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SADDLEPOINT APPROXIMATION BASED SYSTEM RELIABILITY ANALYSIS METHOD WITH LOAD DEPENDENCY

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A new efficient and accurate saddlepoint approximation (SA) based system reliability method that aims at accurately describing the failure correlation and reducing the computational cost is proposed. Firstly, the SA based conditional reliability probability equation of part (CRPEP) with its modeling procedures are introduced. Then SA based conditional reliability probability equation of system (CRPES) is presented. Finally, numerical integration methods are used to calculate system reliability through integrating CRPES in the load distribution range. Feasibility of the proposed method are illustrated.

Keywords: Mechanical system reliability; Failure correlation; Load dependency; Saddlepoint approximation.

1. Introduction

In some mechanical systems, the power source is a single input defined as shared load (such as, torque) distributed on different parts through the assembly relationship. As a result, failures of parts are linked. Additionally, the performance of the system cannot be considered as constant, but possesses the randomness and obeys certain distribution[1-3]. Failure correlations caused by those factors affect the evaluation accuracy of the reliability[4]. Therefore, the correlation caused by load dependency should be considered.

The system failed with load dependency can be seen as one kind of common cause failure (CCF). So far, many investigations have been involved in CCF of redundant systems. Levitin[5] incorporates CCF into multistate series-parallel power system and develops a system failure model by adapting the universal generation function method. Based on the load-strength interference theory, Xie et al[6] presents a system reliability model which is mainly suitable for analyzing the k-out-of-n redundant system. The CCFs mechanism of redundant systems with independent components strength variables and its probability prediction model are illustrated by Zhou et al[7]. The discretely modeling method for system reliability prediction considering CCF is discussed by Xie et al[8]. Considering the load dependency, Zhou et al[9] proposed a system

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reliability model, moreover, the stress and strength are regarded as single random variables without considering the affecting parameters. Bai et al[10] proposed a reliability model of the gear system considering statistically dependent failure based on the theory of the order statistics. System reliability with load dependency can be analyzed by utilizing Nataf transformation as discussed by Zhou et al[11], but the linear correlation coefficient is used and the random parameters affecting the stress and strength are neglected. However, using the linear relation index to describe the failure correlation, is not suitable for some situation especially when the limit state function (LSF) is high nonlinear equation with large dimensions.

In this paper, the concepts of conditional reliability corresponding to part and system are presented. Considering the correlation ship caused by load dependency, random variables determining the stress, and random variables affecting strength distribution, the idea aims to find a simple system reliability analysis method.

2. SA for reliability analysis

Daniels[12] introduced SA technique into the field of approximation distribution of statistics. Then lots of research works have been done by Huang and Du et al[13-15] to expand this method. Huang et al[14] proposed an efficient and accurate mean-value first order saddlepoint approximation (MVFOSA) method to calculate part reliability, and it is proved that a highly accurate reliability value of complex LSF can be obtained only with a small number of equation evaluations. In this paper, MVFOSA is used and its theory is as follows:

Let X be a random variable distributed according to the density function $f_X(x)$. The moment generating function (MGF) of X is defined as

$$M_X(t)=\int_{-\infty}^{+\infty}e^{tx}f_X(x)dx\ , \tag{1}$$

and the cumulant generating function (CGF) of X is defined as

$$K_X(t)=\ln\left[M_X(t)\right]. \tag{2}$$

The two useful properties of CGF which make the reliability analysis realized are as follows:

(1) if $X_1, X_2, \cdots, X_n$ are independent random variables and their CGFs are $K_{X_i}(t)(i=1,2,\cdots,n)$, then the CGF of $Z=\sum_{i=1}^{n}X_i$ is

$$K_Z(t)=\sum_{i=1}^{n}K_{X_i}(t). \tag{3}$$

(2) if X is a random variable with a CGF $K_X(t)$, then the CGF of $Z=aX+b$ is given by

$$K_Z(t)=K_X(at)+bt. \tag{4}$$

Where both a and b are constants. The CGFs of commonly used distributions can be referred to Ref[16].

Let $Z = G(\mathbf{X})$ is the performance function of a mechanical system or a part. $\mathbf{X} = \left[X_1, X_2, \cdots, X_n \right]^{\mathrm{T}}$ is the distribution vector of the random inputs with its mean value vector $\mathbf{u}_{\mathbf{x}} = \left[u_{X_1}, u_{X_2}, \cdots, u_{X_n} \right]^{\mathrm{T}}$. The first order Taylor expansion of $Z = G(\mathbf{X})$ at the mean values of the random variables is given by

$$Z \cong G(\boldsymbol{\mu}_{\mathbf{X}}) + \sum_{i=1}^{n} \left.\frac{\partial G}{\partial X_i}\right|_{\mu_{X_i}} \left(X_i - \mu_{Xi} \right). \tag{5}$$

Based on the two properties of CGF, the CGF of Z is then evaluated by

$$K_Z(t) = \left\{ \mathrm{G}(\boldsymbol{\mu}_{\mathbf{X}}) - \sum_{i=1}^{n} \left.\frac{\partial G}{\partial X_i}\right|_{\boldsymbol{\mu}_{X_i}} \mu_{Xi} \right\} t + \sum_{i=1}^{n} K_{X_i} \left(\left.\frac{\partial G}{\partial X_i}\right|_{\boldsymbol{\mu}_{X_i}} t \right). \tag{6}$$

Once the CGF of Z is obtained as shown in Eq. (6), it is straight forward to apply the SA to probability density function (PDF) estimations. The simple formula for computing the PDF of Z is expressed as[12]

$$f_Z(z) = \left\{ \frac{1}{2\pi K_Z''(t_z)} \right\}^{\frac{1}{2}} \exp\left[K_Z(t_z) - t_z z \right], \tag{7}$$

where K_Z' is the second order derivative of the CGF of Z ; z is the value used to obtain the probability of $Z \le z$; t_z is the saddlepoint, which is the solution of the equation,

$$K_Z'(t_z) = z, \tag{8}$$

where K_Z' is the first order derivative of the CGF Z.

Lugannani and Rice[16] gave a concise formula for calculating the cumulative distribution function (CDF) of Z,

$$F_Z(z) = P\{Z \le z\} = \Phi(w) + \varphi(w)\left(\frac{1}{w} - \frac{1}{v} \right), \tag{9}$$

$$w = \mathrm{sgn}(t_z)\left\{ 2\left[t_z z - K_Z(t_z) \right] \right\}^{\frac{1}{2}}, \tag{10}$$

$$v = t_z \left[K_Z''(t_z) \right]^{\frac{1}{2}}, \tag{11}$$

where $\Phi(\cdot)$ and $\varphi(\cdot)$ are the CDF and PDF of the standard normal distribution, respectively; $\mathrm{sgn}(t_z)$ =1,0 or -1, depending on whether the saddlepoint t_z is positive, negative or zero.

3. SA based system reliability model with load dependency

3.1. SA based reliability model for series system

3.1.1 Conditional reliability based system reliability theory

Case 1: Without considering factors affecting stress and strength

The reliability of the i_{th} part can be calculated by

$$R_i = \int_{-\infty}^{+\infty}\int_{s_i(L)}^{+\infty} f_i(r_i)\cdot f_L(L)\,dr_i dL, i=1,2,\cdots,k \ , \tag{12}$$

where, L represents the shared random load with its PDF as $f_L(L)$; r_i represents random strength of the i_{th} part with its PDF as $f_i(r_i)$; s_i represents the stress of the i_{th} part affected only by L; k is the number of parts.

From Eq.(12), L is the only random variable shared in all the parts's functions. Setting c as an arbitrary value ($c \in (\mu_L - 6\sigma_L, \mu_L + 6\sigma_L)$, μ_L and σ_L are the mean and standard deviation of L), then reliabilities of the parts corresponding to c are independent, where the part's reliability function with c will be named as CRPEP. Then the system reliability equation corresponding to c can be obtained by multiplying all parts's CRPEPs, named as the CRPES, and shown as Eq. (13). The value of CRPES is named as conditional reliability probability of system (CRPS).

$$R_{CRPES}(c) = \prod_{i=1}^{k}\int_{s_{i(c)}}^{+\infty} f_i(r_i)\,dr_i \ . \tag{13}$$

Based on the thoughts of discretely modeling, the reliability shown by CRPES can be seen as the one component of the system reliability corresponding to only one value of the load distribution zone. Then the system reliability can be obtained by integrating CRPES through the load distribution range, expressed by Eq.(14).

$$R=\int_{-\infty}^{+\infty} f_L(L)\{R_{CRPES}(c)\}\Big|_{c\to L} dL=\int_{-\infty}^{+\infty} f_L(L)\prod_{i=1}^{k}\int_{s_{i(L)}}^{+\infty} f_i(r_i)\,dr_i dL \ . \tag{14}$$

Case 2: Considering factors affecting stress and strength

Considering all the random variables affecting the distribution of the performance function, the stress s and the strength r for the i_{th} part are given by

$$s_i = s_i(L,\ \mathbf{X}_i), i=1,2,\cdots k \ , \tag{15}$$

$$r_i = r_i(\mathbf{Y}_i), i=1,2,\cdots k \ , \tag{16}$$

where, $\mathbf{X}_i=[X_{i1},\ X_{i2},\cdots,X_{im_i}]^T$ and $\mathbf{Y}_i=[Y_{i1},\ Y_{i2},\cdots,Y_{in_i}]^T$ are the random variables affecting the stress and strength distribution of the i_{th} part,respectively. Then LSF and reliability of the part can be expressed by Eq.(17) and Eq.(18), respectively.

$$G_i = s_i - r_i, i = 1, 2, \cdots, k \ , \tag{17}$$

$$R_i = P\{G_i < 0\} = \int \cdots \int_{G_i<0} f_i\left(L; X_{i1}, \ X_{i2}, \cdots, X_{im_i}; Y_{i1}, \ Y_{i2}, \cdots, Y_{in_i}\right) dX_{i1} \cdots dX_{im_i} dY_{i1} \cdots dY_{in_i} \cdot dL \tag{18}$$

Where, $f_i(\cdots)$ is the joint PDF of the parameters.

Then the system relibility model with load dependency considering all random variables can be deduced as follows:

(1) Supposing c as an arbitrary value from the distribution interval of L, CRPEP whose solving value is named as conditional reliability probability of part (CRPP).) reads from Eq.(19).

$$R_{ic}(c) = \int \cdots \int_{G_i<0} f_i\left(c; X_{i1}, \ X_{i2}, \cdots, X_{im_i}; Y_{i1}, \ Y_{i2}, \cdots, Y_{in_i}\right) dX_{i1} \cdots dX_{im_i} dY_{i1} \cdots dY_{in_i} \tag{19}$$

(2) Then the conditional joint reliability probability equation of k parts (also known as CRPES) can be defined as

$$R_c(c) = R_{1c}(c) R_{2c}(c) \cdots R_{kc}(c) , \tag{20}$$

substituting $\mathbf{X}_i, i = 1, 2, \cdots, k$ and $\mathbf{Y}_i, i = 1, 2, \cdots, k$ into Eq. (20), we can get

$$R_c(c) = \prod_{i=1}^{k} \int \cdots \int_{G_i<0} f\left(c, \mathbf{X}_i, \mathbf{Y}_i\right) d\mathbf{X}_i d\mathbf{Y}_i \ . \tag{21}$$

(3) Finally, by integrating the CRPS in the whole distribution interval of L, the system reliability with load dependency and all affecting factors can be obtained, and reads as

$$\begin{aligned} R &= \int_{-\infty}^{+\infty} f_L(L) \left\{R_c(c)\right\}\Big|_{c \to L} d\mathbf{X}_i d\mathbf{Y}_i dL \\ &= \int_{-\infty}^{+\infty} f_L(L) \prod_{i=1}^{k} \int \cdots \int_{G_i<0} f\left(L; X_{i1}, \ X_{i2}, \cdots, X_{im_i}; Y_{i1}, \ Y_{i2}, \cdots, Y_{in_i}\right) \\ & \quad dX_{i1} \cdots dX_{im_i} dY_{i1} \cdots dY_{in_i} dL \end{aligned} \ . \tag{22}$$

Based on the presented conditional reliability theory discussed in **Section 3.1.1**, a new SA based system reliability method with consideration of load dependency is developed and will be discussed in the rest contents of this section.

3.1.2. SA based CRPP model corresponding to shared load

The LSF of part contains a large number of random variables resulting in difficulty to direct integrate it, as shown in Eq. (17). $\mathbf{u}_{\mathbf{x}_i} = \left[u_{X_{i1}}, u_{X_{i2}}, \cdots, u_{X_{im_i}}\right]^{\mathrm{T}}$ is the mean value vector of $\mathbf{X}_i = \left[X_{i1}, \ X_{i2}, \cdots, X_{im_i}\right]^{\mathrm{T}}$ and $\mathbf{u}_{\mathbf{Y}_i} = \left[u_{Y_{i1}}, u_{Y_{i2}}, \cdots, u_{Y_{in_i}}\right]^{\mathrm{T}}$ is the mean value of $\mathbf{Y}_i = \left[Y_{i1}, \ Y_{i2}, \cdots, Y_{in_i}\right]^{\mathrm{T}}$.

Supposing c as an arbitrary value from the distribution interval of L,the LSF of the i_{th} part is linearized at mean value vector $\mathbf{u}_{\mathbf{x}_i}, \mathbf{u}_{\mathbf{Y}_i}$ as

$$G_i(c) = G_i\left(\mathrm{c}, \mathbf{u}_{\mathbf{x}_i}, \mathbf{u}_{\mathbf{Y}_i}\right) + \sum_{j=1}^{m_i} \left.\frac{\partial G_i}{\partial X_{ij}}\right|_{c;u_{X_{ij}}} \left(X_{ij} - u_{X_{ij}}\right) + \sum_{h=1}^{n_i} \left.\frac{\partial G_i}{\partial Y_{ih}}\right|_{c;u_{Y_{ih}}} \left(Y_{ih} - u_{Y_{ih}}\right). \tag{23}$$

The CGF of Eq. (23) can be evaluated by

$$K_{G_i}(c,t_i) = \left\{G_i\left(\mathrm{c}, \mathbf{u}_{\mathbf{x}_i}, \mathbf{u}_{\mathbf{Y}_i}\right) - \sum_{j=1}^{m_i} \left.\frac{\partial G_i}{\partial X_{ij}}\right|_{c;u_{X_{ij}}} u_{X_{ij}} - \sum_{h=1}^{n_i} \left.\frac{\partial G_i}{\partial Y_{ih}}\right|_{c;u_{Y_{ih}}} u_{Y_{ih}}\right\} t_i + \left\{\sum_{j=1}^{m_i} K_{X_{ij}} \left(\left.\frac{\partial G_i}{\partial X_{ij}}\right|_{c;u_{X_{ij}}} t_i\right) + \sum_{h=1}^{n_i} K_{X_{ih}} \left(\left.\frac{\partial G_i}{\partial Y_{ih}}\right|_{c;u_{Y_{ih}}} t_i\right)\right\}, \tag{24}$$

Eq. (7-11) then can be transformed into Eq. (25-29), respectively. Then the PDF and CDF of CRPP can be calculated by Eq. (25) and Eq. (27), respectively.

$$f_{G_i}(g_i) = \left\{\frac{1}{2\pi K''_{G_i}(t_i)}\right\}^{\frac{1}{2}} \exp\left[K_{G_i}(t_i) - t_i g_i\right], \tag{25}$$

$$K'_{G_i}(c,t_i) = g_i, \tag{26}$$

$$F_{G_i}(c,g_i) = P\{G_i \le g_i\} = \Phi\left(w_i(c,g_i)\right) + \varphi\left(w_i(c,g_i)\right)\left(\frac{1}{w_i(c,g_i)} - \frac{1}{v_i(c,g_i)}\right), \tag{27}$$

$$w_i(c,g_i) = \operatorname{sgn}(t_i)\left\{2\left[t_i g_i - K_{G_i}(t_i)\right]\right\}^{\frac{1}{2}}, \tag{28}$$

$$v_i(c,g_i) = t_i\left[K''_{G_i}(t_i)\right]^{\frac{1}{2}}, \tag{29}$$

where g_i is the value used to obtain the probability of $G_i \le g_i$ and a threshold value; K'_{G_i} is the first derivative of the CGF of G_i; K''_{G_i} is the second derivative of the CGF of G_i; t_i is the saddlepoint obtained by solving Eq. (26), which is a function of c; $\Phi(\cdot)$ and $\varphi(\cdot)$ are the CDF and PDF of the standard normal distribution, respectively; $\operatorname{sgn}(t_i)$ =1,0 or -1, depending on whether the saddlepoint t_i is positive, negative or zero.

3.1.3. SA based CRPS model corresponding to shared load

When equation $G_i \le g_i, i = 1, 2, \cdots, k$ for the LSFs is fulfilled. Then the CRPEP can be expressed by

$$R_{ic}(c) = F_{G_i}(g_i) = P\{G_i \le g_i\} = \Phi\left(w_i(c,g_i)\right) + \varphi\left(w_i(c,g_i)\right)\left(\frac{1}{w_i(c,g_i)} - \frac{1}{v_i(c,g_i)}\right), \tag{30}$$

and CRPS of the series mechanical system is evaluated by multiplying all of the CRPEPs shown as

$$R_c(c) = R_{1c}(c) R_{2c}(c) \cdots R_{kc}(c) = \prod_{i=1}^{k}\left\{\Phi\left(w_i(c,g_i)\right) + \varphi\left(w_i(c,g_i)\right)\left(\frac{1}{w_i(c,g_i)} - \frac{1}{v_i(c,g_i)}\right)\right\}. \tag{31}$$

3.1.4. SA based Reliability model with load dependency

Transforming c into L, then L is the only random variable included in w_i

and v_i of Eq. (31). Integrating the CRPES in the load range, the system reliability can be obtained by

$$R=\int_{-\infty}^{+\infty} R_c\left(c\right)\Big|_{c\to L} f_L\left(L\right)dL=\int_{-\infty}^{+\infty}\prod_{i=1}^{k}\left\{\Phi\left(w_i\left(L,g_i\right)\right)+\varphi\left(w_i\left(L,g_i\right)\right)\left(\frac{1}{w_i\left(L,g_i\right)}-\frac{1}{v_i\left(L,g_i\right)}\right)\right\}f_L\left(L\right)dL \quad , (32)$$

where, when g_i is set, L is the only random variable in w_i and v_i.

Then the reliability probability R of system is simplified as the definite integration with one dimensional.

3.1.5. Calculation of system reliability by Gauss-Hermite Integration

Eq.(32) only involves one dimensional integrals. Then the numerical methods, Gauss-Hermite Integration method, is used to solve the complex one-dimensional integral equation. First, Eq.(32) should be transformed as:

$$R=\int_{-\infty}^{+\infty}\left(\prod_{i=1}^{k}\left\{\Phi\left(w_i\left(L\right)\right)+\varphi\left(w_i\left(L\right)\right)\left(\frac{1}{w_i\left(L\right)}-\frac{1}{v_i\left(L\right)}\right)\right\}f_L\left(L\right)\exp^{L^2}\right)\exp^{-L^2}dL=\int_{-\infty}^{+\infty}G_{sys}\left(L\right)\exp^{-L^2}dL \quad , \quad (33)$$

GHI approximates the one dimensional integral by summing up several terms of weighted integrand. The weighted integrand is evaluated at, so-called, Gauss points (abscissas)[17]. Then reliability of the system can be calculated by

$$R=\sum_{I=1}^{r} w_I G_{sys}\left(L_I\right), \quad (34)$$

Where r is the quadrature order (the number of abscissas); L_I and w_I are abscissas (Gauss points) and weights (Gauss weights), respectively. For weights and abscissas information of quadrature orders, the readers can refer to[18].

3.2. SA based reliability model for parallel system

Reliability of the parallel system can be deduced as follows:

(1) Eq. (17) is transformed into Eq. (35), named as failure limit state function (FLSF) of the part.

$$G_i^f=r_i\text{-}s_i, i=1,2,\cdots,k \;, \quad (35)$$

Then failure probability of the part is shown as

$$P_i^f=P\left\{G_i^f\le 0\right\}, i=1,2,\cdots,k \;. \quad (36)$$

(2) Then the failure probability of the parallel mechanical system can be evaluated by

$$P^f=\int_{-\infty}^{+\infty}\left\{P_{1c}^f\left(c\right)P_{2c}^f\left(c\right)\cdots P_{kc}^f\left(c\right)\right\}\Big|_{c\to L} f_L\left(L\right)dL=\int_{-\infty}^{+\infty}\prod_{i=1}^{k}\left\{\Phi\left(w_i\left(L\right)\right)+\varphi\left(w_i\left(L\right)\right)\left(\frac{1}{w_i\left(L\right)}-\frac{1}{v_i\left(L\right)}\right)\right\}f_L\left(L\right)dL \;. \quad (37)$$

(3) Then reliability of parallel mechanical system equaling to the difference between 1 and the failure probability reads

$$R=1-P^f. \quad (38)$$

3.3. Procedures of the proposed method to calculate system reliability

Procedures of proposed method consisting five stages list as below:

(1) Considering the assembly relationship, Load applied on the part is acquired by using theoretical mechanics theory. Then correlation (linear or nonlinear) between the part load and the shared load will be found.

(2) Setting c as an arbitrary value from the distribution interval of random load L, the reliability (or failure) saddlepoint approximation equation of the part is expressed by using MVFOSA, which is named as conditional reliability (or failure) probability equation of part (CRPEP or CFPEP).

(3) Multiplying the CRPEPs (or CFPEPs) give the conditional reliability probability equation of series system (or conditional failure probability of parallel system) (short for CRPESS and CFPEPS).

(4) Integrating the product of CRPESS (or CFPEPS) and PDF of the shared random load gives reliability of series mechanical system (or failure probability of parallel mechanical system).

(5) Reliability probability of the parallel system is calculated by Eq. (38).

4. Example and discussions

4.1. A mathematical problem

The first example consists of a series system with three branches with five random distribution variables. $\mathbf{X}=\left[X_1, X_2, X_3, X_4, X_5\right]^{\mathrm{T}}$ is the random variable vector and their distribution parameters are shown in Table 1. The performance functions read in Eq.(39-41), and then the LSF of the system is defined as Eq.(42)

$$g_1(\mathbf{X}) = -0.05(X_1+2)^2(X_2-1)+\sin(2.5X_3)+2\,, \tag{39}$$

$$g_2(\mathbf{X}) = -(X_1+1)^3 + X_4 + 15\,, \tag{40}$$

$$g_3(\mathbf{X}) = -(X_1+1)^4 + 6X_5 + 16\,, \tag{41}$$

$$R_{series} = P\left[g_1(\mathbf{X})<0 \cap g_2(\mathbf{X})<0 \cap g_3(\mathbf{X})<0\right]. \tag{42}$$

Table 1

Distribution information of random variables

Variable	X_1	X_2	X_3	X_4	X_5
Mean	3	5	3	9	7
St.dev	0.5	0.4	0.24	0.72	0.56
Distribution	Gumbel	Gamma	Normal	Normal	Normal

Table 2

Reliability values with different input standard deviations of the shared load

St.dev	0.5	0.7	0.9	1.1	1.3	1.5
MCS	0.99999	0.99739	0.97499	0.93126	0.88239	0.83560
Proposed	0.99552	0.96901	0.91010	0.84537	0.78891	0.74245

Reliability results of the system, computed by the proposed method and

MCS (with 106 simulations), are 0.99552 and 0.99999, respectively. Compared with MCS, error percentage of the proposed method is 0.45%. It is demonstrated that the proposed method is economical and accurate. Then the system reliabilities with a range of standard deviation of the shared random variable from 0.5 to 1.5, and the results are given in Table 2.

Difference between the results of the proposed method and MCS method is the analysis target. From Figure 1, it is noted that the difference increases with the increase of the standard deviation. The difference value increases from 0.00447 at standard deviation 0.5 to 0.09315 at standard deviation 1.5, and the percentage errors are 0.045%, 11.1%, respectively. Just as Huang[14] indicated that, if uncertainty of the random variables is large, linearization at mean values cannot approximate the performance function well, and MVFOSA may consequently results in a large error.

Uncertainty of the random variable can be described by variation coefficient $C = \sigma / \mu$. It is found from Table 3 that the error percentage is 0.045% with C=0.17 (standard deviation as 0.5) and the error percentage is 1.3% with C=0.2 (standard deviation as 0.6). As to the mechanical system, variation coefficient of the random variable is usually less than 0.1, such as, size factor, dimensions, and so on. Therefore, it is believed that the proposed method is accurate and efficient for assessing the reliability of mechanical system.

4.2. A shaft assembly of a wind turbine

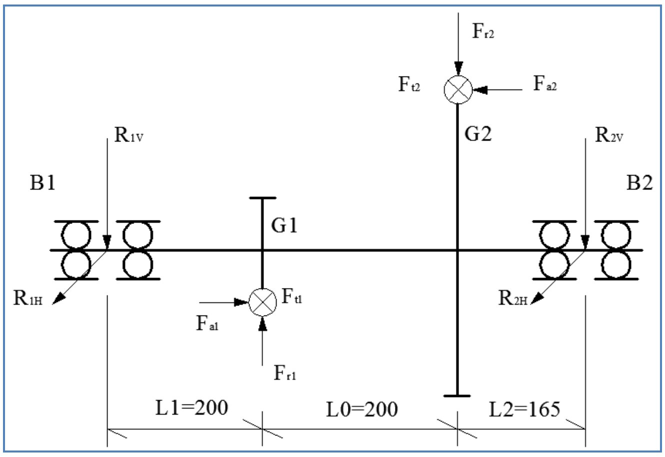

Fig 1. Brief sketch and load analysis of the medium-speed shaft

Considering a shaft assembly of a 1.5 MW wind turbine gear box, which consists of one planetary gear stage and two spur wheel stages. The subsystem consists of a shaft, two spur gear (G1 and G2) and two roller bearings (B1 and B2). The brief sketch and load analysis sketch are shown in Figure 1. The design parameters are: design life of the gear box is 15 years to 20 years; the accuracy grade is 6 class; the material of the shaft is 45steel and the heat treatment is quenching and tempering; the material of other parts is 17CrNiMo6 and the heat treatment is carburizing and quenching. The torque produced by the wind speed is the shared load, with calculation method[19]. Torque $T_L = \left(3.1484\times10^4, 6.2968\times10^3\right) N \bullet m$ applied on the shaft is derived from load analysis of the gear transmission system. T_L is the normal random variable with standard deviation as 6.2968×10^3 and mean value as 3.1484×10^4. Only one single failure mode of each part is considered.

4.2.1. LSF of parts, parameters and load analysis.

Through analysising load distribution, load of each part as a function of torque T_L are: $R_{1H} = 6.5927\cdot T_L$, $R_{1V} = 1.4634\cdot T_L$, $R_{2H} = 5.0220\cdot T_L$, $R_{2V} = 0.9963\cdot T_L$, $F_{t1} = 2000\cdot T_L/d_1$, $F_{r1} = 3.6024\cdot T_L$, $F_{a1} = 1.5854\cdot T_L$, $F_{t2} = 2000\cdot T_L/d_2$, $F_{r2} = 1.1427\cdot T_L$, $F_{a2} = 0.6795\cdot T_L$. Numerous numbers of random variables are included in the LSF of the part, such as, size, size mending factor, surface machining factor, and so on. To obtain the distribution information of these variables, the empirical methods proposed by Zhang[20] are used. Definition of the parameters are shown in the following.

(1) Reliability model and parameter evaluation for the gears

The performance function of contact failure of the gear is calculated by

$$G_{gear} = s1 - r1 = \sigma_H - \sigma'_{Hlim} = Z_H Z_E Z_\varepsilon Z_\beta \sqrt{\frac{K_A K_V K_{H\beta} K_{H\alpha} F_t}{db}\cdot\frac{\mu\pm1}{\mu}} - \sigma_{Hlim} Z_N Z_L Z_V Z_R Z_W Z_X , \tag{43}$$

i=1 and i=2 represent G1 and G2, respectively. Where, Z_H , Z_E , Z_ε , Z_β , K_A , K_V , $K_{H\beta}$, $K_{H\alpha}$, F_t , d , b , μ , σ_{Hlim} , Z_N , Z_L , Z_V , Z_R , Z_W , Z_X are zone factor, elastic coefficient, contact ratio coefficient, helix angle factor, application factor, dynamic factor, load distribution factor, transverse load distribution factor, shear stress, reference diameter, face width, gear ratio, contact strength, life factor, lubricant coefficient, speed factor, finish coefficient, work hardening factor, size factor, respectively. There are eighteen normal random variables, given in Table 3. and denoted as $\mathbf{X}_i = [Z_H, Z_E, Z_\varepsilon, Z_\beta, K_A, K_V, K_{H\beta}, K_{H\alpha}, F_t, d_{gear}, b, \sigma_{Hlim}, Z_N, Z_L, Z_V, Z_R, Z_W, Z_X]^{\mathrm{T}} = \left[X_{i1}, X_{i2}, \cdots, X_{i18}\right]^{\mathrm{T}}, i=1,2$.

Table 3

Parameters of random variables of the gears (G1 and G2)

	Gear (G1)		Gear (G2)			Gear (G1)		Gear (G2)	
Variable	Mean	St.dev	Mean	St.dev	Variable	Mean	St.dev	Mean	St.dev
Z_H (-)	2.31	0.0115	2.36	0.0118	d (mm)	233.92	1.17	652.5	4
Z_E (-)	189.8	5.694	189.8	5.6940	b (mm)	300	1.5	170	1
Z_ε (-)	0.82	0.0041	1.43	0.0072	σ_{Hlim} (MPa)	1235	98.8	1235	98.8
Z_β (-)	0.99	0.0049	0.988	0.0050	Z_N (-)	1.23	0.0492	1.1	0.044
K_A (-)	1.18	0.1298	1.102	0.121	Z_L (-)	0.975	0.0322	0.975	0.0322

K_V (-)	1.15	0.0380	1.04	0.0344	Z_V (-)	0.955	0.0287	1.009	0.0303
$K_{H\beta}$ (-)	1.74	0.0575	1.44	0.0475	Z_R (-)	1.018	0.0305	1.024	0.0307
$K_{H\alpha}$ (-)	1.1	0.0363	1.1	0.0363	Z_W (-)	1.13	0.0339	1	0.03
F_t (N)	$F_{t1}=2000\cdot T_L/d_1$		$F_{t2}=2000\cdot T_L/d_2$		Z_X (-)	1	0.03	1	0.03

(2) Reliability model and parameters for the bearings

The performance function of bearing is transformed from the one expressed by lifetime to the one expressed by basic rated load,

$$G_{bearing}=P\left(\frac{n}{16670}L_h'\right)^{1/\varepsilon}-C_r=f_P P_{bearing}\left(\frac{n}{16670}L_h'\right)^{1/\varepsilon}-C_r, \tag{44}$$

i=3 and i=4 represent bearing B1 and bearing B2, respectively. Where, f_P, $P_{bearing}$, ε, n, L_h', C_r, are dynamic factor, equivalent dynamic load, life factor (ε=10/3 for roller bearing), rotational speed, expected service life, basic dynamic load, respectively. Four random variables which decide the distribution of performance function are $\mathbf{X}_i=\left[f_P,P_{bearing},L_h',C_r\right]^{\mathrm{T}}=\left[X_{i1},\ X_{i2},X_{i3},X_{i4}\right]^{\mathrm{T}}, i=3,4$.

Assuming the working time is 3000h every year combined with the design lifetime, the expected lifetime of bearing is 6000h. The type of roller bearing is 22336. The equivalent load $P_{bearing}$ is computed by considering the axial load and the radial load. Distribution details of the random variables are given in Table 4.

Table 4

Distribution details of random variables of the bearings (B1 and B2)

	Bearing (B1)			Bearing (B2)		
Variable	Mean	St.dev	Distribution	Mean	St.dev	Distribution
f_P (-)	1.15	0.0380	Normal	1.12	0.0370	Normal
$P_{bearing}$ (N)	$P_{bearing}=6.7512\cdot T_L$			$P_{bearing}=6.1610\cdot T_L$		
L_h' (h)	60000	180	Normal	60000	180	Normal
C_r (kN)	3136.6	248	Normal	3136.6	248	Normal

(3) Reliability model and parameters of the shaft

The shaft bears the bend load and the torsion load. Therefore, the third or the fourth strength theory[21] should be applied to analysis the shaft. The performance function of the shaft is represented by

$$G_{shaft}=s3-r3=\sqrt{\sigma^2+4\left(\alpha\tau\right)^2}-\frac{\sigma_{-1}\varepsilon_\alpha\beta}{k_f}=\frac{32}{\pi d_{shaft}^3}\sqrt{M^2+\left(\alpha T_L\right)^2}-\frac{\sigma_{-1}\varepsilon_\alpha\beta}{k_f}, \tag{45}$$

where, σ, τ, α, σ_{-1}, ε_α, β, k_f are bending stress (weakest region), shear stress, conversion coefficient (constant load 0.3; repeated and pulsation load 0.6, repeated and reverse load 1), endurance limit, size factor, surface machining factor, effective stress concentration factor, respectively. Seven normal random variables are $\mathbf{X}_i = \left[d_{shaft}, M, T_L, \sigma_{-1}, \varepsilon_\alpha, \beta, k_f\right]^{\mathrm{T}} = \left[X_{i1}, X_{i2}, X_{i3}, X_{i4}, X_{i5}, X_{i6}, X_{i7}\right]^{\mathrm{T}}, i=5$. Table 5 provides the distribution information of all seven random variables.

The moment M is calculated by equation $M = \sqrt{M_H{}^2 + M_V{}^2}$. Where moment M_H is the load in the horizontal plane and moment M_V is the load in the vertical plane. Then the combined moment $M = 1.3432 \cdot T_L$ of the weakest region is obtained.

Table 5

Information of random variables of the shaft

Variable	d_{shaft} (mm)	M $(N \bullet m)$	T_L $(N \bullet m)$	σ_{-1} (MPa)	ε_α (-)	β (-)	k_f (-)
Mean	180	$1.3432 \cdot T_L$	3.9689×10^4	355	0.60	0.95	1.6
St.dev	1.1		3.9689×10^3	10	0.003	0.0048	0.0912

4.2.2. Reliability analysis of the shaft assembly.

Table 7 gives reliability values of the shaft assembly. The independent reliability probabilities of the part and shaft assembly estimated by MVFOSA and MCS, are also shown in Table 7.

Table 6

Reliability results of the part and the system

	R_{gear5}	R_{gear6}	$R_{bearing6}$	$R_{bearing5}$	R_{shaft}	R_{system}	
MCS	0.97406	0.98498	0.98613	0.98251	0.98665	0.91716	Independent
MVFOSA	0.96900	0.97709	0.98522	0.98313	0.98704	0.90518	
MCS	-	-	-	-	-	0.93550	Dependent
Proposed method	-	-	-	-	-	0.93170	

From Table 6, Probabilities of the shaft assembly without considering the dependency are 0.91716 (for MCS) and 0.90520 (for MVFOSA). Meanwhile, reliability value of the system with dependence is 0.93550 (for MCS). Compared to the result with dependency calculated by MCS, the error percentages of the independent one are 1.960% (for MCS) and 3.241% (for MVFOSA). It is indicated that calculating the system reliability without counting the failure dependency may consequently results in a large error. The proposed method provides an accurate reliability value 0.93170, which is closed to the MCS with the error percentage as 0.406%.

5. Conclusion

In a mechanical system, unlike size related random variables, load random variables are usually distributed with a large variation coefficient, resulting in a large correlation between the parts. Therefore, the purpose of this work is to improve the accuracy of reliability evaluation of the system with load dependency. The advantages of the proposed method are as follows: (1) MVFOSA method improves reliability model of the parts from a complex performance function to a simple formulation. (2) different from the linear correlation coefficient, describing the correlation from the source equation. (3) the complex multi-dimensional ntegral equation of system reliabilty is transformed into a one-dimensional integral function, which is substantially simpler and more efficient. (4) the number of function evaluations is small, indicating the high efficiency of the new proposed method.

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