EXISTENCE AND MULTIPLE SOLUTIONS FOR DISCRETE DIRICHLET BOUNDARY VALUE PROBLEMS VIA VARIATIONAL METHODS

Xia Liu[1], Yuanbiao Zhang[2], Haiping Shi[3]

In this paper, a second order nonlinear difference equation with Jacobi operators is considered. Using the critical point theory, we obtain the existence and multiplicity of solutions of Dirichlet boundary value problems and give some new results.

Keywords: Existence and multiple solutions, boundary value problem, Linking Theorem, critical point theory

MSC2010: 47B36, 39A10

1. Introduction

The second order forward-backward differential-difference equation

$$c^2u''(t) = V'(u(t+1) - u(t)) - V'(u(t) - u(t-1)),\ t \in \mathbf{R} \tag{1}$$

has been studied extensively by many scholars. For example, Smets and Willem [25] have obtained the existence of solitary waves of Eq. (1).

A generalization of Eq. (1) is the following equation

$$Su(t) = f(t, u(t+1), u(t), u(t-1)),\ t \in \mathbf{R}. \tag{2}$$

Here S is the Sturm-Liouville differential expression and $f \in C(\mathbf{R}^4, \mathbf{R})$.

Consider the second order difference equation

$$Lu_n = f(n, u_{n+1}, u_n, u_{n-1}), \tag{3}$$

with boundary value conditions

$$u_0 = A,\ u_{k+1} = B, \tag{4}$$

where the operator L is the Jacobi operator

$$Lu_n = a_n u_{n+1} + a_{n-1}u_{n-1} + b_n u_n,$$

[1]Lecturer, Oriental Science and Technology College, Hunan Agricultural University, Changsha, 410128, China, e-mail: `xia991002@163.com`

[2]Dr., Packaging Engineering Institute, Jinan University, Zhuhai, 519070, China, e-mail: `abiaoa@163.com`

[3]Corresponding author, Dr., Basic Courses Department, Guangdong Construction Vocational Technology Institute, Guangzhou, 510450, China, e-mail: `shp7971@163.com`

This project is supported by Specialized Research Fund for the Doctoral Program of Higher Eduction of China (No: 20114410110002), National Natural Science Foundation of China (No: 11171078) and Scientific Research Fund of Hunan Provincial Education Department (No: 12C0170).

a_n and b_n are real valued for each $n \in \mathbf{Z}$, $f \in C(\mathbf{R}^4, \mathbf{R})$, A and B are constants. Eq. (3) can be considered as a discrete analogue of Eq. (2). L leads to a symmetric matrix representation.

The theory of nonlinear difference equations has been widely used to study discrete models appearing in many fields such as computer science, economics, neural network, ecology, cybernetics, etc. For example, the simple logistic equation

$$u_{n+1} = ru_n$$

is a formula for approximating the evolution of an animal population over time, where u_n is the number of animals this year, u_{n+1} is the number next year and r is the growth rate or fecundity. The the price-demand curve of cobweb phenomenon

$$D_n = -m_d p_n + b_d, \ m_d > 0, \ b_d > 0$$

is the economics application of difference equations, where D_n is the number of units demanded in period n, p_n is the price per unit in period n and m_d represents the sensitivity of consumers to price. Since the last decade, there has been much literature on qualitative properties of difference equations, those studies over many of the branches of difference equations, such as [1,3,11,15,19,24] and the references therein.

Let $\mathbf{N}$, $\mathbf{Z}$ and $\mathbf{R}$ denote the sets of all natural numbers, integers and real numbers respectively. For a, $b \in \mathbf{Z}$, define $\mathbf{Z}(a) = \{a, a+1, \cdots\}$, $\mathbf{Z}(a, b) = \{a, a+1, \cdots, b\}$ when $a \leq b$. Δ is the forward difference operator defined by $\Delta u_n = u_{n+1} - u_n$. k is a positive integer and * is the transpose sign for a vector.

In recent years, the study of boundary value problems for differential equations develops at relatively rapid rate. By using various methods and techniques, such as Schauder fixed point theorem, the cone theoretic fixed point theorem, the method of upper and lower solutions, coincidence degree theory, a series of existence results of nontrivial solutions for differential equations have been obtained in literatures, we refer to [2,4-8,13,26,29]. And critical point theory is also an important tool to deal with problems on differential equations [18,22,29]. Because of applications in many areas for difference equations [1,16,20,21,23], recently, a few authors have gradually paid attention to applying critical point theory to deal with boundary value problems on discrete systems, see [3,27,28,30]. We also refer to [27,28] for the discrete boundary value problems.

Since the last decade, there has been much progress on the qualitative properties of difference equations, which included results on stability and attractivity and results on oscillation and other topics, see [1,3,4,9,12-14,17,27-30]. However, to our best knowledge, no similar results are obtained in the literature for the boundary value problem (BVP) (3) with (4). Since f in Eq. (3) depends on u_{n+1} and u_{n-1}, the traditional ways of establishing the functional in [3,27-29] are inapplicable to our case.

Our aim in this paper is to use the critical point theory to give some sufficient conditions for the existence and multiplicity of the BVP (3) with (4). The main idea in this paper is to transfer the existence of the BVP (3) with (4) into the existence of the critical points of some functional.

Our main results are as follows.

Let

$p_{\max} = \max\{a_n : n \in \mathbf{Z}(0,k)\},\ p_{\min} = \min\{a_n : n \in \mathbf{Z}(0,k)\},$

$p = \max\{|a_n| : n \in \mathbf{Z}(0,k)\},\ q = \max\{|b_n + a_{n-1} + a_n| : n \in \mathbf{Z}(1,k)\}.$

Theorem 1.1. Assume that there exist constants $R_1 > 0,\ \beta > 2$ and a functional $F(n,\cdot) \in C^1(\mathbf{Z} \times \mathbf{R}^2, \mathbf{R})$ with $F(0,\cdot) = 0$ such that for any $n \in \mathbf{Z}(1,k)$,

$$\frac{\partial F(n-1,v_2,v_3)}{\partial v_2} + \frac{\partial F(n,v_1,v_2)}{\partial v_2} = f(n,v_1,v_2,v_3),$$

$$0 < \beta F(n,v_1,v_2) \leq \frac{\partial F(n,v_1,v_2)}{\partial v_1} v_1 + \frac{\partial F(n,v_1,v_2)}{\partial v_2} v_2,\ \forall \sqrt{v_1^2 + v_2^2} \geq R_1. \tag{5}$$

Then the BVP (3) with (4) possesses at least one solution.

Remark 1.1. (5) implies that there exist constants $a_1 > 0$ and $a_2 > 0$ such that

$$F(n,v_1,v_2) \geq a_1 \left(\sqrt{v_1^2 + v_2^2}\right)^\beta - a_2, \forall n \in \mathbf{Z}(1,k).$$

Theorem 1.2. Assume that $B = 0$ and the following hypotheses are satisfied:

(F_1) there exists a functional $F(n,\cdot) \in C^1(\mathbf{Z} \times \mathbf{R}^2, \mathbf{R})$ with $F(0,\cdot) = 0$ such that

$$\lim_{r \to 0} \frac{F(n,v_1,v_2)}{r^2} = 0,\ r = \sqrt{v_1^2 + v_2^2},\ \forall n \in \mathbf{Z}(1,k);$$

(F_2) there exists a constant $\beta > 2$ such that for any $n \in \mathbf{Z}(1,k)$,

$$\frac{\partial F(n-1,v_2,v_3)}{\partial v_2} + \frac{\partial F(n,v_1,v_2)}{\partial v_2} = f(n,v_1,v_2,v_3),$$

$$\frac{\partial F(n,v_1,v_2)}{\partial v_1} v_1 + \frac{\partial F(n,v_1,v_2)}{\partial v_2} v_2 \leq \beta F(n,v_1,v_2) < 0,\ \forall (v_1,v_2) \neq 0; \tag{6}$$

(F_3) $a_n > 0,\ b_n + a_{n-1} + a_n \equiv 0,\ \forall n \in \mathbf{Z}(1,k).$

Then the BVP (3) with (4) possesses at least two nontrivial solutions.

Remark 1.2. (6) implies that there exist constants $a_1 > 0$ and $a_2 > 0$ such that

$$F(n,v_1,v_2) \leq -a_1 \left(\sqrt{v_1^2 + v_2^2}\right)^\beta + a_2,\ \forall n \in \mathbf{Z}(1,k). \tag{7}$$

The rest of the paper is organized as follows. In Section 2 we shall establish the variational framework for the BVP (3) with (4) in order to apply the critical point method and give some useful lemmas. In Section 3 we shall complete the proof of the main results and give an example to illustrate the result.

2. Variational structure and some lemmas

In order to apply the critical point theory, we shall establish the corresponding variational framework for the BVP (3) with (4) and give some basic notations and useful lemmas.

Let $\mathbf{R}^k$ be the real Euclidean space with dimension k. Define the inner product on $\mathbf{R}^k$ as follows:

$$\langle u, v \rangle = \sum_{j=1}^{k} u_j v_j,\ \forall u, v \in \mathbf{R}^k, \tag{8}$$

by which the norm $\|\cdot\|$ can be induced by

$$\|u\| = \left(\sum_{j=1}^{k} u_j^2\right)^{\frac{1}{2}}, \ \forall u \in \mathbf{R}^k. \tag{9}$$

On the other hand, we define the norm $\|\cdot\|_r$ on $\mathbf{R}^k$ as follows:

$$\|u\|_r = \left(\sum_{j=1}^{k} |u_j|^r\right)^{\frac{1}{r}}, \tag{10}$$

for all $u \in \mathbf{R}^k$ and $r > 1$.

Since $\|u\|_r$ and $\|u\|_2$ are equivalent, there exist constants $c_1, \ c_2$ such that $c_2 \geq c_1 > 0$, and

$$c_1\|u\|_2 \leq \|u\|_r \leq c_2\|u\|_2, \ \forall u \in \mathbf{R}^k. \tag{11}$$

Clearly, $\|u\| = \|u\|_2$. For the BVP (3) with (4), consider the functional J on $\mathbf{R}^k$ as follows:

$$J(u) = \frac{1}{2}\sum_{n=0}^{k} a_n(\Delta u_n)^2 - \frac{1}{2}\sum_{n=1}^{k}(b_n + a_{n-1} + a_n)u_n^2 + \sum_{n=1}^{k} F(n, u_{n+1}, u_n), \tag{12}$$

$\forall u = (u_1, u_2, \cdots, u_k)^* \in \mathbf{R}^k, \ u_0 = A, \ u_{k+1} = B.$

Clearly, $J \in C^1(\mathbf{R}^k, \mathbf{R})$ and for any $u = \{u_n\}_{n\in\mathbf{Z}(1,k)} \in \mathbf{R}^k$, by using $u_0 = A, \ u_{k+1} = B$, we can compute the partial derivative as

$$\begin{aligned}\frac{\partial J}{\partial u_n} &= -a_n\Delta u_n + a_{n-1}\Delta u_{n-1} - (b_n + a_{n-1} + a_n)u_n + f(n, u_{n+1}, u_n, u_{n-1}) \\ &= -Lu_n + f(n, u_{n+1}, u_n, u_{n-1}), \ n \in \mathbf{Z}(1, k).\end{aligned}$$

Thus, u is a critical point of J on $\mathbf{R}^k$ if and only if

$$Lu_n = f(n, u_{n+1}, u_n, u_{n-1}), \ \forall n \in \mathbf{Z}(1, k).$$

We reduce the existence of the BVP (3) with (4) to the existence of critical points of J on $\mathbf{R}^k$. That is, the functional J is just the variational framework of the BVP (3) with (4).

Denote

$$W = \{(u_1, u_2, \cdots, u_k)^* \in \mathbf{R}^k | u_n \equiv v, \ v \in \mathbf{R}, \ n \in \mathbf{Z}(1, k)\}$$

and Y be the direct orthogonal complement of $\mathbf{R}^k$ to W, i.e., $\mathbf{R}^k = Y \oplus W$.

Let E be a real Banach space, $J \in C^1(E, \mathbf{R})$, i.e., J is a continuously Fréchet-differentiable functional defined on E. J is said to satisfy the Palais-Smale condition (P.S. condition[10] for short) if any sequence $\left\{u^{(k)}\right\} \subset E$ for which $\left\{J\left(u^{(k)}\right)\right\}$ is bounded and $J'\left(u^{(k)}\right) \to 0 (k \to \infty)$ possesses a convergent subsequence in E.

Let B_ρ denote the open ball in E about 0 of radius ρ and let ∂B_ρ denote its boundary.

Lemma 2.1. (Linking Theorem [18,22]). Let E be a real Banach space, $E = E_1 \oplus E_2$, where E_1 is finite dimensional. Suppose that $J \in C^1(E, \mathbf{R})$ satisfies the P.S. condition and

(J_1) there exist constants $a > 0$ and $\rho > 0$ such that $J|_{\partial B_\rho \cap E_2} \geq a$;

(J_2) there exists an $e \in \partial B_1 \cap E_2$ and a constant $R_0 \geq \rho$ such that $J|_{\partial Q} \leq 0$, where $Q = (\bar{B}_{R_0} \cap E_1) \oplus \{re|0 < r < R_0\}$.
Then J possesses a critical value $c \geq a$, where

$$c = \inf_{h \in \Gamma} \sup_{u \in Q} J(h(u)),$$

and $\Gamma = \{h \in C(\bar{Q}, E) \mid h|_{\partial Q} = id\}$, where id denotes the identity operator.

Lemma 2.2. Assume that $B = 0$ and $(F_1) - (F_3)$ are satisfied. Then the functional J is bounded from above in $\mathbf{R}^k$.

Proof. For any $u \in \mathbf{R}^k$,

$$\begin{aligned} J(u) &= \frac{1}{2} \sum_{n=0}^{k} a_n (\Delta u_n)^2 + \sum_{n=1}^{k} F(n, u_{n+1}, u_n) \\ &\leq p_{\max} \sum_{n=0}^{k} \left(u_{n+1}^2 + u_n^2\right) - a_1 \sum_{n=1}^{k} \left(\sqrt{u_{n+1}^2 + u_n^2}\right)^\beta + a_2 k \\ &\leq 2p_{\max} \sum_{n=0}^{k} u_n^2 - a_1 \sum_{n=1}^{k} |u_n|^\beta + a_2 k \\ &\leq 2p_{\max} \|u\|^2 + 2p_{\max} A^2 - a_1 c_1^\beta \|u\|^\beta + a_2 k. \end{aligned} \tag{13}$$

Since $\beta > 2$, there exists a constant $M > 0$ such that $J(u) \leq M$, $\forall u \in \mathbf{R}^k$. The proof of Lemma 2.2 is complete.

Lemma 2.3. Assume that $B = 0$ and $(F_1) - (F_3)$ are satisfied. Then the functional J satisfies the P.S. condition.

Proof. Let $u^{(l)} \in \mathbf{R}^k$, $l \in \mathbf{Z}(1)$ be such that $\left\{J\left(u^{(l)}\right)\right\}$ is bounded. Then there exists a positive constant M_1 such that

$$-M_1 \leq J\left(u^{(l)}\right) \leq M_1, \ \forall l \in \mathbf{N}.$$

By the proof of Lemma 2.2, it is easy to see that

$$-M_1 \leq J\left(u^{(l)}\right) \leq 2p_{\max} \left\|u^{(l)}\right\|^2 - a_1 c_1^\beta \left\|u^{(l)}\right\|^\beta + 2p_{\max} A^2 + a_2 k.$$

That is,

$$a_1 c_1^\beta \left\|u^{(l)}\right\|^\beta - 2p_{\max} \left\|u^{(l)}\right\|^2 \leq M_1 + 2p_{\max} A^2 + a_2 k.$$

Since $\beta > 2$, there exists a constant $M_2 > 0$ such that

$$\left\|u^{(l)}\right\| \leq M_2, \ \forall l \in \mathbf{N}.$$

Therefore, $\left\{u^{(l)}\right\}$ is bounded on $\mathbf{R}^k$. As a consequence, $\left\{u^{(l)}\right\}$ possesses a convergence subsequence in $\mathbf{R}^k$. And thus the P.S. condition is verified.

3. **Proof of the main results**

Proof of Theorem 1.1. For any $u = (u_1, u_2, \cdots, u_k)^* \in \mathbf{R}^k$, $\sqrt{u_{n+1}^2 + u_n^2} \geq R_1$, we have

$$\begin{aligned}
J(u) &= \frac{1}{2}\sum_{n=0}^{k} a_n(\Delta u_n)^2 - \frac{1}{2}\sum_{n=1}^{k}(b_n + a_{n-1} + a_n)u_n^2 + \sum_{n=1}^{k} F(n, u_{n+1}, u_n) \\
&\geq -\frac{p}{2}\sum_{n=0}^{k}(\Delta u_n)^2 - \frac{q}{2}\sum_{n=1}^{k} u_n^2 + a_1\sum_{n=1}^{k}\left(\sqrt{u_{n+1}^2 + u_n^2}\right)^{\beta} - a_2 k \\
&\geq -p\sum_{n=0}^{k}\left(u_{n+1}^2 + u_n^2\right) - \frac{q}{2}\|u\|^2 + a_1\sum_{n=1}^{k}|u_n|^{\beta} - a_2 k \\
&\geq -2p\sum_{n=1}^{k} u_n^2 - p\left(A^2 + B^2\right) - \frac{q}{2}\|u\|^2 + a_1 c_1^{\beta}\|u\|^{\beta} - a_2 k \\
&= -\left(2p + \frac{q}{2}\right)\|u\|^2 - p\left(A^2 + B^2\right) + a_1 c_1^{\beta}\|u\|^{\beta} - a_2 k \\
&\to +\infty(\|u\| \to +\infty).
\end{aligned}$$

By continuity of J on $\mathbf{R}^k$ and above argument, there exists $\bar{u} \in \mathbf{R}^k$ such that $J(\bar{u}) = \min\left\{J(u) | u \in \mathbf{R}^k\right\}$. Clearly, $\bar{u}$ is a critical point of the functional J. The proof of Theorem 1.1 is finished.

Proof of Theorem 1.2. Assumptions (F_1) and (F_2) imply that $F(n, 0) = 0$ and $f(n, 0) = 0$ for $n \in \mathbf{Z}(1, k)$. Then $u = 0$ is a trivial solution of the BVP (3) with (4).

By Lemma 2.2, J is bounded from the upper on $\mathbf{R}^k$. We define $c_0 = \sup_{u\in\mathbf{R}^k} J(u)$. The proof of Lemma 2.2 implies $\lim_{\|u\|\to+\infty} J(u) = -\infty$. This means that $-J(u)$ is coercive. By the continuity of $J(u)$, there exists $\bar{u} \in \mathbf{R}^k$ such that $J(\bar{u}) = c_0$. Clearly, $\bar{u}$ is a critical point of J.

We claim that $c_0 > 0$. Indeed, by (F_1), for any $\epsilon = \frac{1}{8}p_{\min}\lambda_2$($\lambda_2$ can be referred to (14)), there exists $\rho > 0$, such that

$$|F(n, v_1, v_2)| \leq \frac{1}{8}p_{\min}\lambda_2\left(v_1^2 + v_2^2\right), \forall n \in \mathbf{Z}(1, k),$$

for $\sqrt{v_1^2 + v_2^2} \leq \sqrt{2}\rho$.

For any $u = (u_1, u_2, \cdots, u_k)^* \in Y$ and $\|u\| \leq \rho$, we have $|u_n| \leq \rho,\ n \in \mathbf{Z}(1, k)$.

When $k \geq 2$,

$$\begin{aligned}
J(u) &= \frac{1}{2}\sum_{n=0}^{k} a_n(u_{n+1} - u_n)^2 + \sum_{n=1}^{k} F(n, u_{n+1}, u_n) \\
&\geq \frac{1}{2}p_{\min}\sum_{n=0}^{k}(u_{n+1} - u_n)^2 - \frac{1}{8}p_{\min}\lambda_2\sum_{n=1}^{k}\left(u_{n+1}^2 + u_n^2\right) \\
&\geq \frac{1}{2}p_{\min}(y^* D y) - \frac{1}{4}p_{\min}\lambda_2\|u\|^2,
\end{aligned}$$

where $y^* = (A, u_1, u_2, \cdots, u_k, 0),\ y \in \mathbf{R}^{k+2}$,

$$D = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{pmatrix}_{(k+2)\times(k+2)}.$$

Clearly, $\lambda_1 = 0$ is an eigenvalue of D and $\xi = (v, v, \cdots, v) \in \mathbf{R}^{k+2}(v \neq 0,\ v \in \mathbf{R})$ is an eigenvector of D corresponding to 0. Let $\lambda_2, \lambda_3, \cdots, \lambda_{k+2}$ be the other eigenvalues of D. Applying matrix theory, we know $\lambda_j > 0,\ j = 2, 3, \cdots, k+2$. Without loss of generality, we may assume that

$$0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_{k+2}, \tag{14}$$

then for any $u \in Y$, defining

$$\|y\| = \left(\sum_{i=0}^{k+1} u_i^2\right)^{\frac{1}{2}} = \left(\|u\|^2 + A^2\right)^{\frac{1}{2}},$$

we have

$$\begin{aligned} J(u) &\geq \frac{1}{2} p_{\min} \lambda_2 \|y\|^2 - \frac{1}{4} p_{\min} \lambda_2 \|u\|^2 \\ &\geq \frac{1}{4} p_{\min} \lambda_2 \|u\|^2. \end{aligned}$$

Take $a \triangleq \frac{1}{4} p_{\min} \lambda_2 \|\rho\|^2 > 0$. Therefore,

$$J(u) \geq a > 0,\ \forall u \in Y \cap \partial B_\rho.$$

At the same time, we have also proved that there exist constants $a > 0$ and $\rho > 0$ such that $J|_{Y \cap \partial B_\rho} \geq a$. That is to say, J satisfies the condition (J_1) of the Linking Theorem.

In order to exploit the Linking Theorem in critical point theory, we need to verify other conditions of the Linking Theorem. By Lemma 2.3, J satisfies the P.S. condition. So it suffices to verify the condition (J_2).

Take $e \in \partial B_1 \cap Y$, for any $w \in W$ and $r \in \mathbf{R}$, let $u = re + w$. Then

$$\begin{aligned} J(u) = &\frac{1}{2} \sum_{n=1}^{k-1} a_n (re_{n+1} + w_{n+1} - re_n - w_n)^2 + \frac{a_0}{2} (re_1 + w_1 - A)^2 + \frac{a_k}{2} (0 - re_k - w_k)^2 \\ &+ \sum_{n=1}^{k} F(n, re_{n+1} + w_{n+1}, re_n + w_n) \\ \leq &\frac{p_{\max} r^2}{2} \sum_{n=1}^{k-1} (e_{n+1} - e_n)^2 + \frac{3a_0}{2} \left[(re_1)^2 + w_1^2 + A^2\right] + \frac{3a_k}{2} \left[(re_k)^2 + w_k^2\right] \\ &- a_1 \sum_{n=1}^{k} \left[\sqrt{(re_{n+1} + w_{n+1})^2 + (re_n + w_n)^2}\right]^\beta + a_2 k \end{aligned}$$

$$\leq 2(k-1)p_{\max}r^2 + \frac{3p_{\max}}{2}\left(2r^2 + A^2 + 2w_1^2\right) - a_1 c_1^{\beta}\left(\sum_{n=1}^{k}|re_n + w_n|^2\right)^{\frac{\beta}{2}} + a_2 k$$

$$= \left[2(k-1)p_{\max} + 3p_{\max}\right]r^2 + \frac{3p_{\max}}{2}\left(A^2 + 2w_1^2\right) - a_1 c_1^{\beta}\left(r^2 + \|w\|^2\right)^{\frac{\beta}{2}} + a_2 k$$

$$\leq (2k+1)p_{\max}r^2 + \frac{3p_{\max}}{2}A^2 - a_1 c_1^{\beta} r^{\beta} - a_1 c_1^{\beta}\|w\|^{\beta} + 3p_{\max}\|w\|^2 + a_2 k.$$

Let

$$g_1(r) = (2k+1)p_{\max}r^2 + \frac{3p_{\max}}{2}A^2 - a_1 c_1^{\beta} r^{\beta}, \; g_2(t) = -a_1 c_1^{\beta} t^{\beta} + 3p_{\max}t^2 + a_2 k.$$

Then

$$\lim_{r\to+\infty} g_1(r) = -\infty, \; \lim_{t\to+\infty} g_2(t) = -\infty,$$

$g_1(r)$ and $g_2(t)$ are bounded from above. It is easy to see that there exists a positive constant $R_2 > \rho$ such that for any $u \in \partial Q$, $J(u) \leq 0$, where

$$Q = (\bar{B}_{R_2} \cap W) \oplus \{re | 0 < r < R_2\}.$$

By the Linking Theorem, J possesses a critical value $c \geq a > 0$, where

$$c = \inf_{h\in\Gamma} \sup_{u\in Q} J(h(u)),$$

and $\Gamma = \{h \in C(\bar{Q}, \mathbf{R}^k) \mid h|_{\partial Q} = id\}$.

Let $\tilde{u} \in \mathbf{R}^k$ be a critical point associated to the critical value c of J, i.e., $J(\tilde{u}) = c$. If $\tilde{u} \neq \bar{u}$, then the conclusion of Theorem 1.2 holds. Otherwise, $\tilde{u} = \bar{u}$. Then $c_0 = J(\bar{u}) = J(\tilde{u}) = c$, that is $\sup_{u\in\mathbf{R}^k} J(u) = \inf_{h\in\Gamma}\sup_{u\in Q} J(h(u))$. Choosing $h = id$, we have $\sup_{u\in Q} J(u) = c_0$. Since the choice of $e \in \partial B_1 \cap Y$ is arbitrary, we can take $-e \in \partial B_1 \cap Y$. Similarly, there exists a positive number $R_3 > \rho$, for any $u \in \partial Q_1$, $J(u) \leq 0$, where

$$Q_1 = (\bar{B}_{R_3} \cap W) \oplus \{-re | 0 < r < R_3\}.$$

Again, by the Linking Theorem, J possesses a critical value $c' \geq a > 0$, where

$$c' = \inf_{h\in\Gamma_1} \sup_{u\in Q_1} J(h(u)),$$

and $\Gamma_1 = \{h \in C(\bar{Q}_1, \mathbf{R}^k) \mid h|_{\partial Q_1} = id\}$.

If $c' \neq c_0$, then the proof is finished. If $c' = c_0$, then $\sup_{u\in Q_1} J(u) = c_0$. Due to the fact $J|_{\partial Q} \leq 0$ and $J|_{\partial Q_1} \leq 0$, J attains its maximum at some points in the interior of sets Q and Q_1. However, $Q \cap Q_1 \subset W$ and $J(u) \leq 0$ for any $u \in W$. Therefore, there must be a point $u' \in \mathbf{R}^k$, $u' \neq \tilde{u}$ and $J(u') = c' = c_0$. The above argument implies that the BVP (3) with (4) possesses at least two nontrivial solutions when $k \geq 2$.

In the case $k = 1$, it is easy to complete the proof of Theorem 1.2.

The proof of Theorem 1.2 is complete.

4. **Example**

As an application of Theorem 1.2, finally, we give an example to illustrate our main result.

For $n \in \mathbf{Z}(1,k)$, assume that

$$u_{n+1} + u_{n-1} - 2u_n = -\beta u_n \left[\varphi(n)\left(u_{n+1}^2 + u_n^2\right)^{\frac{\beta}{2}-1} + \varphi(n-1)\left(u_n^2 + u_{n-1}^2\right)^{\frac{\beta}{2}-1}\right] \tag{15}$$

with boundary value conditions

$$u_0 = 6,\ u_{k+1} = 0, \tag{16}$$

where $\beta > 2$, φ is continuously differentiable and $\varphi(n) > 0$, $n \in \mathbf{Z}(1,k)$ with $\varphi(0) = 0$.

We have

$$a_n = a_{n-1} \equiv 1,\ b_n \equiv -2,$$

$$f(n, v_1, v_2, v_3) = -\beta v_2 \left[\varphi(n)\left(v_1^2 + v_2^2\right)^{\frac{\beta}{2}-1} + \varphi(n-1)\left(v_2^2 + v_3^2\right)^{\frac{\beta}{2}-1}\right]$$

and

$$F(n, v_1, v_2) = -\varphi(n)\left(v_1^2 + v_2^2\right)^{\frac{\beta}{2}}.$$

Then

$$\frac{\partial F(n-1, v_2, v_3)}{\partial v_2} + \frac{\partial F(n, v_1, v_2)}{\partial v_2} = -\beta v_2 \left[\varphi(n)\left(v_1^2 + v_2^2\right)^{\frac{\beta}{2}-1} + \varphi(n-1)\left(v_2^2 + v_3^2\right)^{\frac{\beta}{2}-1}\right].$$

It is easy to verify all the assumptions of Theorem 1.2 are satisfied and then the BVP (15) with (16) possesses at least two nontrivial solutions.

Acknowledgement. The authors of this paper would like to thank the anonymous referees and the editors for their helpful remarks and suggestions in improving the presentation and quality of the paper.

REFERENCES

[1] *R.P. Agarwal*, Difference Equations and Inequalities: Theory, Methods and Applications, Marcel Dekker, New York, 1992.

[2] *R.P. Agarwal and D. O'Regan*, Existence criteria for singular boundary value problems with sign changing nonlinearities, J. Differential Equations, **183**(2002), No. 2, 409-433.

[3] *R.P. Agarwal, K. Perera and D. O'Regan*, Multiple positive solutions of singular discrete p-Laplacian problems via variational methods, Adv. Difference Equ., **2005**(2005), 93-99.

[4] *C.D. Ahlbrandt*, Dominant and recessive solutions of symmetric three term recurrences, J. Differential Equations, **107**(1994), No. 2, 238-258.

[5] *V. Anuradha, C. Maya and R. Shivaji*, Positive solutions for a class of nonlinear boundary value problems with Neumann-Robin boundary conditions, J. Math. Anal. Appl., **236**(1994), No. 1, 94-124.

[6] *D. Arcoya*, Positive solutions for semilinear Dirichlet problems in an annulus, J. Differential Equations, **94**(1991), No. 2, 217-227.

[7] *M. Cecchi, M. Marini and G. Villari*, On the monotonicity property for a certain class of second order differential equations, J. Differential Equations, **82**(1989), No. 1, 15-27.

[8] *K.C. Chang*, Infinite Dimensional Morse Theory and Multiple Solution Problems, Birkhäuser, Boston, 1993.

[9] *S. Chen*, Disconjugacy, disfocality, and oscillation of second order difference equations, J. Differential Equations, **107**(1994), No. 2, 383-394.

[10] *K. Deimling*, Nonlinear Functional Analysis, Springer, New York, 1985.

[11] *S. Elaydi*, An Introduction to Difference Equation, Springer, New York, 1996.

[12] *P. Eloe*, A boundary value problem for a system of difference equations, Nonlinear Anal., **7**(1983), No. 8, 813-820.

[13] *J.K. Hale and J. Mawhin*, Coincidence degree and periodic solutions of neutral equations, J. Differential Equations, **15**(1974), No. 2, 295-307.

[14] *J. Henderson and H.B. Thompson*, Existence of multiple solutions for second-order discrete boundary value problems, Comput. Math. Appl., **43**(2002), No. 10-11, 1239-1248.

[15] *W.G. Kelly and A.C. Peterson*, Difference Equations: An Introduction with Applications, Academic Press, New York, 1991.

[16] *V. L. Kocic and G. Ladas*, Global Behavior of Nonlinear Difference Equations of Higher Order with Application, Kluwer Academic Publishers, Boston, 1993.

[17] *H. Matsunaga, Hara T. and S. Sakata*, Global attractivity for a nonlinear difference equation with variable delay, Comput. Math. Appl., **41**(2001), No. 5-6, 543-551.

[18] *J. Mawhin and M. Willem*, Critical Point Theory and Hamiltonian Systems, Springer, New York, 1989.

[19] *R. E. Mickens*, Difference Equations: Theory and Application, Van Nostrand Reinhold, New York, 1990.

[20] *A. Pankov and N. Zakharchenko*, On some discrete variational problems, Acta Appl. Math., **65**(2001), No. 1-3, 295-303.

[21] *A. Peterson*, Boundary value problems for an nth order linear difference equations, SIAM J. Math. Anal., **15**(1984), No. 1, 124-132.

[22] *P.H. Rabinowitz*, Minimax Methods in Critical Point Theory with Applications to Differential Equations, Amer. Math. Soc., Providence, RI, New York, 1986.

[23] *Y. Rodrigues*, On nonlinear discrete boundary value problems, J. Math. Anal. Appl., **114**(1986), No. 2, 398-408.

[24] *A.N. Sharkovsky, Y.L. Maistrenko and E.Y. Romanenko*, Difference Equations and Their Applications, Kluwer Academic Publishers, Dordrecht, 1993.

[25] *D. Smets and M. Willem*, Solitary waves with prescribed speed on infinite lattices, J. Funct. Anal., **149**(1997), No. 1, 266-275.

[26] *H. Wang*, On the existence of positive solutions for semilinear elliptic equations in annulus, J. Differential Equations, **109**(1994), No. 1, 1-7.

[27] *J.S. Yu and Z.M. Guo*, Boundary value problems of discrete generalized Emden-Fowler equation, Sci. China Math, **49**(2006), No. 10, 1303-1314.

[28] *J.S. Yu and Z.M. Guo*, On boundary value problems for a discrete generalized Emden-Fowler equation, J. Differential Equations, **231**(2006), No. 1, 18-31.

[29] *W.M. Zou and M. Schechter*, Critical Point Theory and Its Applications, Springer, New York, 2006.

[30] *Z. Zhou, J.S. Yu and Y. M. Chen*, Homoclinic solutions in periodic difference equations with saturable nonlinearity, Sci. China Math., **54**(2011), No. 1, 83-93.