1,...,\psi_n)$ are the components of X', X'', H_i and Ψ, respectively. In what follows, it is assumed that $J_{H_i}(X'')$ and $J_\Psi(X')$ exist and are symmetric and continuous.

Equation (1) is the vector version for systems of real nonlinear differential equations of the fifth order:

$$x_i^{(5)}(t)+\sum_{k=1}^{n}f_{ik}(x(t),x(t-\tau_1),...,x(t-\tau_n),...,x^{(4)}(t-\tau_1),...,x^{(4)}(t-\tau_n))x_k^{(4)}(t)$$
$$+\sum_{k=1}^{n}g_{ik}(x(t),x(t-\tau_1),...,x(t-\tau_n),...,x^{(4)}(t-\tau_1),...,x^{(4)}(t-\tau_n))x_k^{(3)}(t)$$

$$+h_{1_i}(x_1''(t-\tau_1),...,x_n''(t-\tau_n))+...+h_{n_i}(x_1''(t-\tau_1),...,x_n''(t-\tau_n))$$

$$+\psi_i(x_1'(t-\tau_1),...,x_n'(t-\tau_n))+\sum_{k=1}^{n}a_{ik}x_k(t)=0,$$

$$(i=1,2,...,n).$$

Instead of Eq. (1), we consider the corresponding differential system

$$\dot{X}(t)=Y(t),\ \dot{Y}(t)=Z(t),\quad \dot{Z}(t)=W(t),\quad \dot{W}(t)=U(t),$$

$$\dot{U}(t)=-F(X(t),...,X(t-\tau_n),...,U(t),...,U(t-\tau_n))U(t)$$

$$-G(X(t),...,X(t-\tau_n),...,U(t),...,U(t-\tau_n))W(t)$$

$$-\sum_{i=1}^{n}H_i(Z(t))+\sum_{i=1}^{n}\int_{t-\tau_i}^{t}J_{H_i}(Z(s))W(s)ds$$

$$-\Psi(Y(t))-AX(t), \tag{2}$$

which was obtained by setting $\dot{X}(t)=Y(t)$, $\ddot{X}(t)=Z(t)$, $X^{(3)}(t)=W(t)$ and $X^{(4)}(t)=U(t)$ from Eq. (1).

It is worth mentioning that a review to date of the literature indicates that the instability of solutions of vector differential equations of the fifth order with a deviating argument has not been investigated up to now. This paper is the first known work regarding the instability of solutions for the nonlinear vector delay differential equations of the fifth order with multiple deviating arguments. The motivation of this paper comes from the above papers done on scalar nonlinear differential equations of the fifth order without and with delay and the vector differential equations of the fifth order without delay. Defining a Lyapunov-Krasovskii functional and taking into account the Krasovskii's criteria [3], we prove our main result on the subject.

Note that the instability criteria of Krasovskii [3] can be summarized as the following: According to these criteria, it is necessary to show here that there exists a Lyapunov- Krasovskii functional $V(.)\equiv V(X_t,Y_t,Z_t,W_t,U_t)$ which has Krasovskii properties, say (P_1), (P_2) and (P_3):

(P_1) In every neighborhood of $(0,0,0,0,0)$, there exists a point $(\xi_1,...,\xi_5)$ such that $V(\xi_1,...,\xi_5)>0$,

(P_2) the time derivative $\frac{d}{dt}V(.)$ along solution paths of (2) is positive semi-definite,

(P_3) the only solution $(X,Y,Z,W,U)=(X(t),Y(t),Z(t),W(t),U(t))$ of (2) which satisfies $\frac{d}{dt}V(.)=0$, $(t\ge 0)$, is the trivial solution $(0,0,0,0,0)$.

The symbol $\langle X,Y\rangle$ corresponding to any pair X, Y in $\Re^n$ stands for the usual scalar $\langle X,Y\rangle=\sum_{i=1}^{n} x_i y_i$; thus $\langle X,X\rangle=\|X\|^2$, and $\lambda_i(\Omega)$, $(i=1,2,...,n)$, are the eigenvalues of the real symmetric $n\times n-$ matrix Ω. The matrix Ω is said to be negative semidefinite, when $\langle \Omega X,X\rangle\le 0$ for all nonzero X in $\Re^n$.

By this work, we improve the results in ([4], [14]) to a vector delay differential equation of the fifth order with multiple deviating arguments. The result to be obtained is new and has a contribution to the topic, and may be useful for the researchers working on the qualitative behaviors of solutions of the differential equations.

2. Results used

Before stating the main result, we need the following result.

Lemma. Let A be a real symmetric $n\times n$-matrix and

$$a'\ge \lambda_i(A)\ge a>0,\ (i=1,2,...,n),$$

where a' and a are constants.

Then $$a'\langle X,X\rangle\ge\langle AX,X\rangle\ge a\langle X,X\rangle$$

and $$a'^2\langle X,X\rangle\ge\langle AX,AX\rangle\ge a^2\langle X,X\rangle$$

(Bellman [1]).

Let $r\ge 0$ be given, and let $C=C([-r,0],\ \Re^n)$ with $\|\phi\|=\max_{-r\le s\le 0}|\phi(s)|,\ \phi\in C.$

For $H>0$ define $C_H\subset C$ by $C_H=\{\phi\in C:\|\phi\|<H\}$.

If $x:[-r,A)\to\Re^n$ is continuous, $0<A\le\infty$, then, for each t in $[0,A)$, x_t in C is defined by

$$x_t(s)=x(t+s),\ -r\le s\le 0,\ t\ge 0.$$

Let G be an open subset of C and consider the general autonomous delay differential system with finite delay

$$\dot{x}=F(x_t),\ F(0)=0,\ x_t=x(t+\theta),\ -r\le\theta\le 0,\ t\ge 0,$$

where $F:G\to\Re^n$ is continuous and maps closed and bounded sets into bounded sets. It follows from these conditions on F that each initial value problem

$$\dot{x}=F(x_t),\ x_0=\phi\in G$$

has a unique solution defined on some interval $[0,A)$, $0<A\le\infty$. This solution will be denoted by $x(\phi)(.)$ so that $x(\phi)(0)=\phi.$

Definition. The zero solution, $x = 0$, of $\dot{x} = F(x_t)$ is stable if for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $\|\phi\| < \delta$ implies that $|x(\phi)(t)| < \varepsilon$ for all $t \geq 0$. The zero solution is said to be unstable if it is not stable.

3. Main result

The main result of this paper is given by the following theorem.

Theorem. In addition to the basic assumptions imposed on F, G, H_i, Ψ and A that appear in Eq. (1), we assume that there exist positive constants a_1, a_4, β_i and δ such that the following conditions hold:

$$F(.),\ G(),\ J_{H_i}(.),\ J_{\Psi}(.) \text{ and } A \text{ are symmetric},$$

$$H_i(0) = 0,\ H_i(Z) \neq 0,\ (Z \neq 0),\ \Psi(0) = 0,\ \Psi(Y) \neq 0,\ (Y \neq 0),$$

$$\lambda_i(A)) \leq -a_1,\ \lambda_i(J_{\Psi}(Y)) \geq a_4,\ \left|\lambda_i(J_{H_i}(Z))\right| \leq \beta_i$$

and

$$\frac{1}{4}\lambda_i(F(.))^2 + \lambda_i(G(.)) \leq -\delta.$$

If

$$\tau < \frac{\delta}{\sqrt{n}(\sum_{i=1}^{n}\beta_i)},$$

then the zero solution of Eq. (1) is unstable.

Proof. We define a Lyapunov-Krasovskii functional $V(.) = V(X_t, Y_t, Z_t, W_t, U_t)$:

$$V(.) = \langle W, U\rangle + \langle AX, Z\rangle - \frac{1}{2}\langle AY, Y\rangle + \langle \Psi(Y), Z\rangle$$

$$+\sum_{i=1}^{n}\int_0^1 \langle H_i(\sigma Z), Z\rangle d\sigma - \sum_{i=1}^{n}\mu_i \int_{-\tau_i}^{0}\int_{t+s}^{t}\|W(\theta)\|^2 d\theta ds,$$

where μ_i are certain positive constants and will be determined later in the proof.

It is clear that $V(0,0,0,0,0) = 0$.

Let

$$\bar{\varepsilon} = (\varepsilon_{21}, ..., \varepsilon_{2n}).$$

Using the assumption $\lambda_i(A) \leq -a_1$, we have

$$V(0, \bar{\varepsilon}, 0,0,0) = -\frac{1}{2}\langle A\bar{\varepsilon}, \bar{\varepsilon}\rangle \geq \frac{1}{2}a_1\|\bar{\varepsilon}\|^2 > 0$$

for all arbitrary $\bar{\varepsilon} \neq 0$, $\bar{\varepsilon} \in \Re^n$, which verify the property (P_1) of Krasovskii [3].

Using a basic calculation, the time derivative of $V(.)$ along the solutions of (2) results in

$$\frac{d}{dt}V(.)=\langle U,U\rangle-\langle F(.)W,U\rangle-\langle G(.)W,W\rangle$$
$$+\langle J_{\Psi}(Y)Z,Z\rangle+\sum_{i=1}^{n}<W,\int_{t-\tau_i}^{t}J_{H_i}(Z(s))W(s)ds>$$
$$-<\sum_{i=1}^{n}\mu_i\tau_iW,W>+\sum_{i=1}^{n}\mu_i\int_{t-\tau_i}^{t}\|W(\theta)\|^2d\theta.$$

Under the assumptions of the theorem and the Schwarz inequality, it can be easily seen that

$$<W,\int_{t-\tau_i}^{t}J_{H_i}(Z(s))W(s)ds>\geq-\|W\|\left\|\int_{t-\tau_i}^{t}J_{H_i}(Z(s))W(s)ds\right\|$$
$$\geq-\sqrt{n}\beta_i\|W\|\left\|\int_{t-\tau_i}^{t}W(s)\right\|ds$$
$$\geq-\sqrt{n}\beta_i\|W\|\int_{t-\tau_i}^{t}\|W(s)\|ds$$
$$\geq-\frac{1}{2}\sqrt{n}\beta_i\tau_i\|W\|^2-\frac{1}{2}\sqrt{n}\beta_i\int_{t-\tau_i}^{t}\|W(s)\|^2ds,$$
$$(i=1,2,\dots,n).$$

Hence

$$\dot{V}(.)=\|U-2^{-1}F(.)W\|^2-\frac{1}{4}\langle F(.)W,F(.)W\rangle-\langle G(.)W,W\rangle$$
$$+\langle J_{\Psi}(Y)Z,Z\rangle-(\sum_{i=1}^{n}\mu_i\tau_i)\|W\|^2-\frac{1}{2}(\sqrt{n}\sum_{i=1}^{n}\beta_i\tau_i)\|W\|^2$$
$$+\sum_{i=1}^{n}(\mu_i-\frac{1}{2}\sqrt{n}\beta_i)\int_{t-\tau_i}^{t}\|W(s)\|^2ds.$$

Let $\mu_i=\frac{1}{2}\sqrt{n}\beta_i$. Then, using the assumptions of the theorem and the estimate $\tau=\max\tau_i$, we get

$$\dot{V}(.)=\|U-2^{-1}F(.)W\|^2-\frac{1}{4}\langle F(.)W,F(.)W\rangle-\langle G(.)W,W\rangle$$
$$+\langle J_{\Psi}(Y)Z,Z\rangle-(\sqrt{n}\sum_{i=1}^{n}\beta_i\tau_i)\|W\|^2$$
$$\geq-\frac{1}{4}\langle F(.)W,F(.)W\rangle-\langle G(.)W,W\rangle+\langle J_{\Psi}(Y)Z,Z\rangle$$

$$-(\sqrt{n}\sum_{i=1}^{n}\beta_i\tau_i)\|W\|^2$$

$$\geq a_4\|Z\|^2 + \{\delta - \sqrt{n}(\sum_{i=1}^{n}\beta_i)\tau\}\|W\|^2 > 0.$$

If $\tau < \dfrac{\delta}{\sqrt{n}(\sum_{i=1}^{n}\beta_i)}$, then, for some positive constant k_1, we have

$$\dot{V}(.) \geq a_4\|Z\|^2 + k_1\|W\|^2 > 0,$$

which verifies that V has the property (P_2) of Krasovskii [3].

Besides,

$$\dot{V}(.) = 0 \Leftrightarrow Z = W = U = 0.$$

The substitution of this estimate into system (2) results in

$$\Psi(Y) + AX = 0.$$

That is,

$$\Psi(X') + AX = 0.$$

Because, $Z = X'' = 0$, $X' =$constant vector, for all $t > 0$. Hence, since A is not the zero matrix, we have $X =$constant vector, for all $t > 0$. But, in view of the assumptions of the theorem, this implies that $X' = 0$ and thus also, by $\Psi(X') + AX = 0$, that $X = 0$, for all $t > 0$. These estimates result in $X = Y = Z = W = U = 0$. Hence, the property (P_3) of Krasovskii [3] holds for the Lyapunov -Krasovskii functional $V(.)$.

The proof of the theorem is completed.

4. Conclusion

A non-linear vector delay differential equation of the fifth order with multiple deviating arguments is considered. The instability of the zero solution of this equation is discussed. In proving our result, we employ the Lyapunov-Krasovskii functional approach by defining a new Lyapunov- Krasovskii functional. Comparing with the previous literature, our result is new and complements some known results.

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