WEBBED LOCALLY CONVEX CONES

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In this paper, we introduce the concepts of webs, compatible webs, completing webs and uniformly completing webs in locally convex cones. We obtain some criteria for locally convex cones with completing webs. Finally, we prove a closed graph type theorem in locally convex cones.

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1. Introduction

A *cone* is a set $\mathcal{P}$ endowed with an addition and a scalar multiplication for nonnegative real numbers. The addition is associative and commutative, and $\mathcal{P}$ has a neutral element 0. Also the scalar multiplication has the usual associative and distributive properties, that is $\alpha(\beta a) = (\alpha\beta)a$, $(\alpha+\beta)a = \alpha a + \beta a$, $\alpha(a+b) = \alpha a + \alpha b$, $1a = a$ and $0a = 0$ for all $a, b \in \mathcal{P}$ and $\alpha, \beta \geq 0$.

The theory of locally convex cones was developed in [4] and [6]. In this theory, a topological structure on a cone is introduced with the help of an order theoretical concept or a convex quasiuniform structure. In this paper we use the latter. For recent researches see [1, 3].

Let $\mathcal{P}$ be a cone. A convex quasiuniform structure on $\mathcal{P}$, is a collection $\mathfrak{U}$ of convex subsets $U \subseteq \mathcal{P}^2 = \mathcal{P} \times \mathcal{P}$ if the following properties hold:

(U_1) $\Delta \subseteq U$ for every $U \in \mathfrak{U}$ ($\Delta = \{(a,a): a \in \mathcal{P}\}$);
(U_2) for all $U, V \in \mathfrak{U}$ there is a $W \in \mathfrak{U}$ such that $W \subseteq U \cap V$;
(U_3) $\lambda U \circ \mu U \subseteq (\lambda+\mu)U$ for all $U \in \mathfrak{U}$ and $\lambda, \mu > 0$;
(U_4) $\alpha U \in \mathfrak{U}$ for all $U \in \mathfrak{U}$ and $\alpha > 0$.

We note that for $U, V \subseteq \mathcal{P}^2$,

$$U \circ V = \{(a,b) \in \mathcal{P}^2 : \exists c \in \mathcal{P};\ (a,c) \in U \ \ and \ \ (c,b) \in V\}.$$

Let $\mathcal{P}$ be a cone and $\mathfrak{U}$ be a convex quasiuniform structure on $\mathcal{P}$. The pair $(\mathcal{P}, \mathfrak{U})$ is called a locally convex cone if

(U_5) for each $a \in \mathcal{P}$ and $U \in \mathfrak{U}$ there is some $\rho > 0$ such that $(0,a) \in \rho U$.

With every convex quasiuniform structure $\mathfrak{U}$ on $\mathcal{P}$ we associate two topologies: The neighborhood bases for an element a in the upper and lower topologies are given by the sets

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$$U(a) = \{b \in \mathcal{P}: (b,a) \in U\}, \quad resp. \quad (a)U = \{b \in P: (a,b) \in U\}, \quad U \in \mathfrak{U}.$$

The common refinement of the upper and lower topologies is called symmetric topology. A neighborhood base for $a \in \mathcal{P}$ in this topology is given by the sets

$$U(a)U = U(a) \cap (a)U, \quad U \in \mathfrak{U}.$$

Let $\mathfrak{U}$ and $\mathcal{W}$ be convex quasiuniform structures on $\mathcal{P}$. We say that $\mathfrak{U}$ is finer than $\mathcal{W}$ if for each $W \in \mathcal{W}$ there is $U \in \mathfrak{U}$ such that $U \subseteq W$.

Let $\mathcal{P}$ be a cone and $\mathfrak{U}$ be a convex quasiuniform structure on $\mathcal{P}$. The subset $\mathcal{B}$ of $\mathfrak{U}$ is called a base for $\mathfrak{U}$ if for each $U \in \mathfrak{U}$ there are $n \in \mathbb{N}$, $\lambda_1, \cdots, \lambda_n > 0$ and $B_1, \cdots, B_n \in \mathcal{B}$ such that $\lambda_1 B_1 \cap \cdots \cap \lambda_n B_n \subseteq U$.

In locally convex cone $(\mathcal{P}, \mathfrak{U})$ the *closure* of $a \in \mathcal{P}$ is defined to be the set $\overline{a} = \bigcap_{U \in \mathfrak{U}} U(a)$
(see [4], Chapter I). The locally convex cone $(\mathcal{P}, \mathfrak{U})$ is called *separated* if $\overline{a} = \overline{b}$ implies $a = b$ for $a, b \in \mathcal{P}$. It is proved in [4] that a locally convex cone is separated if and only if its symmetric topology is Hausdorff.

Let $\mathcal{P}$ be a cone. A convex subset B of $\mathcal{P}^2$ is called *uniformly convex* whenever it has the properties (U_1) and (U_3). The locally convex cone $(\mathcal{P}, \mathfrak{U})$ is called a *uc-cone* whenever $\mathfrak{U} = \{\alpha U: \alpha > 0\}$ for some $U \in \mathfrak{U}$ (see [2]).

The extended real number system $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is a cone endowed with the usual algebraic operations, in particular $a + \infty = +\infty$ for all $a \in \overline{\mathbb{R}}$, $\alpha \cdot (+\infty) = +\infty$ for all $\alpha > 0$ and $0 \cdot (+\infty) = 0$. We set $\tilde{\mathcal{V}} = \{\tilde{\varepsilon}: \varepsilon > 0\}$, where $\tilde{\varepsilon} = \{(a,b) \in \overline{\mathbb{R}}^2: a \leq b + \varepsilon\}$.

Then $\tilde{\mathcal{V}}$ is a convex quasiuniform structure on $\overline{\mathbb{R}}$ and $(\overline{\mathbb{R}}, \tilde{\mathcal{V}})$ is a locally convex cone. For $a \in \mathbb{R}$ the intervals $(-\infty, a + \varepsilon]$ are the upper and the intervals $[a - \varepsilon, +\infty]$ are the lower neighborhoods, while for $a = +\infty$ the entire cone $\overline{\mathbb{R}}$ is the only upper neighborhood, and $\{+\infty\}$ is open in the lower topology. The symmetric topology is the usual topology on $\mathbb{R}$ with as an isolated point $+\infty$.

For cones $\mathcal{P}$ and $\mathcal{Q}$, a mapping $T: \mathcal{P} \to \mathcal{Q}$ is called a *linear operator* if $T(a + b) = T(a) + T(b)$ and $T(\alpha a) = \alpha T(a)$ hold for all $a, b \in \mathcal{P}$ and $\alpha \geq 0$. If both $(\mathcal{P}, \mathfrak{U})$ and $(\mathcal{Q}, \mathcal{W})$ are locally convex cones, the operator T is called *(uniformly) continuous* if for every $W \in \mathcal{W}$ one can find $U \in \mathfrak{U}$ such that $(T \times T)(U) \subseteq W$.

A *linear functional* on $\mathcal{P}$ is a linear operator $\mu: \mathcal{P} \to \overline{\mathbb{R}}$. We denote the set of all linear functional on $\mathcal{P}$ by $L(\mathcal{P})$ (the algebraic dual of $\mathcal{P}$). For a subset F of $\mathcal{P}^2$ we define *polar* F° as follows

$$F^\circ = \{\mu \in L(\mathcal{P}): \mu(a) \leq \mu(b) + 1, \forall (a,b) \in F\}.$$

Clearly $(\{(0,0)\})^\circ = L(\mathcal{P})$. A linear functional μ on $(\mathcal{P}, \mathfrak{U})$ is (uniformly) continuous if $\mu \in U^\circ$ for some $U \in \mathfrak{U}$. The *dual cone* $\mathcal{P}^*$ of a locally convex cone $(\mathcal{P}, \mathfrak{U})$ consists of all continuous linear functionals on $\mathcal{P}$ and is the union of all polars U° of neighborhoods $U \in \mathfrak{U}$.

We shall say that the locally convex cone $(\mathcal{P}, \mathfrak{U})$ has the *strict separation property* if the following holds:

(SP) For all $a, b \in \mathcal{P}$ and $U \in \mathfrak{U}$ such that $(a,b) \notin \rho U$ for some $\rho > 1$, there is a linear functional $\mu \in U^\circ$ such that $\mu(a) > \mu(b) + 1$ ([4], II, 2.12).

Also we shall say that the subset U of $\mathcal{P}^2$ has the property (CP) if the following holds:

if $(a,b) \notin U$, then there is $\mu \in \mathcal{P}^*$ such that $\mu(a) > \mu(b) + 1$ and $\mu(c) \leq \mu(d) + 1$ for all $(c,d) \in U$.

Let $(\mathcal{P}, \mathfrak{U})$ be a locally convex cone. The subset B of $\mathcal{P}^2$ is called u-bounded whenever it is absorbed by each $U \in \mathfrak{U}$. The subset A of $\mathcal{P}$ is called bounded above (or below) whenever $A \times \{0\}$ (or $\{0\} \times A$) is u-bounded, respectively (see [2]).

A *dual pair* $(\mathcal{P}, \mathcal{Q})$ consists of two cones $\mathcal{P}$ and $\mathcal{Q}$ with a bilinear mapping

$$(a,x) \to < a,x >: \mathcal{P} \times \mathcal{Q} \to \overline{\mathbb{R}},$$

(see [4]). If $(\mathcal{P}, \mathcal{Q})$ is a dual pair, then every $x \in \mathcal{Q}$ is a linear mapping on $\mathcal{P}$. We denote the coarsest convex quasiuniform structure on $\mathcal{P}$ that makes all $x \in \mathcal{Q}$ continuous by $\mathfrak{U}_\sigma(\mathcal{P}, \mathcal{Q})$. In fact $(\mathcal{P}, \mathfrak{U}_\sigma(\mathcal{P}, \mathcal{Q}))$ is the projective limit of $(\overline{\mathbb{R}}, \tilde{\mathcal{V}})$ by $x \in \mathcal{Q}$ as linear mappings on $\mathcal{P}$ (projective limits of locally convex cones were defined in [5]).

Let $(\mathcal{P}, \mathcal{Q})$ be a dual pair. We shall say that a subset B of $\mathcal{P}$ is $\mathfrak{U}_\sigma(\mathcal{P}, \mathcal{Q})$-bounded below whenever it is bounded below in locally convex cone $(\mathcal{P}, \mathfrak{U}_\sigma(\mathcal{P}, \mathcal{Q}))$. Let $\mathcal{B}$ be a collection of $\mathfrak{U}_\sigma(\mathcal{P}, \mathcal{Q})$-bounded below subsets of $\mathcal{P}$ such that

(a) $\alpha B \in \mathcal{B}$ for all $B \in \mathcal{B}$ and $\alpha > 0$,

(b) For all $X, Y \in \mathcal{B}$ there is $Z \in \mathcal{B}$ such that $X \cup Y \subset Z$.

(c) $\mathcal{P}$ is spanned by $\cup_{B \in \mathcal{B}} B$.

For $B \in \mathcal{B}$ we set

$$U_B = \{(x,y) \in \mathcal{Q}^2 : < b,x > \leq < b,y > +1, \quad \text{for all } b \in \mathcal{B}\}$$

and $\mathfrak{U}_{\mathcal{B}}(\mathcal{Q}, \mathcal{P}) = \{U_B : B \in \mathcal{B}\}$. It is proved in [4], page 37, that $\mathfrak{U}_{\mathcal{B}}(\mathcal{Q}, \mathcal{P})$is a convex quasiuniform structure on $\mathcal{Q}$ and $(\mathcal{Q}, \mathfrak{U}_{\mathcal{B}}(\mathcal{Q}, \mathcal{P}))$ is a locally convex cone.

Let $(\mathcal{P}, \mathfrak{U})$ and $(\mathcal{Q}, \mathcal{W})$ be locally convex cones. The linear operator $T: \mathcal{P} \to \mathcal{Q}$ is called u-bounded whenever for every u-bounded subset B of $\mathcal{P}^2$, $(T \times T)(B)$ is u-bounded. The locally convex cone $(\mathcal{P}, \mathfrak{U})$ is called bornological if every u-bounded linear operator from $(\mathcal{P}, \mathfrak{U})$ into any locally convex cone is continuous. The linear operator $T: \mathcal{P} \to \mathcal{Q}$ is called bounded below whenever for every bounded below subset A of $\mathcal{P}$, $T(A)$ is bounded below. The locally convex cone $(\mathcal{P}, \mathfrak{U})$ is called b-bornological if every bounded below linear operator from $(\mathcal{P}, \mathfrak{U})$ into any locally convex cone is continuous (see [2]). Since every u-bounded linear operator is bounded below, every b-bornological cone is bornological.

For a subset F of $\mathcal{P}^2$ we denote by $uch(F)$, the smallest uniformly convex subset of $\mathcal{P}^2$, which contains F and call it the uniform convex hull of F.

Bornological and b-bornological locally convex cones were studied in [2]. We review the construction of these structures briefly: Let $\mathcal{P}$ be a cone and U be a uniformly convex subset of $\mathcal{P}$. We set $\mathcal{P}_U = \{a \in \mathcal{P} : \exists \lambda > 0,\ (0,a) \in \lambda U\}$ and $\mathfrak{U}_U = \{\alpha(U \cap \mathcal{P}_U^2) : \alpha > 0\}$. Then $(\mathcal{P}_U, \mathfrak{U}_U)$ is a locally convex cone (a uc-cone). In [2], we proved that there is the finest convex quasiuniform structure $\mathfrak{U}_\tau$ (or $\mathfrak{U}_{b\tau}$) on locally convex cone $(\mathcal{P}, \mathfrak{U})$ such that $\mathcal{P}^2$ (or $\mathcal{P}$) has the same u-bounded (or bounded below) subsets under $\mathfrak{U}$ and $\mathfrak{U}_\tau$(or $\mathfrak{U}_{b\tau}$). The locally convex cone $(\mathcal{P}, \mathfrak{U}_\tau)$ is the

inductive limit of the uc-cones $(\mathcal{P}_U, \mathfrak{U}_U)_{U\in\mathcal{B}}$, where $\mathcal{B}$ is the collection of all uniformly convex and u-bounded subsets of $\mathcal{P}^2$. Also $(\mathcal{P}, \mathfrak{U}_{b\tau})$ is the inductive limit of the uc-cones $(\mathcal{P}_U, \mathfrak{U}_U)_{U\in\mathcal{B}}$, where $\mathcal{B} = \{uch(\{0\} \times B): B \text{ is bounded below}\}$. The locally convex cone $(\mathcal{P}, \mathfrak{U})$ is bornological or b-bornological if and only if $\mathfrak{U}$ is equivalent to $\mathfrak{U}_\tau$ or $\mathfrak{U}_{b\tau}$, respectively.

Let $(\mathcal{P}, \mathfrak{U})$ be a locally convex cone. A net $(x_i)_{i\in\mathcal{I}}$ in $(\mathcal{P}, \mathfrak{U})$ is called lower (upper) Cauchy if for every $U \in \mathfrak{U}$ there is some $\gamma_U \in \mathcal{I}$ such that $(x_\beta, x_\alpha) \in U$ (respectively, $(x_\alpha, x_\beta) \in U$) for all $\alpha, \beta \in \mathcal{I}$ with $\beta \geq \alpha \geq \gamma_U$. Also $(x_i)_{i\in\mathcal{I}}$ is called symmetric Cauchy if for each $U \in \mathfrak{U}$ there is some $\gamma_U \in \mathcal{I}$ such that $(x_\beta, x_\alpha) \in U$ for all $\alpha, \beta \in \mathcal{I}$ with $\beta, \alpha \geq \gamma_U$. The locally convex cone $(\mathcal{P}, \mathfrak{U})$ is called lower (upper and symmetric) complete if every lower (respectively, upper and symmetric) Cauchy net converges to an element of $\mathcal{P}$ in the lower (respectively, upper and symmetric) topology.

The locally convex cone $(\mathcal{P}, \mathfrak{U})$ is called sequentially lower (upper and symmetric) complete if every lower (respectively, upper and symmetric) Cauchy sequence converges to an element of $\mathcal{P}$ in the lower (respectively, upper and symmetric) topology.

2. Locally convex cones with webs

Firstly, we define the concept of *web* in cones. We mean a web $\mathcal{W}$ in a cone $\mathcal{P}$ as a countable collection of uniformly convex subsets of $\mathcal{P}^2$, indexed by the finite sequences of positive integers and arranged by *layers*. The first layer consists of the sequence $(W_i)_{i\in\mathbb{N}}$, whose union absorbs each point of $W = uch(\{0\} \times \mathcal{P})$ i.e. for each $(a, b) \in W$ there are $\lambda > 0$ and $m \in \mathbb{N}$ such that $(a, b) \in \lambda W_m$. For each W_i there is a sequence $(W_{ij})_{j\in\mathbb{N}}$ of subsets $\frac{1}{2}W_i$, whose union, i.e. $\cup_{j\in\mathbb{N}} W_{ij}$ absorbs each points of W_i. All the sets W_{ij}, when i and j vary, consist the second layer of $\mathcal{W}$. For each set W_{ij} there is a sequence $(W_{ijk})_{k\in\mathbb{N}}$ of subsets of $\frac{1}{2}W_{ij}$, whose union, i.e. $\cup_{k\in\mathbb{N}} W_{ijk}$ absorbs each point of W_{ij}. By the induction, the sets $W_{i_1 i_2 \cdots i_n}$ can be defined. A *strand* of the web $\mathcal{W}$ is any sequence $W_i, W_{ij}, W_{ijk}, \ldots$, one from each layer. Thus each infinite sequence $(i_n)_{n\in\mathbb{N}}$ of positive integers determines the strand $W_{i_1}, W_{i_1 i_2}, W_{i_1 i_2 i_3}, \ldots, W_{i_1 i_2 \cdots i_n}, \ldots$. Then we can denote a typical strand by $(W_n)_{n\in\mathbb{N}}$. Therefore $W_{n+1} \subseteq \frac{1}{2}W_n$ for each $n \in \mathbb{N}$.

Definition 2.1. Let $(\mathcal{P}, \mathfrak{U})$ be a locally convex cone. We shall say that the web $\mathcal{W}$ is compatible with the convex quasiuniform structure of $\mathcal{P}$, i.e. $\mathfrak{U}$, whenever for each strand $(W_n)_{n\in\mathbb{N}}$ and each $U \in \mathfrak{U}$, there is $n \in \mathbb{N}$ such that $W_n \subseteq U$.

For a subset F of $\mathcal{P}^2$ and $a \in \mathcal{P}$, we set $F(a) = \{b \in \mathcal{P}: (b, a) \in F\}$, $(a)F = \{b \in \mathcal{P}: (a, b) \in \mathcal{P}\}$ and $F(a)F = F(a) \cap (a)F$.

Lemma 2.2. *Let $(\mathcal{P},\mathfrak{U})$ be a locally convex cone. Then the web $\mathcal{W}$ is compatible with the convex quasiuniform structure $\mathfrak{U}$ if and only if for each $x\in\mathcal{P}$ and every strand (W_n) and each choice $x_n\in W_n(x)$(or $x_n\in(x)W_n$) the sequence $(x_n)_{n\in\mathbb{N}}$ is upper (or lower) convergent to x.*

Proof. Let $x\in\mathcal{P}$ be arbitrary and (W_n) be a strand of $\mathcal{W}$. If the web $\mathcal{W}$ is compatible with the convex quasiuniform structure $\mathfrak{U}$ and $x_n\in W_n(x)$ (or $x_n\in(x)W_n$) for each $n\in\mathbb{N}$, then for every $U\in\mathfrak{U}$, there is $m\in\mathbb{N}$ such that $W_m\subseteq U$. Then $W_n\subseteq U$ for all $n\geq m$ by the definition of a web. This implies that $W_n(x)\subseteq U(x)$ (or $(x)W_n\subseteq(x)U$). This shows that $x_n\in U(x)$ (or $x_n\in(x)U$) for all $n\geq m$ i.e. $(x_n)_{n\in\mathbb{N}}$ is upper (or lower) convergent to x. For the converse let $U\in\mathfrak{U}$ and (W_n) be a strand. For arbitrary $x\in\mathcal{P}$ and $(x_n)_{n\in\mathbb{N}}\in(W_n)_{n\in\mathbb{N}}$ there is $N\in\mathbb{N}$ such that for each $n\geq N$ we have $(x_n,x)\in U$ (i.e. $(x_n)_{n\in\mathbb{N}}\in(W_n)_{n\in\mathbb{N}}$ is upper convergent to x). Now, since x_N is an arbitrary element of $W_N(x)$, we have $W_N(x)\subseteq U(x)$, for all $x\in\mathcal{P}$. Now, suppose $(a,b)\in W_N$. Then $a\in W_n(b)\subset U(b)$. Thus $(a,b)\in U$. Then the web $\mathcal{W}$ is compatible. Similarly, if $(x_n)_{n\in\mathbb{N}}\in(W_n)_{n\in\mathbb{N}}$ is lower convergent to x, then we can prove that the web $\mathcal{W}$ is compatible.

Corollary 2.3. *Let $(\mathcal{P},\mathfrak{U})$ be a locally convex cone. If the web $\mathcal{W}$ is compatible with the convex quasiuniform structure $\mathfrak{U}$, then for each $x\in\mathcal{P}$ and every strand (W_n) and each choice $x_n\in W_n(x)W_n$ the sequence (x_n) is symmetric convergent to x.*

Lemma 2.4. *Let $(\mathcal{P},\mathfrak{U})$ be a locally convex cone and the web $\mathcal{W}$ is compatible with the convex quasiuniform structure $\mathfrak{U}$. For the strand $(W_n)_{n\in\mathbb{N}}$and each choice $x_n\in W(0)$(or $x_n\in(0)W$), the sequence of partial sums of the series $\sum x_n$ is lower (or upper) Cauchy.*

Proof. Let $U\in\mathfrak{U}$ and for each $n\in\mathbb{N}$, $x_n\in W_n(0)$. Since $\mathcal{W}$ is compatible with the convex quasiuniform structure $\mathfrak{U}$, there is $N\in\mathbb{N}$ such that $W_N\subseteq\frac{1}{2}U$. Let $s_n=\sum_{i=1}^{n}x_i$. For $m\geq n\geq N$ we have

$$\begin{aligned}(s_m,s_n)=(s_n+\sum_{i=n+1}^{m}x_i,s_n)&=(s_n,s_n)+(\sum_{i=n+1}^{m}x_i,0)\in\Delta+W_{n+1}+\cdots+W_m\\&\subseteq\Delta+\tfrac{1}{2}W_n+\cdots+\tfrac{1}{2^{m-n}}W_n\\&\subseteq\Delta+W_n\\&\subseteq\Delta+W_N\subseteq\tfrac{1}{2}U+\tfrac{1}{2}U\subseteq U.\end{aligned}$$

Then (s_n) is lower Cauchy. Similarly, if $x_n\in(0)W_n$, then we can prove the sequence of partial sums of the series $\sum x_n$ is upper Cauchy.

Corollary 2.5. *Let $(\mathcal{P},\mathfrak{U})$ be a locally convex cone and the web $\mathcal{W}$ is compatible with the convex quasiuniform structure $\mathfrak{U}$. For the strand $(W_n)_{n\in\mathbb{N}}$ and*

each choice $x_n \in W(0)W$, the sequence of partial sums of the series $\sum x_n$ is lower, upper and symmetric Cauchy.

Corollary 2.6. *Let $(\mathcal{P},\mathfrak{U})$ be a sequentially symmetric complete locally convex cone and the web $\mathcal{W}$ is compatible with the convex quasiuniform structure $\mathfrak{U}$. For the strand $(W_n)_{n\in\mathbb{N}}$ and each choice $x_n \in W(0)W$ the series $\sum x_n$ is symmetric convergent.*

Definition 2.7. *Let $(\mathcal{P},\mathfrak{U})$ be a locally convex cone. The compatible web $\mathcal{W}$ in $(\mathcal{P},\mathfrak{U})$ is called completing if for each strand $(W_n)_{n\in\mathbb{N}}$ the series $\sum x_n$ is symmetric convergent for every choice $x_n \in W_n(0)W_n$.*

By Corollary 2.6, in every sequentially symmetric complete locally convex cone, each compatible web is completing.

Example 2.8. *Let E endowed with the topology τ be a locally convex space and $\mathcal{V}$ be a neighborhood base of convex and balanced sets for τ. For each $V \in \mathcal{V}$ we set $\tilde{V} = \{(a,b) \in E^2 : a - b \in V\}$. Then the collection $\tilde{\mathcal{V}} = \{\tilde{V} : V \in \mathcal{V}\}$ is a convex quasiuniform structure on E^2 and $(E,\tilde{\mathcal{V}})$ is a locally convex cone. The upper, lower and symmetric topologies of $(E,\tilde{\mathcal{V}})$ is identical with the topology τ. Now, let $\mathcal{M}$ be a web in $(E,\tilde{\mathcal{V}})$ as a locally convex cone. Then the collection $\mathcal{W} = \{W(0)W : W \in \mathcal{M}\}$ is a web in E as a locally convex space. Also, if $\mathcal{N}$ is a web in E as a locally convex space, then the collection $\tilde{\mathcal{N}} = \{\tilde{N} : N \in \mathcal{N}\}$, where $\tilde{N} = \{(a,b) \in E^2 : a - b \in N\}$ is a web in $(E,\tilde{\mathcal{V}})$ as a locally convex space. If $\mathcal{N}$ is completing or compatible in E as a locally convex space, then $\tilde{\mathcal{N}}$ is completing or compatible in $(E,\tilde{\mathcal{V}})$, since the upper, lower and symmetric topologies of $(E,\tilde{\mathcal{V}})$ is identical with τ. Therefore the concept of web in locally convex cones is an extension of the concept of web in locally convex spaces.*

Example 2.9. *We consider the cone $\mathcal{M} = \{[a,b] : a,b \in \mathbb{R} \text{ and } a \le b\}$ endowed with the usual algebraic operations. We set $B = [-1,1]$ and*

$$\tilde{B} = \{([a,b],[c,d]) \in \mathcal{M}^2 : [a,b] \subseteq [c,d] + [-1,1]\}$$
$$= \{([a,b],[c,d]) \in \mathcal{M}^2 : c - 1 \le a \le b \le d + 1\}.$$

Then $\mathfrak{U} = \{\alpha\tilde{B} : \alpha > 0\}$ is a convex quasiuniform structure on $\mathcal{M}$ and $(\mathcal{M},\mathfrak{U})$ is a locally convex cone. We prove that $(\mathcal{M},\mathfrak{U})$ is symmetric complete. Since $(\mathcal{M},\mathfrak{U})$ is a uc-cone, it is enough to show that every symmetric Cauchy sequence is symmetric convergent. Let $([a_n,b_n])_{n\in\mathbb{N}}$ is a symmetric Cauchy sequence in $\mathcal{M}$. Then for each $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that for each $m,n \ge n_0$, $([a_n,b_n],[a_m,b_m]) \in \varepsilon\tilde{B}$. Then for each $m \ge n \ge n_0$, $a_m - \varepsilon \le a_n \le b_n \le b_m + \varepsilon$ and $a_n - \varepsilon \le a_m \le b_m \le b_n + \varepsilon$. This shows that $|a_n - a_m| \le \varepsilon$ and $|b_n - b_m| \le \varepsilon$ for each $m \ge n \ge n_0$. Therefore the sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are Cauchy sequences in $\mathbb{R}$ with respect to the usual topology. Since $\mathbb{R}$ is complete with respect to the usual topology, there are $a,b \in \mathbb{R}$ such that $a_n \to a$ and $b_n \to b$. It is clear that $a \le b$. We

prove that $[a_n, b_n] \to [a, b]$ with respect to the symmetric topology of $\mathcal{M}$. For $\varepsilon > 0$ there are $k_1, k_2 \in \mathbb{N}$ such that for each $n \geq k_1$, $|a_n - a| \leq \varepsilon$ and for each $n \geq k_2$, $|b_n - b| \leq \varepsilon$. Now, for each $n \geq \max\{k_1, k_2\}$, we have $a - \varepsilon \leq a_n \leq b_n \leq b + \varepsilon$ and $a_n - \varepsilon \leq a \leq b \leq b_n + \varepsilon$. Therefore $[a_n, b_n] \to [a, b]$ with respect to the symmetric topology and $(\mathcal{M}, \mathfrak{U})$ is symmetric complete. We construct the web $\mathcal{W}$ on $\mathcal{M}$ as follows:

$$W_{i_1} = \tilde{B}, W_{i_1 i_2} = \frac{1}{2}\tilde{B}, \dots, W_{i_1 i_2 \cdots i_n} = \frac{1}{2^{n-1}}\tilde{B}.$$

Clearly, $\mathcal{W}$ is a compatible web on $\mathcal{P}$. Now since $(\mathcal{M}, \mathfrak{U})$ is symmetric complete, $\mathcal{W}$ is completing by Corollary 2.6.

Proposition 2.10. *Let $(\mathcal{P}, \mathfrak{U})$ be a sequentially symmetric complete locally convex cone. If $\mathfrak{U}$ has a countable base, then $(\mathcal{P}, \mathfrak{U})$ has a completing web.*

Proof. We choose the base $(U_n)_{n\in\mathbb{N}}$ for $\mathfrak{U}$ such that $U_{n+1} \subseteq U_n$ for each $n \in \mathbb{N}$. Let $\mathcal{W}$ is the web formed by taking every set in the nth layer to be U_n. Clearly, $\mathcal{W}$ is a compatible web. Now, since $(\mathcal{P}, \mathfrak{U})$ is sequentially symmetric complete, then $\mathcal{W}$ is completing.

Corollary 2.11. *Let $(\mathcal{P}, \mathfrak{U})$ be a sequentially symmetric complete uc-cone. Then $(\mathcal{P}, \mathfrak{U})$ has a completing web.*

The concept of completion for locally convex cones has been established in [3]. For a locally convex cone $(\mathcal{P}, \mathfrak{U})$ with (SP), the completion $\hat{\mathcal{P}}$ of $\mathcal{P}$, is the subcone $\bigcap_{U\in\mathfrak{U}} (\mathcal{P} + (\{0\} \times U^\circ)^\circ)$ of $L(\mathcal{P}^*)$ endowed with the convex convex quasiuniform structure $\hat{\mathfrak{U}} = \mathfrak{U}_{\mathcal{B}}(\hat{\mathcal{P}}, \mathcal{P}^*)$, where $\mathcal{B} = \{U^\circ : U \in \mathfrak{U}\}$. For details see [3].

Lemma 2.12. *Let $(\mathcal{P}, \mathfrak{U})$ be a locally convex cone with (SP) and $\mathcal{B}$ be a collection of $U_\sigma(\mathcal{P}, \mathcal{P}^*)$-bounded subsets of $\mathcal{P}$ with the properties (a), (b) and (c). If every linear functional on $\mathcal{P}$ which is continuous on each $(\mathcal{P}_{A'}, \mathfrak{U}_{A'})$, where $A' = uch(\{0\} \times A)$ for $A \in \mathcal{B}$, is continuous on $\mathcal{P}$, then the locally convex cone $(\mathcal{P}^*, \mathfrak{U}_{\mathcal{B}}(\mathcal{P}^*, \mathcal{P}))$ is complete with respect to upper and symmetric topology.*

Proof. We prove that the completion of $\mathcal{P}^*$ under $\mathfrak{U}_{\mathcal{B}}(\mathcal{P}^*, \mathcal{P})$ is identical with $\mathcal{P}$. Let $\varphi \in \widehat{\mathcal{P}^*} = \bigcap_{B\in\mathcal{B}} (\mathcal{P}^* + (\{0\} \times U_B^\circ)^\circ) = \bigcap_{B\in\mathcal{B}} (\bigcup_{u\in\mathfrak{U}} U^\circ + (\{0\} \times U_B^\circ)^\circ)$. Then for each $B \in \mathcal{B}$ there is $U \in \mathfrak{U}$ such that $\varphi \in U^\circ + (\{0\} \times U_B)^\circ$. Then there exist $\mu \in U^\circ$ and $\vartheta \in U_B^\circ$ such that $\varphi = \mu + \vartheta$. For $A \in \mathcal{B}$, we have $A \subseteq U_A^\circ$ and then $\{0\} \times A \subseteq \{0\} \times U_A^\circ$. This shows that $(\{0\} \times U_A^\circ)^\circ \subseteq (\{0\} \times A)^\circ = (A')^\circ \subseteq \mathcal{P}_{A'}^*$. This yields that ϑ is continuous on each $\mathcal{P}_{A'}$. Now, the assumption shows that ϑ is continuous on $\mathcal{P}$ and then $\vartheta \in \mathcal{P}^*$. Therefore $\varphi = \mu + \vartheta \in \mathcal{P}^* + \mathcal{P}^* = \mathcal{P}^*$. Then $\widehat{\mathcal{P}^*} = \mathcal{P}^*$. This shows that $(\mathcal{P}^*, \mathfrak{U}_{\mathcal{B}}(\mathcal{P}^*, \mathcal{P}))$ is complete with respect to upper and symmetric topology.

Proposition 2.13. *Let $(\mathcal{P}, \mathfrak{U})$ be a b-bornological locally convex cone with (SP) and $\mathcal{B}$ be a collection of $U_\sigma(\mathcal{P}, \mathcal{P}^*)$-bounded subsets of $\mathcal{P}$ with the properties (a), (b) and (c). If every lower compact subset K of $\mathcal{P}$ is contained in some $A \in \mathcal{B}$, then $(\mathcal{P}^*, \mathfrak{U}_{\mathcal{B}}(\mathcal{P}^*, \mathcal{P}))$ is complete with respect to upper and symmetric topology.*

Proof. Let μ be a linear functional on $\mathcal{P}$ which is continuous on each $(\mathcal{P}_{A'}, \mathfrak{U}_{A'})$, where $A' = uch(\{0\} \times A)$ for $A \in \mathcal{B}$. We prove that μ is bounded below on $\mathcal{P}$. If it is not true, then there are points x_n in some bounded below subset B of $\mathcal{P}$ such that $\mu(x_n) \leq -n^2$, $n \in \mathbb{N}$. Since B is bounded below, the sequence $(\frac{x_n}{n})_{n\in\mathbb{N}}$ is lower convergent to 0 in $\mathcal{P}$. This shows that the set $\{\frac{x_n}{n} : n \in \mathbb{N}\} \cup \{0\}$ is lower compact and then it is contained in some $A \in \mathcal{B}$. Since $\mu(\frac{x_n}{n}) \leq -n$, we conclude that μ is not lower bounded on A. This contradiction shows that μ is bounded below on $\mathcal{P}$. Since $(\mathcal{P}, \mathfrak{U})$ is a b-bornological locally convex cone, μ is continuous. Now Lemma 2.12 shows that $(\mathcal{P}^*, \mathfrak{U}_{\mathcal{B}}(\mathcal{P}^*, \mathcal{P}))$ is complete with respect to upper and symmetric topology.

Suppose $(\mathcal{P}, \mathfrak{U})$ is a locally convex cone. We set $\mathfrak{U}_{b\beta}(\mathcal{P}^*, \mathcal{P}) = \mathfrak{U}_{\mathcal{B}}(\mathcal{P}^*, \mathcal{P})$, where $\mathcal{B}$ is the collection of all bounded below subsets of $\mathcal{P}$.

Corollary 2.14. *Let $(\mathcal{P}, \mathfrak{U})$ be a b-bornological locally convex cone with (SP). Then $(\mathcal{P}^*, \mathfrak{U}_{b\beta}(\mathcal{P}^*, \mathcal{P}))$ is upper and symmetric complete.*
Proof. For, every lower compact subset of $\mathcal{P}$ is bounded below.

Theorem 2.15 *Let $(\mathcal{P}, \mathfrak{U})$ be a b-bornological locally convex cone with (SP). If $\mathfrak{U}$ has a countable base, then $(\mathcal{P}^*, \mathfrak{U}_{b\beta}(\mathcal{P}^*, \mathcal{P}))$ has a completing web.*
Proof. Let (U_n) be a base for $\mathfrak{U}$ and B be the collection of all bounded below subsets of $\mathcal{P}$. For $U \in \mathfrak{U}$ we set $\widetilde{U^\circ} = \{(\mu, \psi) \in \mathcal{P}^* \times \mathcal{P}^* : \psi \in \mu + U^\circ\}$. We define $\mathcal{W}$ as follows:

$$W_{i_1} = \widetilde{U}_{i_1^\circ}, W_{i_1 i_2} = \frac{1}{2}\widetilde{U^\circ}_{i_1}, \dots, W_{i_1 i_2 \cdots i_n} = \frac{1}{2^{n-1}}\widetilde{U^\circ}_{i_1}.$$

Then clearly $\mathcal{W}$ is a web. Now, let $A \in B$ be arbitrary and $(W_n)_{n\in\mathbb{N}}$ be an strand of $\mathcal{W}$. Then $W_n = \frac{1}{2^{n-1}}\widetilde{U^\circ}_i$ for some $i \in \mathbb{N}$. Since A is bounded below, there is $\lambda > 0$ such that $\{0\} \times A \subseteq \lambda U_i$. Then $\frac{1}{\lambda}U_{i_1}^\circ \subseteq (\{0\} \times A)^\circ$. This shows that

$$\lambda\widetilde{U_i^\circ} = \frac{1}{\lambda}\widetilde{U_i^\circ} \subseteq (\widetilde{\{0\} \times A})^\circ \subseteq U_A.$$

Now there is $m \in \mathbb{N}$ such that $\frac{1}{2^{m-1}} \leq \lambda$. Therefore

$$W_m = \frac{1}{2^{m-1}}\widetilde{U^\circ}_i \subseteq \lambda\widetilde{U_i^\circ} \subseteq U_A.$$

Then $\mathcal{W}$ is a compatible web. Now, since $(\mathcal{P}^*, \mathfrak{U}_{b\beta}(\mathcal{P}^*, \mathcal{P}))$ is symmetric complete by the Corollary 2.14, then $\mathcal{W}$ is a completing web by Lemma 2.6.

Corollary 2.16. *Let $(\mathcal{P}, \mathfrak{U})$ be a b-bornological uc-cone with (SP). Then $(\mathcal{P}^*, \mathfrak{U}_{b\beta}(\mathcal{P}^*, \mathcal{P}))$ has a completing web by Theorem 2.15. In fact every uc-cone has a countable base and $\{\frac{1}{n}U : n \in \mathbb{N}\}$ is a base for $\mathfrak{U}$.*

Now, we turn to study some stability properties of the classes of locally convex cones with webs. If a locally convex cone $(\mathcal{P}, \mathfrak{U})$ has a compatible web, then clearly every subcone of $\mathcal{P}$ has a compatible web. Also, if $(\mathcal{P}, \mathfrak{U})$ has a completing

web, then every sequentially symmetric complete subcone of $\mathcal{P}$ has a completing web.

Proposition 2.17. *Let $(\mathcal{P},\mathfrak{U})$ and $(\mathcal{Q},\mathcal{V})$ be separated locally convex cones and $T:\mathcal{P}\to\mathcal{Q}$ be a continuous linear operator.*

(i) If $(\mathcal{P},\mathfrak{U})$ has a compatible web, then $(T(\mathcal{P}),\mathcal{M})$ has a compatible web, where

$$\mathcal{M}=\{V\cap(T(\mathcal{P}))^2: V\in\mathcal{V}\}.$$

(ii) If $(\mathcal{P},\mathfrak{U})$ has a completing web, then $(T(\mathcal{P}),\mathcal{M})$ has a completing web.

Proof. For (i), let $\mathcal{W}$ be a compatible web on $\mathcal{P}$. We define $\mathcal{F}$ as follows: $F_{i_1i_2\cdots i_k}=(T\times T)(W_{i_1i_2\cdots i_k})$. Then it is clear that $\mathcal{F}$ is a web in $(T(\mathcal{P}),\mathcal{M})$. We prove $\mathcal{F}$ it is compatible. Let $V\in\mathcal{M}$ and $(F_n)_{n\in\mathbb{N}}$ be a strand of $\mathcal{F}$, then there is a strand $(W_n)_{n\in\mathbb{N}}$ of $\mathcal{W}$ such that $F_n=(T\times T)(W_n)$. Also, there is $U\in\mathfrak{U}$ such that $(T\times T)(U)\subseteq V$, since T is continuous. Now, since $\mathcal{W}$ is a compatible web, there is $m\in\mathbb{N}$ such that $W_m\subseteq U$. Then $F_m=(T\times T)(W_m)\subseteq(T\times T)(U)\subseteq V$. This shows that $\mathcal{F}$ is compatible.

For (ii), let $\mathcal{W}$ be a completing web on $\mathcal{P}$. We show that $\mathcal{F}$ as defined above is completing. Let $x_n\in F_n(0)F_n$. Then $T^{-1}(\{x_n\})\subseteq W_n(0)W_n$. Let $a_n\in T^{-1}(\{x_n\})$ for each $n\in\mathbb{N}$. Since $\mathcal{W}$ is completing, $\sum a_n$ is symmetric convergent. Now, the continuity of T shows that $\Sigma x_n=\Sigma T(a_n)$ is symmetric convergent.

Theorem 2.18. *Let $(\mathcal{P},\mathfrak{U})$ be the projective limit of separated locally convex cones $(\mathcal{P}_n,\mathfrak{U}_n)$ by the linear mappings $\vartheta_n:\mathcal{P}\to\mathcal{P}_n$. If each $(\mathcal{P}_n,\mathfrak{U}_n)$ has a compatible web $\mathcal{W}^n$, then $(\mathcal{P},\mathfrak{U})$ has a compatible web $\mathcal{W}$. Also suppose that*

(1) whenever (x_r) is a sequence in $\mathcal{P}$ such that $(\vartheta_n(x_r))$ is symmetric convergent in $\mathcal{P}_n$ for each $n\in\mathbb{N}$, then (x_r) is symmetric convergent in $\mathcal{P}$.

Then $(\mathcal{P},\mathfrak{U})$ has a completing web $\mathcal{W}$ if each $(\mathcal{P}_n,\mathfrak{U}_n)$ has a completing web $\mathcal{W}^n$.

Proof. For the first layer of $\mathcal{W}$, we consider the sets $(\vartheta_1\times\vartheta_1)^{-1}(W_i^1)$, where $(W_i^1)_{i\in\mathbb{N}}$ is the first layer of $\mathcal{W}^1$. For the sets of the second layer of $\mathcal{W}$ which is determined by $(\vartheta_1\times\vartheta_1)^{-1}(W_i^1)$, we take the sets

$$(\vartheta_1\times\vartheta_1)^{-1}(W_{ij}^1)\cap(\vartheta_2\times\vartheta_2)^{-1}(W_r^2)\quad(j,r=1,2,\cdots),$$

where (W_r^2) is the first layer of $\mathcal{W}^2$. If we continue this way, then we can determine the strand $\mathcal{W}_n$ of $\mathcal{W}$ by

$$W_n=(\vartheta_1\times\vartheta_1)^{-1}(W_n^1)\cap(\vartheta_2\times\vartheta_2)^{-1}(W_{n-1}^2)\cap\cdots\cap(\vartheta_n\times\vartheta_n)^{-1}(W_1^n),$$

where (W_n^i) is a strand in the web $\mathcal{W}^i$. Now, let $U\in\mathfrak{U}$. Since $(\mathcal{P},\mathfrak{U})$ is the projective limit of separated locally convex cones $(\mathcal{P}_n,\mathfrak{U}_n)$, then there are $n\in\mathbb{N}$ and $U_i\in\mathfrak{U}_i$, $i=1,\dots,n$ such that $\bigcap_{i=1}^n(\vartheta_i\times\vartheta_i)^{-1}(U_i)\subseteq U$. Since $\mathcal{W}^i$ is a compatible web for each $i\in\{1,\dots,n\}$, there is $m\geq n$ such that

$$W_m^1\subseteq U_1, W_{m-1}^2\subseteq U_2,\dots,W_{m-n+1}^n\subseteq U_n.$$

This shows that

$$W_m=\bigcap_{i=1}^m(\vartheta_i\times\vartheta_i)^{-1}(W_{m-i+1}^i)\subseteq\bigcap_{i=1}^n(\vartheta_i\times\vartheta_i)^{-1}(W_{m-i+1}^i)$$

$$\subseteq \bigcap_{i=1}^{n} (\vartheta_i \times \vartheta_i)^{-1} U_i \subseteq U.$$

Therefore $\mathcal{W}$ is compatible.

Now suppose (1) holds and $a_n \in W_n$ for each $n \in \mathbb{N}$. Then for $i, n \in \mathbb{N}$ with $n \geq i$, we have $\vartheta_i(a_n) \in W_n^i$. Since $\mathcal{W}^i$ is completing, $\sum_{n \geq i} \vartheta_i(a_n)$ is symmetric convergent. Then for $b_r = \Sigma_{n=1}^{r} a_n$, $(v_i(b_r))_{n \in \mathbb{N}}$ is symmetric convergent in $(\mathcal{P}_i, \mathfrak{U}_i)$ for each $i \in \mathbb{N}$. Now (1) shows that $\Sigma_{n=1}^{\infty} a_n$ is symmetric convergent in $(\mathcal{P}, \mathfrak{U})$.

Corollary 2.19. *If $(\mathcal{P}, \mathfrak{U})$ is sequentially symmetric complete and is the projective limit of sequences of locally convex cones with compatible webs then $(\mathcal{P}, \mathfrak{U})$ has a completing web.*

Corollary 2.20. *If $(\mathcal{P}, \mathfrak{U}) = \prod_{n \in \mathbb{N}} (\mathcal{P}_n, \mathfrak{U}_n)$ and each $(\mathcal{P}_n, \mathfrak{U}_n)$ has a completing web, then $(\mathcal{P}, \mathfrak{U})$ has a completing web. In fact, for every $n \in \mathbb{N}$, $\vartheta_n = p_n$ is the projection map form $\mathcal{P}$ into $\mathcal{P}_n$ which satisfies condition* (1).

Example 2.21. Consider the cone $\mathcal{P} = \prod_{n=1}^{\infty} (\overline{\mathbb{R}}_+, \tilde{\mathcal{V}})$ i.e. the cone all sequences in $\overline{\mathbb{R}}_+$. We consider the projective limit convex quasiuniform structure on $\mathcal{P}$ and denote it by $\mathfrak{U}$. This structure consists of the sets

$$U_\varepsilon = \{((x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}}) \in \mathcal{P}^2 : \forall n \in \mathbb{N};\ x_n \leq y_n + \varepsilon\}.$$

Since $(\overline{\mathbb{R}}_+, \tilde{\mathcal{V}})$ is a symmetric complete uc-cone, it has a completing web by Corollary 2.11. Now, Corollary 2.20, shows that $(\mathcal{P}, \mathfrak{U})$ has a completing web.

2. The closed graph theorem

The main aim of this section is to prove a closed graph type theorem for a linear operator from a complete separated bornological locally convex cone into a locally convex cone with some properties. Suppose $(\mathcal{P}, \mathfrak{U})$ and $(\mathcal{Q}, \mathcal{V})$ are locally convex cones. The graph of T is the set $graph(T) = \{(x, T(x)) : x \in \mathcal{P}\}$. If T is symmetric continuous, then its graph is symmetric closed in $(\mathcal{P} \times \mathcal{Q}, \mathcal{M})$, where $\mathcal{M}$ is the projective limit convex quasiuniform structure on $\mathcal{P} \times \mathcal{Q}$. But the converse is not true in general. In this section we find some conditions under which the converse holds.

For the proving the main results of this section i.e. the the closed graph theorem, we need some useful topological results which will be established in the following three Lemmas.

Lemma 3.1. *Let $(\mathcal{P}, \mathfrak{U})$ be a separated locally convex cone. Then $\mathcal{P}$ endowed with the symmetric topology has a base of symmetric closed neighborhoods.*

Proof. Let $a \in \mathcal{P}$. It is enough to show that $\overline{(\frac{1}{2}U)(a)(\frac{1}{2}U)}^{s} \subseteq U(a)U$, where $\overline{(\frac{1}{2}U)(a)(\frac{1}{2}U)}^{s}$ denotes the closure of $(\frac{1}{2}U)(a)(\frac{1}{2}U)$ with respect to the symmetric topology. Let $b \in \overline{(\frac{1}{2}U)(a)(\frac{1}{2}U)}^{s}$. Then

$$(\frac{1}{2}U)(b)(\frac{1}{2}U) \cap (\frac{1}{2}U)(a)(\frac{1}{2}U) \neq \emptyset.$$

Let $c \in (\frac{1}{2}U)(b)(\frac{1}{2}U) \cap (\frac{1}{2}U)(a)(\frac{1}{2}U)$. Then $(b,c) \in (\frac{1}{2}U)$,$(c,b) \in (\frac{1}{2}U)$, $(c,a) \in (\frac{1}{2}U)$ and $(a,c) \in (\frac{1}{2}U)$. This shows that $(a,b) \in U$ an $(b,a) \in U$ by (U_3). Then $b \in U(a)U$.

Lemma 3.2. *Let $(\mathcal{P},\mathfrak{U})$ be a Baire space with respect to its symmetric topology. Then the followings hold.*

(a) If $\mathcal{W}$ is any web in $(\mathcal{P},\mathfrak{U})$, then there is a strand $(W_n)_{n\in\mathbb{N}}$ of $\mathcal{W}$ such that $\overline{(0)W_n}^s$ is a symmetric neighborhood (i.e. contains a symmetric neighborhood of an element of $\mathcal{P}$) for each $n \in \mathbb{N}$,

(b) if $\mathcal{V}$ is a web in $(\mathcal{P},\mathfrak{U})$ such that $\cup_{i_1=1}^{\infty} V_{i_1}(0)V_{i_1}$ absorbs each point of $\mathcal{P}$ i.e. for each $a \in \mathcal{P}$ there is $m \in \mathbb{N}$ and $\lambda > 0$ such that $a \in \lambda(V_m(0)V_m)$, then there is a strand $(V_n)_{n\in\mathbb{N}}$ of $\mathcal{V}$ such that $\overline{V_n(0)V_n}^s$ is a symmetric neighborhood for each $n \in \mathbb{N}$.

Proof. For (a), we define a strand $(W_n)_{n\in\mathbb{N}}$ for which each set $(0)W_n$ is not meagre (the set of interior points of $\overline{(0)W_n}^s$ is nonempty): the union of the sets of the first layer absorbs each point of $uch(\{0\} \times \mathcal{P})$. This shows that $\cup_{i_1=1}^{\infty} (0)W_{i_1}$ absorbs each point of $\mathcal{P}$. Since $(\mathcal{P},\mathfrak{U})$ is a Baire space with respect to its symmetric topology, there is $m_1 \in \mathbb{N}$ such that $(0)W_{m_1}$ is not meagre. Now, since each point of $(0)W_m$ absorbed by the union of the sets $(0)W_{m_1 j}$, $j \in \mathbb{N}$, there is $m_2 \in \mathbb{N}$ such that $(0)W_{m_1 m_2}$ is not meagre in $\mathcal{P}$ with respect to symmetric topology. By continuing this way we obtain the non meagre sets $(0)W_{m_1 m_2 \cdots m_n}$. Now, we set $W_n = W_{m_1 m_2 \cdots m_n}$. Since for each $n \in \mathbb{N}$, $(0)W_n$ is not meagre, $\overline{(0)W_n}^s$ has an interior point $a_n \in \mathcal{P}$. Then for each $n \in \mathbb{N}$, there is $U_n \in \mathfrak{U}$ and $b_n \in \mathcal{P}$ such that $U_n(b_n)U_n \subseteq \overline{(0)W_n}^s$. Then $\overline{(0)W_n}^s$ is a symmetric neighborhood of b_n for each $n \in \mathbb{N}$.

For (b), let $\mathcal{V}$ be a web in $(\mathcal{P},\mathfrak{U})$ such that $\cup_{i_1=1}^{\infty} V_{i_1}(0)V_{i_1}$ absorbs each point of $\mathcal{P}$. Then there is $m_1 \in \mathbb{N}$ such that $V_{m_1}(0)V_{m_1}$ is not meagre, since $\mathcal{P}$ is Baire space. Now since each point of $V_{m_1}(0)V_{m_1}$ is absorbed by the union of the sets $V_{m_1 j}(0)V_{m_1 j}$, $j \in \mathbb{N}$ there is $m_2 \in \mathbb{N}$ such that $V_{m_1 m_2}(0)V_{m_1 m_2}$ is not meagre in $\mathcal{P}$ with respect to symmetric topology. By continuing this way we can determine the non-meagre sets $V_{m_1 m_2 \cdots m_n}(0)V_{m_1 m_2 \cdots m_n}$. We set $V_n = V_{m_1 m_2 \cdots m_n}$. Since for each $n \in \mathbb{N}$, $V_n(0)V_n$ is not meagre, $\overline{V_n(0)V_n}^s$ has an interior point $a_n \in \mathcal{P}$. Then for each $n \in \mathbb{N}$, there is $U_n \in \mathfrak{U}$ such that $U_n(a_n)U_n \subseteq \overline{V_n(0)V_n}^s$.

Lemma 3.3. *Let $(\mathcal{P},\mathfrak{U})$ be a symmetric complete separated uc-cone. Then $(\mathcal{P},\mathfrak{U})$ is a Baire space endowed with the symmetric topology.*

Proof. We prove that the intersection of any countable collection of open dense subsets of $\mathcal{P}$ is dense. Let $\mathfrak{U} = \{\alpha U : \alpha > 0\}$ and $\{O_n\}_{n\in\mathbb{N}}$ be a collection of open subsets of $\mathcal{P}$ which are dense in $\mathcal{P}$ with respect to symmetric topology. It is

enough to show that the intersection of any symmetric neighborhood O with $\bigcap_{n=1}^{\infty} O_n$ is nonempty. Since O_1 is dense, then $O \cap O_1 \neq \emptyset$ and hence there is $x_1 \in O \cap O_1$. Now since $O \cap O_1$ is a symmetric neighborhood there is $r_1 > 0$ such that $\overline{(r_1U)(x_1)(r_1U)}^{s} \subseteq O \cap O_1$ by Lemma 3.1. Since $(r_1U)(x_1)(r_1U)$ is a symmetric neighborhood and O_2 is dense there is $x_2 \in (r_1U)(x_1)(r_1U) \cap O_2$ and $r_2 > 0$ such that $\overline{(r_2U)(x_2)(r_2U)}^{s} \subseteq (r_1U)(x_1)(r_1U) \cap O_2$. By the induction there are $0 < r_n < \frac{1}{n}$ and x_n, $n \in \mathbb{N}$ such that $\overline{(r_nU)(x_n)(r_nU)}^{s} \subseteq (r_{n-1}U)(x_{n-1})(r_{n-1}U) \cap O_n$, since O_n is symmetric neighborhood and dense. The above steps show that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is symmetric Cauchy in $\mathcal{P}$. Then $\{x_n\}_{n\in\mathbb{N}}$ is convergent to some $x \in \mathcal{P}$ by symmetric completeness of $\mathcal{P}$. Now closedness of $\overline{(r_nU)(x_n)(r_nU)}^{s}$ shows that $x \in \overline{(r_nU)(x_n)(r_nU)}^{s} \subseteq O$ for all $n \in \mathbb{N}$. On the other hand, we have $x \in O_n$ for each $n \in \mathbb{N}$. Therefore $x \in O \cap (\bigcap_{n=1}^{\infty} O_n)$ and $O \cap (\bigcap_{n=1}^{\infty} O_n) \neq \emptyset$.

Corollary 3.4. *Let $(\mathcal{P}, \mathfrak{U})$ be a symmetric complete separated uc-cone. Then by Lemma 3.3, $(\mathcal{P}, \mathfrak{U})$ is a Baire space with respect to symmetric topology. Now Lemma 3.2 shows that for any web $\mathcal{W}$ in $(\mathcal{P}, \mathfrak{U})$ there is a strand (W_n) such that $\overline{(0)W_n}^{s}$ is a symmetric neighborhood for each $n \in \mathbb{N}$.*

Definition 3.5. *We say that the locally convex cone $(\mathcal{P}, \mathfrak{U})$ is symmetric quasi-full whenever for $U \in \mathfrak{U}$, $c \in \mathcal{P}$ and $a \in U(c)U$ there is $t \in U(0)U$ such that $a = c + t$.*

Example 3.6. The locally convex cone $(\overline{\mathbb{R}}, \tilde{\mathcal{V}})$, where $\tilde{\mathcal{V}} = \{\tilde{\varepsilon}: \varepsilon > 0\}$ and $\tilde{\varepsilon} = \{(a, b) \in \overline{\mathbb{R}}^2 : a \leq b + \varepsilon\}$, is a symmetric quasi-full locally convex cone. For $a \in \overline{\mathbb{R}}$ and $\varepsilon > 0$ we have $\tilde{\varepsilon}(a)\tilde{\varepsilon} = [a - \varepsilon, a + \varepsilon] = a + [-\varepsilon, -\varepsilon] = a + \tilde{\varepsilon}(0)\tilde{\varepsilon}$. For $+\infty \in \overline{\mathbb{R}}$, we have

$$\tilde{\varepsilon}(\infty)\tilde{\varepsilon} = \infty + \tilde{\varepsilon}(0)\tilde{\varepsilon} = \{+\infty\}.$$

Example 3.7. Let $\mathcal{F}(X, \overline{\mathbb{R}})$ be the cone of all $\overline{\mathbb{R}}$-valued functions on nonempty set X. For the constant positive real function $\varepsilon: X \to \overline{\mathbb{R}}$, we set $\tilde{\varepsilon} = \{(f, g) \in (\mathcal{F}(X, \overline{\mathbb{R}}))^2 : f(x) \leq g(x) + \varepsilon\}$. Then $\hat{\mathcal{V}} = \{\tilde{\varepsilon}: \varepsilon > 0\}$ is a convex quasiuniform structure $\mathcal{F}(X, \overline{\mathbb{R}})$. We denote by $\mathcal{F}_b(X, \overline{\mathbb{R}})$ the subcone of $\mathcal{F}(X, \overline{\mathbb{R}})$ which contains all uniformly bounded below $\overline{\mathbb{R}}$-valued functions on X with respect to $\hat{\mathcal{V}}$. Then $(\mathcal{F}_b(X, \overline{\mathbb{R}}), \hat{\mathcal{V}})$ is a locally convex cone. We prove that $(\mathcal{F}_b(X, \overline{\mathbb{R}}), \hat{\mathcal{V}})$ is symmetric quasi-full. Let $f \in \tilde{\varepsilon}(g)\tilde{\varepsilon}$. Then $f(x) \leq g(x) + \varepsilon$ and $g(x) \leq f(x) + \varepsilon$ for all $x \in X$. If for every $x \in X$, $g(x) < \infty$, then $f(x) < +\infty$ for all $x \in X$. Therefore $(f - g)(x) \leq \varepsilon$ and $0 \leq (f - g)(x) + \varepsilon$ for all $x \in X$ in this case. Then $(f - g) \in \tilde{\varepsilon}(0)\tilde{\varepsilon}$. Now, since $f = g + (f - g)$, the assertion holds in this case. On the other hand, if $g(y) = +\infty$, for some $y \in X$ and $g(x) < +\infty$ for all $x \neq y$, then $f(y) = +\infty$, and $f(x) < +\infty$ for all $x \neq y$. Now, we define the function h as following

$$h(x) = \begin{cases} (f - g)(x) & x \neq y \\ 0 & x = y. \end{cases}$$

We have $f = g + h$ and $h \in \tilde{\varepsilon}(0)\tilde{\varepsilon}$. Then the locally convex cone $(\mathcal{F}_b(X, \overline{\mathbb{R}}), \hat{\mathcal{V}})$ is symmetric quasi-full.

Lemma 3.8. *Let $(\mathcal{P}, \mathfrak{U})$ be a symmetric quasi-full locally convex cone and $(\mathcal{Q}, \mathcal{V})$ be another locally convex cone. Then the linear operator $T: \mathcal{P} \to \mathcal{Q}$ is symmetric continuous on $\mathcal{P}$ if and only if it is symmetric continuous at* 0.

Proof. If $T: \mathcal{P} \to \mathcal{Q}$ is symmetric continuous on $\mathcal{P}$, then clearly it is symmetric continuous at 0. We prove the converse. Let T be symmetric continuous at 0 and let $V \in \mathcal{V}$ and $b \in \mathcal{P}$. There is $U \in \mathfrak{U}$ such that $T(U(0)U) \subseteq V(T(0))V = V(0)V$. We prove that $T(U(b)U) \subseteq V(T(b))V$. Let $a \in U(b)U$. Since $(\mathcal{P}, \mathfrak{U})$ is symmetric quasi-full, there is $s \in U(0)U$ such that $a = b + s$. Then

$$T(a) = T(b) + T(s) \in T(b) + T(U(0)U) \subseteq T(b) + V(0)V \subseteq V(T(b))V.$$

Therefore T is symmetric continuous at b.

Definition 3.9. *Let $(\mathcal{P}, \mathfrak{U})$ be a locally convex cone. The web $\mathcal{W}$ on $\mathcal{P}$ is called uniformly completing if it has the following properties*

(c_1) $\mathcal{W}$ is completing,

(c_2) $\cup_{i_1}^{\infty} W_{i_1}(0)W_{i_1}$ absorbs each point of $\mathcal{P}$, where $\{W_{i_1}: i_1 \in \mathbb{N}\}$ is the first layer of $\mathcal{W}$,

(c_3) for a strand (W_n) of $\mathcal{W}$ and $m \in \mathbb{N}$, $c + d \in \overline{W_m(0)W_m}^s$ and $c \in \overline{W_m(0)W_m}^s$ imply that $c \in 2\overline{W_m(0)W_m}^s$.

Example 3.10. We consider the locally convex cone $(\mathcal{M}, \mathfrak{U})$ from the Example 2.9. We have

$$\tilde{B}(\{0\})\tilde{B} = \{[c,d] \in \mathcal{M}: [c,d] \subseteq \{0\} + [-1,1], \{0\} \subseteq [c,d] + [-1,1]\}$$
$$= \{[c,d]: -1 \le c \le d \le 1\}.$$

We prove that the neighborhood $\widetilde{B}(\{0\})\widetilde{B}$ is closed with respect to the symmetric topology: If $[a,b] \in \overline{\widetilde{B}(\{0\})\widetilde{B}}^s$, then there is a sequence $([c_n, d_n])_{n\in\mathbb{N}} \subseteq [-1,1]$ which is symmetric convergent to $[a,b]$. Then the sequences $(c_n)_{n\in\mathbb{N}}$ and $(d_n)_{n\in\mathbb{N}}$ are convergent to a and b respectively. This shows that $-1 \le a \le b \le 1$. Then $[a,b] \in \widetilde{B}(\{0\})\widetilde{B}$. We claim that the web $\mathcal{W}$, which was defined in Example 2.9, is uniformly completing: The condition (c_1) holds by Example 2.9. For (c_2), we have $W_{i_1}(0)W_{i_1} = \widetilde{B}(\{0\})\widetilde{B} = \{[c,d]: -1 \le c \le d \le 1\}$ for each $i_1 \in \mathbb{N}$. For $[a,b] \in \mathcal{M}$, if $[a,b] = \{0\}$ then clearly $[a,b] \in \widetilde{B}(\{0\})\widetilde{B} = W_{i_1}(0)W_{i_1}$ for each $i_1 \in \mathbb{N}$. Let $[a,b] \in \mathcal{M}$ and $[a,b] \neq \{0\}$. We set $\lambda = \max\{|a|, |b|\}$. Then we have $[\frac{a}{\lambda}, \frac{b}{\lambda}] \in W_{i_1}(0)W_{i_1}$ and $[a,b] = \lambda[\frac{a}{\lambda}, \frac{b}{\lambda}] \in c(W_{i_1}(0)W_{i_1})$. This shows that $W_{i_1}(0)W_{i_1}$ absorbs each point of $\mathcal{M}$. For (c_3), let (W_n) be a strand of $\mathcal{W}$ and $m \in \mathbb{N}$. Then $W_m = \frac{1}{2^{m-1}}\widetilde{B}$ and $W_m(0)W_m = (\frac{1}{2^{m-1}}\widetilde{B})(\{0\})(\frac{1}{2^{m-1}}\widetilde{B}) = \frac{1}{2^{m-1}}(\widetilde{B}(\{0\})\widetilde{B})$. This shows that

$$\overline{W_m(0)W_m}^s = \overline{\frac{1}{2^{m-1}}(\widetilde{B}(\{0\})\widetilde{B})}^s = \frac{1}{2^{m-1}}\overline{(\widetilde{B}(\{0\})\widetilde{B})}^s = \frac{1}{2^{m-1}}(\widetilde{B}(\{0\})\widetilde{B}).$$

Let $[a,b] + [e,f] \in \overline{W_m(0)W_m}^s$ and $[e,f] \in \overline{W_m(0)W_m}^s$. Then $[a+e, b+f], [e,f] \in \frac{1}{2^{m-1}}(\widetilde{B}(\{0\})\widetilde{B})$. This shows that $[-f,-e] \in \frac{1}{2^{m-1}}(\widetilde{B}(\{0\})\widetilde{B})$. Then $a+e, b+f, -e, -f \in$

$[-\frac{1}{2^{m-1}}, \frac{1}{2^{m-1}}]$. Therefore $a = (a + e) + (-e) \in [-\frac{2}{2^{m-1}}, \frac{2}{2^{m-1}}]$ and $b = (b + f) + (-f) \in [-\frac{2}{2^{m-1}}, \frac{2}{2^{m-1}}]$. This implies

$$[a, b] \in \frac{2}{2^{m-1}}(\widetilde{B}(\{0\})\widetilde{B}) = 2W_m(0)W_m = 2\overline{W_m(0)W_m}^{s}.$$

Then $\mathcal{W}$ is a uniformly completing web on $\mathcal{M}$.

Theorem 3.11. *Let $(\mathcal{P}, \mathfrak{U})$ be a separated symmetric complete and symmetric quasi-full uc-cone and $(\mathcal{Q}, \mathcal{Y})$ be a separated locally convex cone with a uniformly completing web $\mathcal{W}$. Then every linear operator $T: \mathcal{P} \to \mathcal{Q}$ with symmetric closed graph is symmetric continuous.*

Proof. The inverse image of the sets of $\mathcal{W}$ by $T \times T$, form a web in $\mathcal{P}$ and therefore by Lemma 3.2 there is a strand (W_n) of $\mathcal{W}$ such that $\overline{T^{-1}(W_n(0)W_n)}^{s}$ is a symmetric neighborhood for each $n \in \mathbb{N}$. Then for every $n \in \mathbb{N}$ there is $U_n \in \mathfrak{U}$ and $a_n \in \mathcal{P}$ such that $U_n(a_n)U_n \subseteq \overline{T^{-1}(W_n(0)W_n)}^{s}$. Without loss of generality we suppose $a_n \in T^{-1}(W_n(0)W_n)$ and $\mathcal{B} = \{U_n : n \in \mathbb{N}\}$ is decreasing base for $\mathfrak{U}$ (since $(\mathcal{P}, \mathfrak{U})$ is a uc-cone, $\mathfrak{U}$ has a such base). Since $(\mathcal{P}, \mathfrak{U})$ is quasi-full, we have

$$\overline{T^{-1}(W_n(0)W_n)}^{s} \subseteq T^{-1}(W_n(0)W_n) + U(0)U,$$

for each $U \in \mathfrak{U}$. This implies

$$U_n(a_n)U_n \subseteq \overline{T^{-1}(W_n(0)W_n)}^{s} \subseteq T^{-1}(W_n(0)W_n) + U_{n+1}(0)U_{n+1}$$

Therefore $T(U_n(a_n)U_n) \subseteq W_n(0)W_n + T(U_{n+1}(0)U_{n+1})$ for each $n \in \mathbb{N}$. Since $\mathcal{P}$ is symmetric quasi-full, we have $U_n(a_n)U_n = a_n + U_n(0)U_n$. Then

$$T(a_n + U_n(0)U_n) \subseteq W_n(0)W_n + T(U_{n+1}(0)U_{n+1}),$$

for each $n \in \mathbb{N}$. Now let V be a symmetric closed neighborhood of 0 in $(\mathcal{Q}, \mathcal{Y})$. Then there is $m \in \mathbb{N}$ such that $2(W_{m-1}(0)W_{m-1}) \subseteq V$. We shall prove that $T(U_m(0)U_m) \subseteq V$. Let $x_0 \in U_m(0)U_m$. Then there is $x_1 \in U_{m+1}(0)U_{m+1}$ and $c_0 \in W_m(0)W_m$ such that $T(a_m) + T(x_0) = c_0 + T(x_1)$. Also, there is $x_2 \in U_{m+2}(0)U_{m+2}$ and $c_1 \in W_{m+1}(0)W_{m+1}$ such that $T(a_{m+1}) + T(x_1) = c_1 + T(x_2)$. Then

$$T(a_m) + T(a_{m+1}) + T(x_0) = c_0 + c_1 + T(x_2).$$

By continuing this way we obtain $x_{t+1} \in U_{m+t+1}(0)U_{m+t+1}$ and $c_t \in W_{m+t}(0)W_{m+t}$ such that

$$\sum_{i=0}^{t} T(a_{m+i}) + T(x_0) = \sum_{i=0}^{t} c_i + T(x_{t+1}).$$

Also, we have

$$\begin{aligned}
\textstyle\sum_{i=0}^{t} T(a_{m+i}) + T(x_0) &= \textstyle\sum_{i=0}^{t} c_i + T(x_{t+1}) \\
&\in W_m(0)W_m + W_{m+1}(0)W_{m+1} + \cdots + W_{m+t}(0)W_{m+t} + T(x_{t+1}) \\
&\subseteq W_m(0)W_m + \tfrac{1}{2}(W_m(0)W_m) + \cdots + \tfrac{1}{2^t}(W_m(0)W_m) + T(x_{t+1}) \\
&\subseteq 2(1 - \tfrac{1}{2^{t+1}})(W_m(0)W_m) \\
&\subseteq 2(W_m(0)W_m) \\
&\subseteq W_{m-1}(0)W_{m-1}. \qquad (3.1)
\end{aligned}$$

Since $\mathcal{W}$ is a completing web and $T(a_{m+i}), c_i \in W_{m+i}(0)W_{m+i}$, the series $\sum_{i=0}^{\infty} c_i$ and $\sum_{i=0}^{\infty} T(a_{m+i})$ are symmetric convergent. Let their limit be a and c respectively. Now if $t \to \infty$, by (3.1) we have

$$a + T(x_0) \in \overline{W_{m-1}(0)W_{m-1}}^{s}.$$

Also, since

$$\sum_{i=0}^{t} T(a_{m+i}) \in W_m(0)W_m + W_{m+1}(0)W_{m+1} + \cdots + W_{m+t}(0)W_{m+t} \subseteq W_{m-1}(0)W_{m-1},$$

we have $a \in \overline{W_{m-1}(0)W_{m-1}}^{s}$. Therefore $T(x_0) \in 2\overline{W_{m-1}(0)W_{m-1}}^{s} \subseteq \overline{V}^{s} = V$, since $\mathcal{W}$ is uniformly completing. This shows that T is symmetric continuous at 0. Now since $(\mathcal{P}, \mathfrak{U})$ is symmetric quasi-full, Lemma 3.8 shows that T is symmetric continuous at each $a \in \mathcal{P}$.

Theorem 3.12 (Closed Graph Theorem). *Let $(\mathcal{P}, \mathfrak{U})$ be a separated bornological locally convex cone which is symmetric complete and symmetric quasi-full and $(\mathcal{Q}, \mathcal{Y})$ be a separated locally convex cone with a uniformly completing web $\mathcal{W}$. Then every linear operator $T: \mathcal{P} \to \mathcal{Q}$ with symmetric closed graph is symmetric continuous.*

Proof. By Proposition 2.12 from [1], $(\mathcal{P}, \mathfrak{U})$ is the inductive limit of complete separated uc-subcones $(\mathcal{P}_B, \mathfrak{U}_B)_{B\in\mathcal{B}}$, where $\mathcal{B}$ is the collection of all uniformly convex u-bounded subsets of $\mathcal{P}^2$, under the inclusion mappings $I_B: \mathcal{P}_B \to \mathcal{P}$, $B \in \mathcal{B}$. Since $\mathfrak{U}$ induces $\mathfrak{U}_B$ on $\mathcal{P}_B$ for each $B \in \mathcal{B}$, the uc-cones $(\mathcal{P}_B, \mathfrak{U}_B)_{B\in\mathcal{B}}$ are symmetric quasi-full. Also, for each $B \in \mathcal{B}$, I_B is symmetric continuous, then it has closed graph. Then for each $B \in \mathcal{B}$, $T \circ I_B$ has closed graph. Now, by Theorem 3.11, each $T \circ I_B$ is symmetric continues. This shows that T is symmetric continuous.

Example 3.13. We consider the locally convex cones $(\mathcal{F}_b(X, \overline{\mathbb{R}}), \tilde{\mathcal{V}})$ from Example 3.7 and $(\mathcal{M}, \mathfrak{U})$ from Example 3.10. The first one i.e. $(\mathcal{F}_b(X, \overline{\mathbb{R}}), \tilde{\mathcal{V}})$ is separated, bornological (a uc-cone), symmetric complete and symmetric quasi-full. Also, $(\mathcal{M}, \mathfrak{U})$ is separated and has a uniformly completing web (see Example 3.10). Now, every linear operator from $(\mathcal{F}_{\mathrm{b}}(\mathrm{X}, \overline{\mathbb{R}}), \tilde{\mathcal{V}})$ into $(\mathcal{M}, \mathfrak{U})$ with closed graph is symmetric continuous by Theorem 3.12. If we consider the subcone $(\mathcal{F}_b(X, \mathbb{R}), \tilde{\mathcal{V}})$ of $(\mathcal{F}_b(X, \overline{\mathbb{R}}), \tilde{\mathcal{V}})$, then for every $\mathrm{x} \in \mathrm{X}$, the linear operator $T_x: (\mathcal{F}_b(X, \mathbb{R}), \tilde{\mathcal{V}}) \to (\mathcal{M}, \mathfrak{U})$ defined by $T_x(f) = [\min f, f(x)]$, has closed graph with respect to symmetric topology. Then it is symmetric continuous by Theorem 3.12.

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