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APPLICATION OF THE HOMOTOPY PERTURBATIONS METHOD IN APPROXIMATION PROBABILITY DISTRIBUTIONS OF NON-LINEAR TIME SERIES

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In this paper, the application of homotopy perturbations method (HPM) in approximation the probability distribution of some non-linear stochastic models that have probability densities functions (PDFs) with no closed form is described. The main result is based on HPM approximations the PDFs of some non-linear autoregressive time series, which are widely used in modeling actual data, especially in econometrics. It is formally confirmed the convergence and efficiency of HPM approximations, and illustrated also with certain examples of non-linear stochastic models of autoregressive time series.

Keywords: Homotopy perturbation, non-linear time series, approximation, probability distribution

MSC 2010: 62E17, 60E10, 41A46, 42B10.

1. Introduction

The homotopy perturbation method (HPM), firstly introduced by He [1]–[3], represents a very general mathematical approach that combines homotopy in topology with perturbation techniques in order to obtain (aproximate) solutions of non-linear equations of different types. In following years, the HPM has been the subject of extensive studies [4]-[6] and applied in solving various problems. These are, first of all, the different types of non-linear differential and partial-differential equations [7]-[9], the different kinds of physical problems [10]-[12], as well as Fredholm and Volterra integral equations [13]-[14]. On the other hand, the application of HPM in stochastic theory is still not significantly represented. We point out [15] where the HPM has been applied in determination the invariant PDFs of non-linear dynamical systems with chaotic behaviour. In research of discrete-time non-linear stochastic models one of the main (and often intractable) problem is determination of their probability distributions [16]-[18]. Here, we describe one of the possible ways of solving this problem, based on HPM technique.

2. Stochastic assumptions. Formulation of the problem

Let $(\Omega, \mathscr{F}, P)$ be the probability space and (Z_t), $t = 0, \pm 1, \pm 2, \ldots$, the series of independent identically distributed (i.i.d.) and non-negative random variables (RVs). In addition, we suppose that RVs (Z_t), which are usually called *the noise-series*, have an absolutely-continuous probability density function (PDF) $f_Z(x)$, such that:

$$E(Z_t) = P(Z_t > 0) = 1. \tag{1}$$

Also, let $\mathscr{F}_t = \sigma\{Z_s : s \leq t\}$ be the filtration of non-decreasing σ–algebras on Ω, generated by the series (Z_t), and (X_t) be the series of $\mathscr{F}_t$ adaptive RVs, functionally depended on (Z_t). In various practical interpretations, (X_t) represents the realizations of some actual time series (financial index, for example), with an unknown PDF $f_X(x)$. Therefore, one of the main goals in stochastic analysis of series (X_t) is to

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determine its probability distribution, based on the known PDF $f_Z(x)$. In that way, the behaviour of the series (X_t) is completely described in a stochastic sense.

Among the most popular non-linear econometric models, which were successfully applied in many analysis of financial markets, is the *AutoRegressive Conditional Heteroscedastic* (*ARCH*) *model*, firstly introduced by Engle [19], and the *AutoRegressive Conditional Duration* (*ACD*) *model*, which was introduced by Engle and Russel [20]. Both of these discrete-time stochastic models can be defined recursively, in the general form, as

$$\begin{cases} X_t &= V_t Z_t \\ V_t &= \phi(X_{t-1}, \ldots, X_{t-p}), \end{cases} \tag{2}$$

where $p \in \mathbb{N}$ is the order of the model. Here, (V_t) is series of the $\mathscr{F}_{t-1}$ adaptive RVs which is usually called *the volatility*, and it represents the measure of uncertainty (or dispersion) of the actual series (X_t). According to Eqs. (1) and (2), as well as assumptions introduced above, it follows:

$$E(X_t|\mathscr{F}_{t-1}) = V_t E(Z_t|\mathscr{F}_{t-1}) = V_t. \tag{3}$$

In other words, Eq. (3) points to that all "information" about X_t, at the time moment $t-1$, are based on the known values of the volatility V_t. That is why the main goal is determination the common PDFs of the RVs (X_t, V_t), according to the known PDF of the noise series (Z_t).

In the following, we consider the simplest form of the stochastic model given by Eqs. (2), where the volatility (V_t) has a linear form of order $p = 1$:

$$V_t = a_0 + a_1 X_{t-1}. \tag{4}$$

Additionally, we will assume that inequalities $a_0 \geq 0$ and $0 < a_1 < 1$ hold. In that way, the non-negative and stationary conditions of (X_t, V_t) are fulfilled (see, for instance [21]). Now, according to the first equality in Eqs. (2), we can define the following transformation:

$$\begin{cases} X_t &= V_t Z_t \\ Y_t &= Z_t \end{cases} \iff \begin{cases} V_t &= \dfrac{X_t}{Y_t} \\ Z_t &= Y_t \end{cases},$$

whose Jacobian is

$$\Im = \begin{vmatrix} \dfrac{\partial v_t}{\partial x_t} & \dfrac{\partial v_t}{\partial y_t} \\ \dfrac{\partial z_t}{\partial x_t} & \dfrac{\partial z_t}{\partial y_t} \end{vmatrix} = \frac{1}{y_t} > 0.$$

Using the conventional method, as well as the stochastic independence of the RVs V_t and Z_t, we can obtain the common PDF of the RVs (X_t, Y_t) as follows:

$$f_{(X,Y)}(x_t, y_t) = f_V\left[v_t(x_t, y_t)\right] f_Z\left[z_t(x_t, y_t)\right] |\Im| = \frac{f_V\left(\dfrac{x_t}{y_t}\right) f_Z(y_t)}{y_t}.$$

where $f_V(v)$ and $f_X(x)$ are PDFs of the RVs (V_t) and (X_t), respectively. From here, the marginal PDF of the series (X_t) is as follows:

$$f_X(x) = \int_0^{+\infty} \frac{f_Z(y)}{y} f_V\left(\frac{x}{y}\right) \mathrm{d}y. \tag{5}$$

In the other way, Eq. (4) gives the following expression of the same PDF:

$$f_X(x) = a_1 f_V(a_0 + a_1 x). \tag{6}$$

Thus, by equalizing right-hand sides of Eqs. (5) and (6), we obtain the following equation:

$$a_1 f_V(a_0 + a_1 x) = \int_0^{+\infty} \frac{f_Z(y)}{y} f_V\left(\frac{x}{y}\right) \mathrm{d}y, \tag{7}$$

which represents an integral equation on unknown PDF $f_V(x)$.

Let us notice that, by using substitution $v := x/y$, Eq. (7) can be equivalently written as:

$$a_1 f_V(a_0 + a_1 x) = \int_0^{+\infty} \frac{f_V(v)}{v}\, f_Z\Big(\frac{x}{v}\Big)\,\mathrm{d}v. \tag{8}$$

Then, by substituting $u := a_0 + a_1 x$, Eq. (8) becomes:

$$a_1 f_V(u) = \int_0^{+\infty} \frac{f_V(v)}{v}\, f_Z\Big(\frac{u-a_0}{a_1\, v}\Big)\,\mathrm{d}v. \tag{9}$$

In addition, when $u < a_0$, the equality $f_Z((u-a_0)/(a_1 v)) = 0$ holds. Therefore, Eq. (9) can be written as:

$$a_1 f_V(u) = \int_{a_0}^{+\infty} \frac{f_V(v)}{v}\, f_Z\Big(\frac{u-a_0}{a_1\, v}\Big)\,\mathrm{d}v, \tag{10}$$

where $u \geq a_0$. According to translation $(u,v) \longmapsto (u+a_0, v+a_0)$, Eq. (10) becomes:

$$a_1 f_V(u+a_0) = \int_0^{+\infty} \frac{f_V(v+a_0)}{v+a_0} f_Z\Big(\frac{u}{a_1(v+a_0)}\Big)\,\mathrm{d}v. \tag{11}$$

Finally, if we introduce the new unknown function $g_V(x)$ as:

$$g_V(x) = \begin{cases} f_V(x+a_0), & x \geq 0 \\ 0, & x < 0 \end{cases},$$

then Eq. (11) becomes:

$$g_V(x) = \frac{1}{a_1} \int_0^{+\infty} K(x,y) g_V(y)\,\mathrm{d}y, \tag{12}$$

with the kernel:

$$K(x,y) = \frac{1}{y+a_0} f_Z\Big(\frac{x}{a_1(y+a_0)}\Big). \tag{13}$$

Note that function $K(x,y)$ depends, also, on parameters a_0, a_1. Therefore, even though closely, Eq. (12) is not a typical, homogenous Fredholm integral equation of the second kind, and it can not be solved in the conventional way.

3. **Approximation of PDFs with HPM**

In order to solve Eq. (12), we apply the HPM technique. For this purpose, we firstly introduce *the homotopy equation*:

$$(1-p)\mathscr{L}[\widetilde{g}_V(x;p)] = p\,\mathscr{N}[\widetilde{g}_V(x;p)], \tag{14}$$

where $p \in (0,1)$ is the so-called *embedding parameter*,

$$\mathscr{L}[\widetilde{g}_V(x;p)] = \widetilde{g}_V(x;p) - g_Z(x)$$

is *the linear part*, and

$$\mathscr{N}[\widetilde{g}_V(x;p)] = \frac{1}{a_1} \int_0^{+\infty} K(x,y) \widetilde{g}_V(y;p)\,\mathrm{d}y - \widetilde{g}_V(x;p)$$

is the *non-linear* or *"true" part* of homotopy Eq. (14). Obviously, when $p = 0$, Eq. (14) becomes $\mathscr{L}[\widetilde{g}_V(x;0)] = 0$, with the unique solution:

$$\widetilde{g}_V(x;0) = g_Z(x) \stackrel{def}{=} \begin{cases} f_Z(x+a_0), & x \geq 0 \\ 0, & x < 0. \end{cases}$$

This solution is usually called *an initial solution* or *an initial approximation* of the homotopy Eq. (14). Otherwise, when $p = 1$, Eq. (14) becomes $\mathscr{N}[\widetilde{g}_V(x,0)] = 0$, and it is the equivalent to Eq. (12), with the main solution $\widetilde{g}_V(x;1) \equiv g_V(x)$.

The HPM is based on the assumption that solution of Eq. (14) can be express as the power series (in p):

$$\widetilde{g}_V(x;p) = \sum_{k=0}^{+\infty} p^k g_k(x). \tag{15}$$

From here, when $p = 0$, we get as initial solution of Eq. (14):

$$g_0(x) := \widetilde{g}_V(x;0) = g_Z(x).$$

Otherwise, the main solution of Eq. (14), obtained when $p = 1$, will be:

$$g_V(x) = \lim_{p\to 1^-} \widetilde{g}_V(x;p) = g_Z(x) + \sum_{k=1}^{\infty} g_k(x), \tag{16}$$

on the condition that series in Eq. (16) converges. Substituting Eq. (15) in Eq. (14), we obtain:

$$(1-p)\sum_{k=1}^{+\infty} p^k g_k(x) = \frac{1}{a_1}\sum_{k=1}^{+\infty} p^k \int_0^{+\infty} K(x,y) g_{k-1}(y)\,\mathrm{d}y - \sum_{k=1}^{+\infty} p^k g_{k-1}(x), \tag{17}$$

and if we introduce the sequence of integrals:

$$I_k(x) = \int_0^{+\infty} K(x,y) g_k(y)\,\mathrm{d}y, \qquad k = 0,1,2,\dots,$$

then Eq. (17) becomes:

$$\sum_{k=1}^{+\infty} p^k g_k(x) = \frac{1}{a_1}\sum_{k=1}^{+\infty} p^k I_{k-1}(x) - p\,g_0(x).$$

Now, equating expressions with the identical powers p^k, where $k = 1,2,\dots$, gives the following recurrence relations:

$$\begin{cases} g_1(x) &= \dfrac{1}{a_1} I_0(x) - g_0(x), \\ g_k(x) &= \dfrac{1}{a_1} I_{k-1}(x), \qquad k = 2,3,\dots \end{cases} \tag{18}$$

In that way, the functions $g_k(x)$, $k = 0,1,2,\dots$ can be obtained. Moreover, if we introduce another recurrence sequence:

$$\begin{cases} J_0(x) &= g_0(x) = g_Z(x), \\ J_k(x) &= \displaystyle\int_0^{+\infty} K(x,y) J_{k-1}(y)\,\mathrm{d}y, \qquad k = 1,2,\dots, \end{cases} \tag{19}$$

then functions $\{g_k(x)\}_{k=1}^{\infty}$ can be expressed in the following way:

$$\begin{aligned} g_1(x) &= \frac{1}{a_1}\int_0^{+\infty} K(x,y) g_0(y)\,\mathrm{d}y - g_0(x) \\ &= \frac{1}{a_1} J_1(x) - J_0(x) \\ g_2(x) &= \frac{1}{a_1}\int_0^{+\infty} K(x,y) g_1(y)\,\mathrm{d}y = \frac{1}{a_1^2}\int_0^{+\infty} K(x,y) J_1(y)\,\mathrm{d}y - \frac{1}{a_1}\int_0^{+\infty} K(x,y) J_0(y)\,\mathrm{d}y \\ &= \frac{1}{a_1^2} J_2(x) - \frac{1}{a_1} J_1(x), \text{ etc.} \end{aligned}$$

It can be easily shown, by using the induction method, that:

$$g_k(x) = \frac{1}{a_1^k} J_k(x) - \frac{1}{a_1^{k-1}} J_{k-1}(x), \qquad k = 1,2,\dots, \tag{20}$$

and Eqs. (19) and (20) give the appropriate *HPM-approximations* (or *HPM-estimates*) of $g_V(x)$:

$$\begin{aligned} \widehat{g}_0(x) &= g_0(x) = g_Z(x) \\ \widehat{g}_1(x) &= \widehat{g}_0(x) + g_1(x) = \frac{1}{a_1} J_1(x) \\ \widehat{g}_2(x) &= \widehat{g}_1(x) + g_2(x) = \frac{1}{a_1^2} J_2(x), \text{ etc.} \end{aligned}$$

Thus, the unique expression of HPM-approximations of the unknown function $g_V(x)$ is obtained as:

$$\widehat{g}_k(x) = \sum_{j=0}^{k} g_j(x) = \frac{1}{a_1^k} J_k(x), \qquad k = 0, 1, 2, \ldots \tag{21}$$

Obviously, the solution of main Eq. (12) is the limit of HPM-approximations $\widehat{g}_k(x)$, when $k \to +\infty$. Under certain sufficient conditions, this convergence can be shown as follows.

Theorem 3.1. Let $\{\widehat{g}_k(x)\}_{k=0}^{\infty}$ be the sequence of HPM-approximations, defined by Eqs. (21). In addition, assume that there exist $C > 0$, $\alpha > 0$ and $0 \le \beta < 1$ such that for each $x > 0$ the PDF $f_Z(x)$ of noise-series (Z_t) satisfies inequality:

$$f_Z(x) \le C \, \frac{\mathrm{e}^{-\alpha x}}{x^{\beta}}. \tag{22}$$

Then, the series $\{\widehat{g}_k(x)\}_{k=0}^{\infty}$ converges to the function $g_V(x)$, i.e. it converges to the solution of Eq. (12).

Proof. According to assumption of the theorem, for fixed but an arbitrary $x > 0$ the kernel $K(x,y)$, given by Eq. (13), satisfies inequality:

$$K(x,y) \le C \left(\frac{a_1}{x}\right)^{\beta} (y + a_0)^{\beta - 1} \exp\left(\frac{-\alpha x}{a_1(y + a_0)}\right). \tag{23}$$

Notice that function:

$$h(u) = u^{\beta - 1} \exp\left(-\frac{\alpha x}{a_1 u}\right)$$

is positive for any $u > 0$ and $\lim_{u \to 0^+} h(u) = 0^+$ holds [22]. Also, at the point:

$$u_0 = \frac{\alpha x}{a_1(1 - \beta)} > 0$$

it has a maximum (Fig. 1):

$$h_{\max} = h(u_0) = \left[\frac{\alpha \, \mathrm{e} x}{a_1 \, (1 - \beta)}\right]^{\beta - 1}.$$

Thus, substitution $h_{\max}$ in Eq. (23) gives:

$$K(x,y) \le \frac{a_1 C}{x} \left(\frac{1 - \beta}{\alpha \, \mathrm{e}}\right)^{1 - \beta}. \tag{24}$$

On the other hand, as the PDF $f_Z(x)$ satisfies the normality condition:

$$\int_0^{+\infty} f_Z(x) \, \mathrm{d}x = 1,$$

for the kernel $K(x,y)$ is valid:

$$\int_0^{+\infty} K(x,y) \, \mathrm{d}x = \int_0^{+\infty} f_Z\left(\frac{x}{a_1(y + a_0)}\right) \frac{\mathrm{d}x}{y + a_0} = a_1. \tag{25}$$

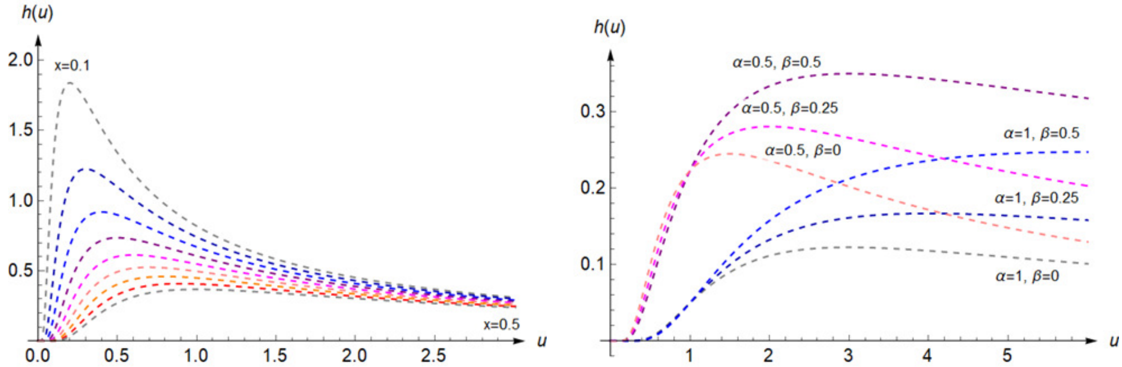

FIGURE 1. Graphics of the function $h(u)$ for various values of x-variable when $\alpha = 1$, $\beta = 0$ (panel left), and various values of parameters α, β when $x = 1.5$ (panel right).

According to Eqs. (24) and (25), for the sequence $\{g_k(x)\}_{k=1}^{\infty}$, defined by Eqs. (18), it follows [23]:

$$
\begin{aligned}
|g_1(x)| &\leq \frac{1}{a_1}\int_0^{+\infty} K(x,y)\,g_0(y)\,\mathrm{d}y \leq \frac{C}{x}\left(\frac{1-\beta}{\alpha\,\mathrm{e}}\right)^{1-\beta}\int_0^{+\infty} f_Z(x+a_0)\,\mathrm{d}x\\
&\leq \frac{C}{x}\left(\frac{1-\beta}{\alpha\,\mathrm{e}}\right)^{1-\beta}\big[1-F_Z(a_0)\big],\\
|g_2(x)| &\leq \frac{1}{a_1}\int_0^{+\infty} K(x,y)\,|g_1(y)|\,\mathrm{d}y \leq \frac{C}{x}\left(\frac{1-\beta}{\alpha\,\mathrm{e}}\right)^{1-\beta}\int_0^{+\infty} |g_1(y)|\,\mathrm{d}y\\
&\leq \frac{C}{x}\left(\frac{1-\beta}{\alpha\,\mathrm{e}}\right)^{1-\beta}\int_0^{+\infty}\frac{\mathrm{d}y}{a_1}\int_0^{+\infty} K(y,z)\,g_0(z)\,\mathrm{d}z\\
&\leq \frac{C}{x}\left(\frac{1-\beta}{\alpha\,\mathrm{e}}\right)^{1-\beta}\int_0^{+\infty}\frac{g_0(z)\,\mathrm{d}z}{a_1}\int_0^{+\infty} K(y,z)\,\mathrm{d}y\\
&\leq \frac{C}{x}\left(\frac{1-\beta}{\alpha\,\mathrm{e}}\right)^{1-\beta}\int_0^{+\infty} g_0(z)\,\mathrm{d}z\\
&\leq \frac{C}{x}\left(\frac{1-\beta}{\alpha\,\mathrm{e}}\right)^{1-\beta}\big[1-F_Z(a_0)\big], \text{ etc.}
\end{aligned}
$$

Obviously, for each $k = 1,2,\ldots$ the following inequalities are valid:

$$
|g_k(x)| \leq \frac{C}{x}\left(\frac{1-\beta}{\alpha\,\mathrm{e}}\right)^{1-\beta}\big[1-F_Z(a_0)\big], \tag{26}
$$

where we denoted as $F_Z(x) := P\{Z_t < x\} = \int_0^x f_Z(y)\,\mathrm{d}y$ the cumulative distribution function (CDF) of RVs (Z_t).

Further, denote with $r(x)$ the radius of convergence of the power series in Eq. (15). Then, by applying the Cauchy-Hadamard theorem and Eq. (26) it follows:

$$
r(x) = \left[\limsup_{k\to\infty} |g_k(x)|^{1/k}\right]^{-1} \geq 1.
$$

Thus, the power series in Eq. (15) converges at $p = 1$. In the same way as above, according to Eqs. (19), for the HPM-estimates $\{\widehat{g}_k(x)\}_{k=1}^{\infty}$ we obtain:

$$|\widehat{g}_k(x)| \leq \frac{C}{x}\left(\frac{1-\beta}{\alpha\,\mathrm{e}}\right)^{1-\beta}\big[1-F_Z(a_0)\big], \quad k = 1,2,\ldots \tag{27}$$

Thus, when $k \to \infty$, it follows:

$$\lim_{k\to\infty}|\widehat{g}_k(x)| = \left|\sum_{j=0}^{\infty} g_j(x)\right| < +\infty,$$

i.e. this power series is absolutely convergent at $p = 1$. By applying the Abel theorem, it follows that function $\widetilde{g}(x,p)$ is continuous from the left at point $p = 1$, i.e. Eq. (16) holds. So, the series $\sum_{j=0}^{\infty} g_j(x)$ is a solution of the homotopy Eq. (14), when $p = 1$, and it is also a solution of Eq. (12). □

Remark 3.1. The boundary condition (BC) given by Eq. (22) is usually fulfilled, especially for the well-known *exponential family of PDFs* (see, for instance [24]). In the next section, practical application of the previously mentioned HPM procedure will be described, and this BC will be examined in more detail.

Remark 3.2. If we introduce the so-called L^1-norm $\|g(x)\|_1 := \int_0^{+\infty} |g(x)|\,\mathrm{d}x$, then for the HPM-estimates $\{\widehat{g}_k(x)\}_{k=0}^{\infty}$ it follows:

$$\begin{aligned}
\|\widehat{g}_0(x)\|_1 &= \int_0^{+\infty} f_Z(x+a_0)\,\mathrm{d}x = 1-F_Z(a_0),\\
\|\widehat{g}_1(x)\|_1 &= \frac{1}{a_1}\int_0^{+\infty} J_1(x)\,\mathrm{d}x = \frac{1}{a_1}\int_0^{+\infty}\mathrm{d}x\int_0^{+\infty} K(x,y)\,g_0(y)\,\mathrm{d}y\\
&= \frac{1}{a_1}\int_0^{+\infty} g_0(y)\,\mathrm{d}y\int_0^{+\infty} K(x,y)\,\mathrm{d}x = \int_0^{+\infty} g_0(y)\,\mathrm{d}y\\
&= 1-F_Z(a_0), \text{ etc.}
\end{aligned}$$

In general, equalities:

$$\|\widehat{g}_k(x)\|_1 = 1-F_Z(a_0)$$

hold for each $k = 0,1,2,\ldots$ Furthermore, let us notice that functions $\widehat{f}_{V,k}(x) := \widehat{g}_k(x-a_0)$ represent the appropriate approximations of the unknown PDF $f_V(x)$. In the same way as before, it can be easily shown, for any $k = 0,1,2,\ldots$, the normality conditions:

$$\left\|\widehat{f}_{V,k}(x)\right\|_1 = 1.$$

Finally, estimates $\widehat{f}_{X,k}(x)$ of the unknown PDF $f_X(x)$ can be obtained according to Eq. (6) as:

$$\widehat{f}_{X,k}(x) = a_1\widehat{f}_{V,k}(a_0+a_1x) = a_1\,\widehat{g}_k(a_1x), \quad k = 0,1,2,\ldots. \tag{28}$$

4. Examples and applications

As we have already pointed out in the introductory part, there are two by far the most important stochastic models in which the aforementioned HPM procedure would find application. Both of these models can be described by a unique autoregressive model, given by the following equations:

$$\begin{cases} X_t &= V_t Z_t\\ V_t &= a_0 + a_1 X_{t-1}. \end{cases}$$

The only difference between these two models is in the stochastic distribution of noise series (Z_t), that is, its PDF $f_Z(x)$. In the following, both of them will be considered in more detail, whereby, for the simplicity, we shall suppose that $a_0 = 0$, $a_1 = 1/2$.

4.1. ACD model

The times of changes of some financial index (stock prices, for example) can be modelled by *the Autoregressive Conditional Duration (ACD) model*, introduced by Engle and Russel [20]. In this stochastic model, as the known PDF of RVs $Z_t \geq 0$ is usually taken the so-called *standardized exponential distribution*, with PDF given as:

$$f_Z(x) := \begin{cases} e^{-x}, & x \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

Obviously, the condition $E(Z_t) = 1$ holds, and the PDF $f_Z(x)$ satisfies the BC given by Eq. (22), when $C = \alpha = 1$ and $\beta = 0$. Thereby, under the condition $x \geq 0$ and by using the previously described HPM procedure, i.e. Eqs. (19)–(21), we obtained the series of HPM-estimates of PDF $f_V(x)$:

$$\widehat{g}_0(x) = \widehat{f}_{V,0}(x) = \exp(-x),$$

$$\widehat{g}_1(x) = \widehat{f}_{V,1}(x) = 2\int_0^\infty K(x,y)\,J_0(y)\,\mathrm{d}y = 2\int_0^\infty \frac{\mathrm{e}^{-\frac{2x}{y}-y}}{y}\,\mathrm{d}y = 4K_0(2\sqrt{2x}),$$

and, in general:

$$\widehat{g}_k(x) = \widehat{f}_{V,k}(x) = 2^k\int_0^\infty K(x,y)\,J_{k-1}(y)\,\mathrm{d}y = 2^k\int_0^\infty \frac{\mathrm{e}^{-\frac{2x}{y}}}{y}J_{k-1}(y)\,\mathrm{d}y, \quad k = 1,2,\ldots$$

Here, $K_0(x)$ is the hyperbolic Bessel function of the second kind, and $\{J_k(x)\}_{k=0}^{\infty}$ is the recurrence sequence of integrals defined by Eqs. (19). Thereafter, estimates $\widehat{f}_{X,k}(x)$ of the unknown PDF $f_X(x)$ can be easily obtained by using Eqs. (28). In Fig. 2 are shown the graphics of both of these estimates, when $k = 0,1,2,\ldots,7$.

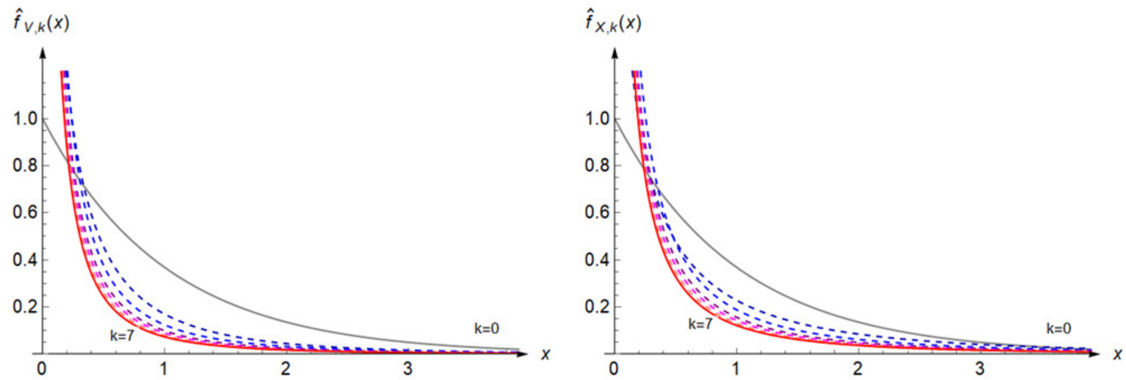

FIGURE 2. HPM-approximations of unknown PDFs of the ACD model: the volatility series (V_t) (panel left) and the basic series (X_t) (panel right).

4.2. ARCH model

The Autoregressive Conditional Heteroscedastic (ARCH) models, introduced by Engle [19], were successfully applied in many aspects of the financial markets analysis. They explain a lot of the properties of financial indexes dynamics such as a non-linear behaviour of the volatility, heavy tails distributions and clustering. In this case, as the PDF of non-negative noise-series (Z_t) is usually taken *the squared Gaussian distribution*:

$$f_Z(x) := \begin{cases} \frac{1}{\sqrt{2\pi x}} e^{-x/2}, & x \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

Notice that this PDF also satisfies $E(Z_t) = 1$, as well as the BC given by Eq. (22), when $C = (2\pi)^{-1/2}$ and $\alpha = \beta = 1/2$. Analogous to the previous one, the sequence of HPM-estimates of the unknown PDF $f_V(x)$ is as follows:

$$\widehat{g}_0(x) = \widehat{f}_{V,0}(x) = (2\pi x)^{-1/2} \exp\left(-\frac{x}{2}\right),$$

$$\widehat{g}_1(x) = \widehat{f}_{V,1}(x) = \frac{1}{\pi\sqrt{2x}} \int_0^{+\infty} \frac{\mathrm{e}^{-\frac{x}{y}-\frac{y}{2}}}{y}\,\mathrm{d}y = \frac{\sqrt{2}K_0\left(\sqrt{2x}\right)}{\pi\sqrt{x}}, \text{ etc.}$$

In general, for any $k = 1, 2, \ldots$, HPM-estimates can be obtained by the recurrence relation:

$$\widehat{g}_k(x) = \widehat{f}_{V,k}(x) = 2^k \int_0^{\infty} K(x,y)\, J_{k-1}(y)\,\mathrm{d}y = \frac{2^{k-1}}{\sqrt{\pi x}} \int_0^{+\infty} \frac{e^{-\frac{x}{y}}}{\sqrt{y}}\, J_{k-1}(y)\,\mathrm{d}y,$$

where $\{J_k(x)\}_{k=0}^{\infty}$ is defined by Eqs. (19). After that, in the same way as in previously, the PDF $f_X(x)$ can be estimated according to Eqs. (28). Graphics of estimates $\widehat{f}_{V,k}(x)$ and $\widehat{f}_{X,k}(x)$, when $k = 0, 1, 2, \ldots, 7$, are shown in Fig. 3.

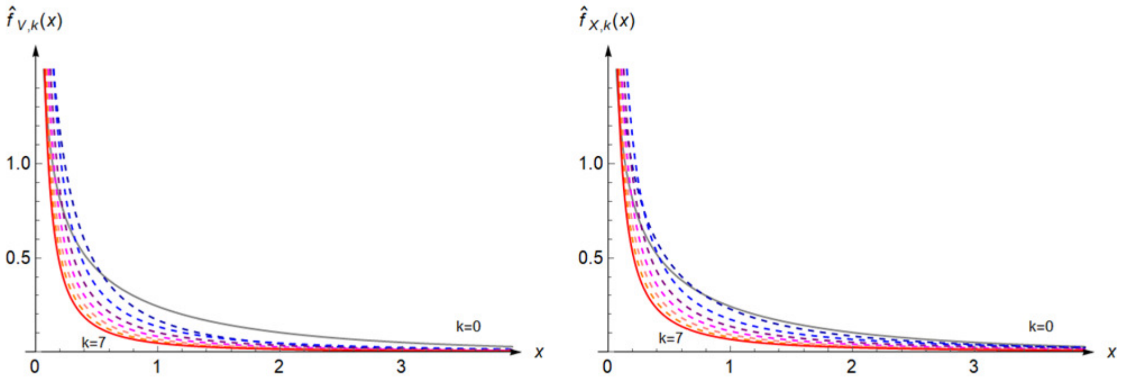

FIGURE 3. HPM-approximations of unknown PDFs of the ARCH model: the volatility series (V_t) (panel left) and the basic series (X_t) (panel right).

5. **Conclusion**

The Homotopy Perturbation Method (HPM) has been proposed in order to (approximate) solving non-linear equation which enables finding the probability density functions (PDFs) of some important classes of stochastic models, primarily used in econometrics. Here presented theoretical and practical results indicate that thus obtained HPM-approximations converge to the exact solution of this equation. Therefore, this method could find application in solving some similar and related stochastic problems.

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REFERENCES

[1] *J.-H. He*, Homotopy perturbation technique, Comput. Methods Appl. Mech. Engrg. **178**(1999) 257–262.
[2] *J.-H. He*, Asymptology by homotopy perturbation method, Appl. Math. Comput. **156**(2004) 591–596.
[3] *J.-H. He*, Notes on the optimal variational iteration method, Appl. Math. Lett. **25**(2012) 1579–1581.
[4] *Z.M. Odibat*, A new modification of the homotopy perturbation method for linear and nonlinear operators, Appl. Math. Comput. **189**(2007) 746–753.
[5] *X. Jiao, Y. Zheng and B. Wu*, Approximate homotopy symmetry and infinite series solutions to the perturbed mKdV equation, Appl. Math. Comput. **218**(2012) 8486–8491.
[6] *A. Nazari-Golshan, S.S. Nourazar, H. Ghafoori-Fard, A. Yildirim and A.Campo*, A modified homotopy perturbation method coupled with the Fourier-Stieltjes transform for nonlinear and singular Lane-Emden equations, Appl. Math. Lett. **26:10**(2013) 1018–10025.
[7] *S. Momani, Z. Odibat*, Homotopy perturbation method for nonlinear partial differential equations of fractional order, Physics Letters A, **365:(5-6)**(2007), 345–350.
[8] *S.T. Mohyud-Din, M.A. Noor*, Homotopy perturbation method for solving partial differential equations. Zeitschrift für Naturforschung A – J. Phys. Sci., **64:(3-4)**(2009), 157–170.
[9] *H. Aminikhah, J. Biazar*, A new HPM for ordinary differential equations, Numerical Methods for Partial Differential Equations: An International Journal, **26:2**(2010), 480–489.
[10] *T. Kevkić, V. Stojanović, D. Petković*, Solving Schrödinger equation for a particle in one-dimensional lattice: An homotopy perturbation approach, Rom. Rep. Phys. **71:1**(2019), Article ID: 101.
[11] *T. Kevkić, V. Stojanović*, Approximate solution of coupled Schrödinger and Poisson equation in inversion layer problem: An approach based on homotopy perturbations, Zeitschrift für Naturforschung A – J. Phys. Sci. **74:6**(2019), 457–467.
[12] *A. R. Appadu, A. S. Kelil*, On Semi-Analytical Solutions for Linearized Dispersive KdV Equations. Mathematics, **8:10**(2020), Article ID: 1769.
[13] *J. Biazar, H. Ghazvini*, Numerical solution for special non-linear Fredholm integral equation by HPM, Appl. Math. Comput. **195:2**(2008), 681–687.
[14] *S.-M. Jung, Ş. Sevgin and H. Şevli*, On the perturbation of Volterra integro-differential equations, Appl. Math. Lett. **26**(2013) 665–669.
[15] *V. Stojanović, T. Kevkić, G. Jelić, D. Randjelović*, Determination of invariant measures: An approach based on homotopy perturbations, U.P.B. Sci. Bull., Series A, **80:2**(2018), 119–128.
[16] *S. C. Ciucu, V.-P. Paun*, Financial data analysis using nonlinear time series methods. Fluctuations interpretation of foreign currency exchange rates, U.P.B. Sci. Bull., Series A. **77:1**(2015), 235–248.
[17] *V. Stojanović, T. Kevkić, E. Ljajko, G. Jelić*, Noise-Indicator ARMA model with application in fitting physically–based time series, U.P.B. Sci. Bull., Series A, **81:2**(2019), 257–264.
[18] *M. Ranđelović, V. Stojanović, T. Kevkić*, Noise-Indicator autoregressive conditional heteroskedastic process with application in modeling actual time series, U.P.B. Sci. Bull., Series A, **81:3**(2019), 77–84.
[19] *R.F. Engle*, Autoregressive conditional heteroscedasticity with estimates of the variance of United Kingdom inflation, Econom. **50:4**(1982) 987–1007.
[20] *R.F. Engle and J. R. Russel*, Autoregressive Conditional Duration: A New Model for Irregularly Spaced Transaction Data, Econom. **66**(1998) 1127–1162.
[21] *H. Kantz, T. Schreiber*, Nonlinear time series analysis, Cambridge University Press, UK (2003).
[22] *G. Jelić*, Gamma-functions and beta-function, Proceed. Inter. Confer. RADMI (2014), 854–860.
[23] *G. Jelić*, Riemann integral on a compact sum, Proceed. Inter. Confer. RADMI (2013), 793–800.
[24] *R.W. Keener*, Theoretical statistics: Topics for a core course, Springer, New York (2006).