A SELF-ADAPTIVE SUBGRADIENT EXTRAGRADIENT ALGORITHM WITH INERTIAL EFFECTS FOR VARIATIONAL INEQUALITIES

Qiao-Li Dong[1], Xiao-Huan Li[2], Tzu-Chien Yin[3]

In this paper, we introduce a new self-adaptive subgradient extragradient algorithm with inertial effects for variational inequality, for which the stepsize is chosen by a new way. The weak convergence of the algorithm is established. The numerical examples are given which illustrate the efficiency and advantage of the proposed algo-rithms.

Keywords: Variational inequality, Extragradient algorithm, Subgradient extragradient algorithm, Inertial type algorithm, Weak convergence.

MSC2020: 90C47, 49J35.

1. Introduction

Let C be a nonempty, closed and convex set in a real Hilbert space H and $f : H \to H$ be a given mapping. In this article, we consider the classical variational inequality $(VI(C, f))$

$$\text{Find } x^* \in C, \text{ such that } \langle f(x^*), x - x^* \rangle \geq 0, \quad \forall x \in C. \tag{1}$$

The theory of variational inequality has applications in many fields such as mathematical economy, physics, society, and engineering, and provides a simple, natural and unified framework for many issues, such as the minimization problems, fixed point problems, equilibrium problems and so on, see [1, 3, 5, 6, 11, 21, 24–26, 29, 30, 32–37, 42–48, 50].

A great deal of projection methods for solving $VI(C, f)$ have been studied (see, e.g., [4, 8, 10, 13, 14, 17, 27, 38–41, 49]), where the simplest one is

$$x^{k+1} = P_C(x^k - \tau f(x^k)), \ k \geq 0 \tag{2}$$

where τ is some positive real number and P_C is the metric projection onto C (see its definition in Definition 2.1).

If f is Lipschitz continuous and strongly monotone, then the sequence $\{x^k\}_{k \in \mathbb{N}}$ generated by (2) converges to the solution of the problem (1). However, if the strong monotonicity hypothesis reduces to the plain monotonicity, then the sequence may be divergent.

In order to deal with this situation, Korpelevich [16] proposed the well-known extragradient algorithm

$$\begin{cases} y^k = P_C(x_k - \tau f(x^k)), \\ x^{k+1} = P_C(x_k - \tau f(y^k)), \end{cases} \tag{3}$$

which is convergent when f is Lipschitz continuous and monotone. If C is a general closed and convex set, then a minimal distance problem has to be solved (twice) in order to obtain the next iterate. This might seriously affect the efficiency of the extragradient method.

[1]College of Science and Tianjin Key Lab for Advanced Signal Processing, Civil Aviation University of China,, Tianjin 300300, China, e-mail: `dongql@lsec.cc.ac.cn`

[2]College of Science, Civil Aviation University of China, Tianjin 300300, China, e-mail: `xiaohuanlimath@163.com`

[3]Corresponding author. Research Center for Interneural Computing, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan, e-mail: `yintzuchien@mail.cmuh.org.tw`

To overcome this problem, Censor et al. [4] introduced a subgradient extragradient algorithm which replaces the second projection onto C of the extragradient method (3) by a projection onto a specific constructible half-space T_k. The subgradient extragradient algorithm is defined by

$$\begin{cases} y^k = P_C(x^k - \tau f(x^k)), \\ T^k := \{w \in H \,|\, \langle x^k - \tau f(x^k) - y^k, w - y^k \rangle \leq 0\}, \\ x^{k+1} = P_{T_k}(x^k - \tau f(y^k)), \end{cases} \tag{4}$$

where τ is a positive real number.

To accelerate the speed of the extragradient-type algorithms, the inertial extrapolation technique was combined with the projection methods by some authors [7, 9, 31] for solving the variational inequality problems. These inertial projection algorithms have excellent numerical performance.

Recently, Thong and Hieu [27] proposed the following inertial subgradient extragradient algorithm:

$$\begin{cases} w^k = x^k + \alpha_n(x^k - x^{k-1}), \\ y^k = P_C(w^k - \tau f(w^k)), \\ T_n := \{x \in H \,|\, \langle w^k - \tau f(w^k) - y^k, x - y^k \rangle \leq 0\}, \\ x^{k+1} = P_{T_n}(w^k - \tau f(y^k)), \end{cases} \tag{5}$$

where τ is a positive real number.

Thong and Hieu [28] also applied the self-adaptive technique to give the stepsize τ_k which is the largest $\tau \in \{\gamma, \gamma l, \gamma l^2, ...\}$ satisfying $\tau \|f(w^k) - f(y^k)\| \leq \mu \|w^k - y^k\|$, $l, \mu \in (0, 1)$.

Yang [31] introduced another self-adaptive technique as follows:

$$\tau_{k+1} = \begin{cases} \min\left\{ \dfrac{\mu(\|w^k - y^k\|^2 + \|x^{k+1} - y^k\|^2)}{2\langle f(w^k) - f(y^k), x^{k+1} - y^k \rangle}, \tau_n \right\}, & \text{if } \langle f(w^k) - f(y^k), x^{k+1} - y^k \rangle > 0, \\ \tau_n, & \text{otherwise,} \end{cases} \tag{6}$$

where $\mu \in (0, 1)$.

The main purpose of this article is to introduce a new inertial subgradient extragradient algorithm, for which the stepsize is chosen through a different way from (6) and that in [28].

This paper is organized as follows. In Section 2, we recall some definitions and preliminary results used in the proof of the main results. Section 3 introduces an inertial subgradient extragradient algorithm and shows its weak convergence. In section 4, we provide two numerical experiments to illustrate the behaviors of the proposed algorithm by comparing with other methods.

2. Preliminaries

We use $x_k \rightharpoonup x$ ($x_k \to x$) to indicate that the sequence $(x_k)_{k \in \mathbb{N}}$ converges weakly (strongly) to x.

Let C be a closed convex subset of real Hilbert space H. Denote by $N_C(v)$ the normal cone ([19], p.76) of C at $v \in C$, i.e.,

$$N_C(v) := \{d \in H | \langle d, y - v \rangle \leq 0, \quad \forall y \in C\}.$$

Recall that in a Hilbert space H

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda \|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2, \tag{7}$$

for all $x, y \in H$ and $\lambda \in \mathbb{R}$ (see Corollary 2.14 in [2]). There also holds

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in H. \tag{8}$$

Definition 2.1. *Let C be a closed convex subset of real Hilbert space H. P_C is called the* `(metric or nearest point) projection` *from H onto C if for $x \in H$, $P_C x$ is the unique point in C such that*

$$\|x - P_C x\| = \inf\{\|x - z\| : z \in C\}.$$

Lemma 2.1. *Given $x \in H$ and $z \in C$. Then $z = P_C x$ if and only if there holds the relation:*

$$\langle x - z, y - z\rangle \le 0, \quad \text{for all } y \in C.$$

Lemma 2.2 ([4]). *For any $x, y \in H$ and $z \in C$, it holds*
  (i) $\|P_C(x) - P_C(y)\| \le \|x - y\|$;
  (ii) $\|P_C(x) - z\|^2 \le \|x - z\|^2 - \|P_C(x) - x\|^2$.
  (iii) $\|P_C(x) - P_C(y)\|^2 \le \langle P_C(x) - P_C(y), x - y\rangle$.

Lemma 2.3 ([2]). *Assume that $\{u_k\}$, $\{v_k\}$ and $\{\alpha_k\}$ are three sequences in $[0, \infty)$ such that*
  (i) $u_{k+1} - u_k \le \alpha_k(u_k - u_{k_1}) + v_k$ *for all $k \ge 0$;*
  (ii) $0 \le \alpha_k \le \alpha(\forall k \ge 0)$ *for some $\alpha \in (0, 1)$;*
  (iii) $\sum_{k=0}^{\infty} v_k < \infty$.
Then $\lim_{k \to \infty} u_k$ exists.

Lemma 2.4 ([20]). *Let $f : H \to H$ be a monotone and L-Lipschitz continuous mapping. Assume that the sequence $\{x^k\} \subset H$ satisfies $x^k \rightharpoonup u^\dagger$ and $x^k - P_C(I - \tau f)x^k \to 0(\tau > 0)$. Then $u^\dagger \in SOL(f, C)$.*

Lemma 2.5. *(Opial's lemma) Let C be a nonempty set of H and $\{x^k\}$ be a sequence in H such that the following two conditions hold:*
  (i) *for every $u \in C$, $\lim_{k \to \infty} \|x^k - u\|$ exists;*
  (ii) $\omega_w(x^k) \subset C$.
Then $\{x^k\}$ converges weakly to a point in C.

3. The main results

In this section, we introduce a new self-adaptive inertial subgradient extragradient algorithm and establish its weak convergence.

We firstly impose the following assumptions on the variational inequality.
Condition 3.1 The solution set of (1), denoted by $SOL(C, f)$, is nonempty.
Condition 3.2 The mapping f is monotone on H, i.e.

$$\langle f(x) - f(y), x - y\rangle \ge 0, \quad \forall x, y \in H.$$

Condition 3.3 The mapping f is Lipschitz continuous on H with constant $L > 0$, that is,

$$\|f(x) - f(y)\| \le L\|x - y\|, \quad \forall x, y \in H.$$

3.1. Algorithm

Now we present the self-adaptive inertial subgradient extragradient algorithm.

Algorithm 3.1.

  Initialization: Let $\{\alpha_k\}$ be a sequence in $[0, \infty)$. Let $\mu, l \in (0, 1)$, $\sigma > 0$ and $x_0, x_{-1} \in H$ be arbitrary.

  Iterative Steps: Assume that x^k has been given. Calculate x^{k+1} as follows:

Step 1. Compute

$$w^k = x^k + \alpha_k(x^k - x^{k-1}),$$

and

$$y^k = P_C(w^k - \tau_k f(w^k)), \tag{9}$$

where τ_k is the largest $\tau \in \{\sigma, \sigma l, \sigma l^2, ...\}$ satisfying

$$\tau \|f(w^k) - f(y^k)\| \le \mu \|w^k - y^k\|. \tag{10}$$

If $y^k = w^k$ then stop. Otherwise, go to **Step 2**.

Step 2. Construct the half-space

$$T_k := \{z \in H : \langle w^k - \gamma\tau_k\beta_k f(w^k) - y^k, z - y^k \rangle \le 0\}$$

and compute

$$x^{k+1} = P_{T_k}(w^k - \gamma\tau_k\beta_k f(y^k)), \tag{11}$$

where γ is a positive real number and

$$\beta_k := \begin{cases} \varphi(w^k, y^k)/\|d(w^k, y^k)\|^2, & \text{if } d(w^k, y^k) \ne 0, \\ \dfrac{1}{(1-\mu)^2}, & \text{if } d(w^k, y^k) = 0, \end{cases} \tag{12}$$

and

$$\varphi(w^k, y^k) := \langle w^k - y^k, d(w^k, y^k) \rangle,$$

and

$$d(w^k, y^k) = (w^k - y^k) - \tau_k(f(w^k) - f(y^k)).$$

Let $k := k+1$ and return to **Step 1**.

Remark 3.1. We have the following remarks for the stepsize and the inertial parameters α_k.

(i) The stepsizes in y^k and x^{k+1} in Algorithm 3.1 are different and the choice of the stepsizes of our algorithm is different from (6) and that in (5).

(ii) By (10), it is easy to show that $\dfrac{\mu l}{L} \le \tau_k \le \sigma$.

(iii) In Algorithm 3.1, $\{\alpha_k\}_{k \in \mathbb{N}}$ is assumed to be nondecreasing with $\alpha_1 = 0$ and satisfy $0 \le \alpha_k \le \alpha < \sqrt{5} - 2$. The parameters μ and α satisfy the following inequality

$$0 < \gamma \le \frac{(1-\mu)^2(1 - 4\alpha - \alpha^2 - 2\delta)}{\mu(1-\alpha)^2}. \tag{13}$$

where $\delta \in \left(0, \dfrac{1 - 4\alpha - \alpha^2}{2}\right)$.

From (13), it is easy to verify that

$$0 < \frac{\gamma\mu}{(1-\mu)^2} \le \frac{(1 - 4\alpha - \alpha^2 - 2\delta)}{(1-\alpha)^2} < 1. \tag{14}$$

Remark 3.2. By Lemma 2.1, we know that $w^\dagger \in SOL(C, f) \Leftrightarrow w^\dagger = P_C(w^\dagger - \tau f(w^\dagger))$ for some $\tau > 0$. Thus, if at some iterative step $w^k = y^k = P_C(w^k - \tau_k f(w^k))$, then $w^k \in SOL(C, f)$.

Remark 3.3. We assume that f is L-Lipschitz continuous. However, the information of L is not necessary priority to be known. That is, we need not to estimate the value of L.

Lemma 3.1. Let $\{\beta_k\}_{k \in \mathbb{N}}$ be generated by (12). Under the Condition 3.2, it holds

$$\beta_k \le \frac{1}{(1-\mu)^2}. \tag{15}$$

Proof. By the monotonicity of f, we have

$$\langle f(w^k) - f(y^k), w^k - y^k \rangle \geq 0.$$

Using Condition 3.2 and the definition of $d(w^k, y^k)$, we obtain

$$
\begin{aligned}
\varphi(w^k, y^k) &= \langle d(w^k, y^k), w^k - y^k \rangle \\
&= \langle w^k - y^k, w^k - y^k \rangle - \tau_k \langle f(w^k) - f(y^k), w^k - y^k \rangle \\
&\leq \|w^k - y^k\|^2,
\end{aligned}
\tag{16}
$$

and

$$
\begin{aligned}
\|d(w^k, y^k)\|^2 &= \|(w^k - y^k) - \tau_k(f(w^k) - f(y^k))\|^2 \\
&= \|w^k - y^k\|^2 + \tau_k^2 \|f(w^k) - f(y^k)\|^2 - 2\tau_k \langle f(w^k) - f(y^k), w^k - y^k \rangle \\
&\geq \|w^k - y^k\|^2 + \tau_k^2 \|f(w^k) - f(y^k)\|^2 - 2\tau_k \|f(w^k) - f(y^k)\| \|w^k - y^k\| \\
&= (\|w^k - y^k\| - \tau_k \|f(w^k) - f(y^k)\|)^2 \\
&\geq (1 - \mu)^2 \|w^k - y^k\|^2.
\end{aligned}
\tag{17}
$$

From (16) and (17), (15) follows. $\qquad \square$

3.2. The convergence analysis

We present some convergence results for Algorithm 3.1 as follows.

Lemma 3.2. *Let $u \in SOL(C, f)$ and $\{x^k\}_{k \in \mathbb{N}}$ be the sequence generated by Algorithm 3.1. Then, under Conditions 3.1, 3.2 and 3.3, we have*

$$\|x^{k+1} - u\|^2 \leq \|w^k - u\|^2 - \frac{1}{2}\left(1 - \frac{\gamma\mu}{(1-\mu)^2}\right)\|x^{k+1} - w^k\|^2.$$

Proof. Applying Lemma 2.2(ii) to (11), we obtain

$$
\begin{aligned}
\|x^{k+1} - u\|^2 &\leq \|w^k - \gamma\tau_k\beta_k f(y^k) - u\|^2 - \|w^k - \gamma\tau_k\beta_k f(y^k) - x^{k+1}\|^2 \\
&= \|w^k - u\|^2 - 2\gamma\tau_k\beta_k \langle w^k - u, f(y^k) \rangle + \gamma^2\tau_k^2\beta_k^2 \|f(y^k)\|^2 \\
&\quad - \|w^k - x^{k+1}\|^2 + 2\gamma\tau_k\beta_k \langle w^k - x^{k+1}, f(y^k) \rangle - \gamma^2\tau_k^2\beta_k^2 \|f(y^k)\|^2 \\
&= \|w^k - u\|^2 + 2\gamma\tau_k\beta_k \langle u - x^{k+1}, f(y^k) \rangle - \|w^k - x^{k+1}\|^2 \\
&= \|w^k - u\|^2 + 2\gamma\tau_k\beta_k \langle u - y^k, f(y^k) - f(u) \rangle - \|w^k - x^{k+1}\|^2 \\
&\quad + 2\gamma\tau_k\beta_k \langle u - y^k, f(u) \rangle + 2\gamma\tau_k\beta_k \langle y^k - x^{k+1}, f(y^k) \rangle.
\end{aligned}
\tag{18}
$$

Since f is monotone, $\langle u - y^k, f(y^k) - f(u) \rangle \leq 0$. Noting that $u \in SOL(C, f)$ and $y^k \in C$, we deduce that $\langle u - y^k, f(u) \rangle \leq 0$. It follows from (18) that

$$
\begin{aligned}
\|x^{k+1} - u\|^2 &\leq \|w^k - u\|^2 - \|w^k - x^{k+1}\|^2 + 2\gamma\tau_k\beta_k \langle y^k - x^{k+1}, f(y^k) \rangle \\
&= \|w^k - u\|^2 - \|w^k - y^k + y^k - x^{k+1}\|^2 + 2\gamma\tau_k\beta_k \langle y^k - x^{k+1}, f(y^k) \rangle \\
&= \|w^k - u\|^2 - \|w^k - y^k\|^2 - 2\langle w^k - y^k, y^k - x^{k+1} \rangle - \|y^k - x^{k+1}\|^2 \\
&\quad + 2\gamma\tau_k\beta_k \langle y^k - x^{k+1}, f(y^k) \rangle.
\end{aligned}
\tag{19}
$$

By the definition of T_k and $x^{k+1} \in T_k$, we have

$$\langle w^k - \gamma\tau_k\beta_k f(w^k) - y^k, x^{k+1} - y^k \rangle \leq 0,$$

which implies that

$$\langle w^k - y^k, y^k - x^{k+1} \rangle \geq \gamma\tau_k\beta_k \langle f(w^k), y^k - x^{k+1} \rangle. \tag{20}$$

By virtue of (19) and (20), we obtain

$$
\begin{aligned}
\|x^{k+1} - u\|^2 \leq{} & \|w^k - u\|^2 - \|w^k - y^k\|^2 - 2\gamma\tau_k\beta_k\langle f(w^k), y^k - x^{k+1}\rangle \\
& - \|y^k - x^{k+1}\|^2 + 2\gamma\tau_k\beta_k\langle y^k - x^{k+1}, f(y^k)\rangle \\
\leq{} & \|w^k - u\|^2 + 2\gamma\tau_k\beta_k\langle y^k - x^{k+1}, f(y^k) - f(w^k)\rangle \\
& - \|w^k - y^k\|^2 - \|y^k - x^{k+1}\|^2.
\end{aligned}
\tag{21}
$$

In the light of (10), we deduce

$$
\begin{aligned}
2\gamma\tau_k\beta_k\langle y^k - x^{k+1}, f(y^k) - f(w^k)\rangle & \leq 2\gamma\beta_k\tau_k\|f(y^k) - f(w^k)\|\|y^k - x^{k+1}\| \\
& \leq 2\gamma\beta_k\mu\|y^k - w^k\|\|y^k - x^{k+1}\| \\
& \leq \gamma\beta_k\mu\|y^k - w^k\|^2 + \gamma\beta_k\mu\|y^k - x^{k+1}\|^2.
\end{aligned}
$$

This together with (21) implies that

$$
\begin{aligned}
\|x^{k+1} - u\|^2 \leq{} & \|w^k - u\|^2 + \gamma\beta_k\mu\|y^k - w^k\|^2 + \gamma\beta_k\mu\|y^k - x^{k+1}\|^2 \\
& - \|w^k - y^k\|^2 - \|y^k - x^{k+1}\|^2 \\
={} & \|w^k - u\|^2 - (1 - \gamma\beta_k\mu)\|w^k - y^k\|^2 - (1 - \gamma\beta_k\mu)\|x^{k+1} - y^k\|^2.
\end{aligned}
\tag{22}
$$

By the mean value inequality and the trigonometric inequality,

$$
\begin{aligned}
\|x^{k+1} - u\|^2 \leq{} & \|w^k - u\|^2 - \frac{(1 - \gamma\beta_k\mu)}{2}(\|w^k - y^k\| + \|x^{k+1} - y^k\|)^2 \\
\leq{} & \|w^k - u\|^2 - \frac{(1 - \gamma\beta_k\mu)}{2}\|x^{k+1} - w^k\|^2.
\end{aligned}
\tag{23}
$$

Combining (15) with (23), we have

$$
\|x^{k+1} - u\|^2 \leq \|w^k - u\|^2 - \frac{1}{2}\left(1 - \frac{\gamma\mu}{(1-\mu)^2}\right)\|x^{k+1} - w^k\|^2.
\tag{24}
$$

The proof is complete. $\square$

Theorem 3.1. *Assume that Conditions 3.1, 3.2 and 3.3 hold. Then the sequence $\{x^k\}_{k\in\mathbb{N}}$ generated by Algorithm 3.1 converges weakly to $SOL(C, f)$.*

Proof. From the definition of w^k, we have

$$
\begin{aligned}
\|x^{k+1} - w^k\|^2 ={} & \|x^{k+1} - (x^k + \alpha_k(x^k - x^{k-1}))\|^2 \\
={} & \|x^k - x^{k+1}\|^2 + \alpha_k^2\|x^k - x^{k+1}\|^2 + 2\alpha_k\langle x^k - x^{k+1}, x^k - x^{k-1}\rangle \\
\geq{} & (1 - \alpha_k)\|x^{k+1} - x^k\|^2 + (\alpha_k^2 - \alpha_k)\|x^k - x^{k-1}\|^2.
\end{aligned}
$$

Using (7), we obtain

$$
\|w^k - u\|^2 = (1 + \alpha_k)\|x^k - u\|^2 - \alpha_k\|x^{k-1} - u\|^2 + \alpha_k(1 + \alpha_k)\|x^k - x^{k-1}\|^2.
$$

Then, from (14) and Lemma 3.2, we get

$$
\begin{aligned}
& \|x^{k+1} - u\|^2 \\
\leq{} & (1 + \alpha_k)\|x^k - u\|^2 - \alpha_k\|x^{k-1} - u\|^2 + \alpha_k(1 + \alpha_k)\|x^k - x^{k-1}\|^2 \\
& - \frac{1}{2}\left(1 - \frac{\gamma\mu}{(1-\mu)^2}\right)\left[(1 - \alpha_k)\|x^{k+1} - x^k\|^2 + (\alpha_k^2 - \alpha_k)\|x^k - x^{k-1}\|^2\right] \\
\leq{} & (1 + \alpha_k)\|x^k - u\|^2 - \alpha_k\|x^{k-1} - u\|^2 - \zeta_k\|x^{k+1} - x^k\|^2 + \xi_k\|x^k - x^{k-1}\|^2,
\end{aligned}
\tag{25}
$$

where

$$
\zeta_k := \frac{1}{2}\left(1 - \frac{\gamma\mu}{(1-\mu)^2}\right)(1 - \alpha_k) \geq 0,
\tag{26}
$$

and

$$\xi_k := \alpha_k(1 + \alpha_k) - \frac{1}{2}\left(1 - \frac{\gamma\mu}{(1-\mu)^2}\right)(\alpha_k^2 - \alpha_k) \geq 0. \tag{27}$$

Let $h = \frac{1}{2}\left(1 - \frac{\gamma\mu}{(1-\mu)^2}\right)$. By (13), it is easy to show

$$
\begin{aligned}
\zeta_k - \xi_{k+1} =& h(1 - \alpha_k) - \alpha_{k+1}(1 + \alpha_{k+1}) + h(\alpha_{k+1}^2 - \alpha_{k+1}) \\
\geq& h(1 - \alpha_{k+1}) - \alpha_{k+1}(1 + \alpha_{k+1}) + h(\alpha_{k+1}^2 - \alpha_{k+1}) \\
=& h(1 - \alpha_{k+1})^2 - \alpha_{k+1} - \alpha_{k+1}^2 \\
\geq& h(1 - \alpha)^2 - \alpha - \alpha^2 \\
\geq& \left(\frac{1}{2} - 2\alpha - \frac{1}{2}\alpha^2\right) - \frac{1}{2}(1 - \alpha)^2\frac{\gamma\mu}{(1-\mu)^2} \\
\geq& \delta.
\end{aligned}
\tag{28}
$$

We define the sequence, for all $k \geq 1$,

$$\Gamma^k := \|x^k - u\|^2 - \alpha_k\|x^{k-1} - u\|^2 + \xi_k\|x^k - x^{k-1}\|^2. \tag{29}$$

Using the monotonicity of $\{\alpha_k\}_{k\geq 1}$, we get

$$
\begin{aligned}
\Gamma^{k+1} - \Gamma^k \leq& \|x^{k+1} - u\|^2 - \alpha_k\|x^k - u\|^2 + \xi_{k+1}\|x^{k+1} - x^k\|^2 \\
& - \|x^k - u\|^2 + \alpha_k\|x^{k-1} - u\|^2 - \xi_k\|x^k - x^{k-1}\|^2 \\
=& \|x^{k+1} - u\|^2 - (1 + \alpha_k)\|x^k - u\|^2 + \alpha_k\|x^{k-1} - u\|^2 \\
& + \xi_{k+1}\|x^{k+1} - x^k\|^2 - \xi_k\|x^k - x^{k-1}\|^2 \\
\leq& -\zeta_k\|x^{k+1} - x^k\|^2 + \xi_{k+1}\|x^{k+1} - x^k\|^2 \\
=& -(\zeta_k - \xi_{k+1})\|x^{k+1} - x^k\|^2 \\
\leq& -\delta\|x^{k+1} - x^k\|^2 \leq 0.
\end{aligned}
\tag{30}
$$

Hence, $\{\Gamma^k\}$ is a monotone decreasing sequence.

According to (29), we get

$$
\begin{aligned}
\|x^k - u\|^2 \leq& \alpha_k\|x^{k-1} - u\|^2 + \Gamma^k \\
\leq& \alpha\|x^{k-1} - u\|^2 + \Gamma^1 \\
\leq& \dots \\
\leq& \alpha^k\|x^0 - u\|^2 + (1 + \alpha + \dots + \alpha^{k-1})\Gamma^1 \\
\leq& \alpha^k\|x^0 - u\|^2 + \frac{1}{1-\alpha}\Gamma^1.
\end{aligned}
\tag{31}
$$

Note that

$$
\begin{aligned}
\Gamma^{k+1} =& \|x^{k+1} - u\|^2 - \alpha_{k+1}\|x^k - u\|^2 + \xi_{k+1}\|x^{k+1} - x^k\|^2 \\
\geq& -\alpha_{k+1}\|x^k - u\|^2.
\end{aligned}
\tag{32}
$$

Combining (31) and (32), we deduce

$$-\Gamma^{k+1} \leq \alpha_{k+1}\|x^k - u\|^2 \leq \alpha\|x^k - u\|^2 \leq \alpha^{k+1}\|x^0 - u\|^2 + \frac{\alpha}{1-\alpha}\Gamma^1.$$

This together with (30) implies that

$$\delta \sum_{k=1}^{i} \|x^{k+1} - x^k\|^2 \leq \sum_{k=1}^{i} (\Gamma^k - \Gamma^{k+1}) = \Gamma^1 - \Gamma^{k+1}$$

$$\leq \alpha^{i+1} \|x^0 - u\|^2 + \frac{1}{1-\alpha} \Gamma^1$$

$$\leq \|x^0 - u\|^2 + \frac{1}{1-\alpha} \Gamma^1.$$

which implies that

$$\sum_{k=1}^{\infty} \|x^{k+1} - x^k\|^2 < \infty. \tag{33}$$

Therefore,

$$\lim_{k \to \infty} \|x^{k+1} - x^k\| = 0. \tag{34}$$

Note that

$$\|w^k - x^k\| = \|\alpha_k(x^k - x^{k-1})\| \leq \alpha_k \|x^k - x^{k-1}\| \leq \alpha \|x^k - x^{k-1}\|. \tag{35}$$

By (34) and (35), we derive

$$\lim_{k \to \infty} \|x^k - w^k\| = 0. \tag{36}$$

By the definition (27) of ξ_k, there exists a positive constant $\overline{\xi}$ such that $\xi_k \leq \overline{\xi}$ for all $k \geq 0$. From (25), we have

$$\|x^{k+1} - u\|^2 \leq (1 + \alpha_k)\|x^k - u\|^2 - \alpha_k \|x^{k-1} - u\|^2 + \xi_k \|x^k - x^{k-1}\|^2$$

$$\leq (1 + \alpha_k)\|x^k - u\|^2 - \alpha_k \|x^{k-1} - u\|^2 + \overline{\xi}\|x^k - x^{k-1}\|^2.$$

It follows that

$$\|x^{k+1} - u\|^2 - \|x^k - u\|^2 \leq \alpha_k (\|x^k - u\|^2 - \|x^{k-1} - u\|^2) + \overline{\xi}\|x^k - x^{k-1}\|^2. \tag{37}$$

Applying Lemma 2.3 to (37), we deduce that the limit $\lim_{k \to \infty} \|x^k - u\|$ exists. Thus, the sequence $\{x^k\}$ is bounded.

From (22), we obtain

$$(1 - \gamma\mu\beta_k)\|w^k - y^k\|^2 \leq \|w^k - u\|^2 - \|x^{k+1} - u\|^2$$

$$\leq \|w^k - x^{k+1}\|(\|w^k - u\| + \|x^{k+1} - u\|). \tag{38}$$

By (14) and (15), $\gamma\mu\beta_k \leq \frac{(1 - 4\alpha - \alpha^2 - 2\delta)}{(1-\alpha)^2} < 1$. Thus, $1 - \gamma\mu\beta_k > 1 - \frac{(1 - 4\alpha - \alpha^2 - 2\delta)}{(1-\alpha)^2} > 0$. Thanks to (34) and (36), $\lim_{k \to \infty} \|w^k - x^{k+1}\| = 0$. At the same time, $\{\|w^k - u\|\}$ and $\{\|x^k - u\|\}$ are all bounded. Therefore, from (38), we get

$$\lim_{k \to \infty} \|w^k - y^k\| = 0. \tag{39}$$

Let $u^\dagger \in \omega_w(x_n)$. There exists a subsequence $\{x^{k_i}\} \subset \{x^k\}$ such that $x^{k_i} \rightharpoonup u^\dagger$. Thus, $w^{k_i} \rightharpoonup u^\dagger$ due to (36). Applying Lemma 2.4 to (39), we conclude that $u^\dagger \in SOL(f, C)$. Hence, $\lim_{k \to \infty} \|x^k - u^\dagger\|$ exists. Therefore, by Lemma 2.5, we deduce that x^k weakly converges to an element in $SOL(f, C)$. The proof is completed. $\square$

4. **Numerical illustrations**

In this section, we provide two numerical examples to show the practicability and the advantage of our proposed algorithm by comparing it with Algorithm 1 in [22] and Algorithm 3.1 in [23].

In numerical results listed in the following tables, 'Iter.' and 'Sec.' denote the number of iterations and the cpu time in seconds, respectively.

We take $\alpha_k = 0.1$ in these Algorithms and $\sigma = 0.2$ in Algorithm 3.1 and Algorithm 1 in [22]. Take $\gamma = 0.9$ and $\mu = 0.5$ in Algorithm 3.1 and choose $\mu = 0.5$ in Algorithm 3.1 in [23]. Set $l = 0.01$ in Example 4.1 and $l = 0.01$ in Example 4.2, respectively.

Example 4.1. Let the operator $f(x) := Mx + q$, $x \in R^m$. This example is taken from [12] and has been considered by many authors for numerical experiments (see, for example, [15, 18, 23]), where

$$M = BB^T + F + D,$$

and B is an $m \times m$ matrix, F is an $m \times m$ skew-symmetric matrix, D is an $m \times m$ diagonal matrix, whose diagonal entries are nonnegative (so M is positive semidefinite), q is a vector in R^m. The feasible set $C \subset R^m$ is a closed and convex subset defined by $C := \{x \in R^m : Qx \le b\}$, where Q is an $l \times m$ matrix and b is a nonnegative vector. It is clear that f is monotone and L-Lipschitz-continuous with $L = \|M\|$. Let $q = 0$. Then, the solution set $\Gamma := \{0\}$.
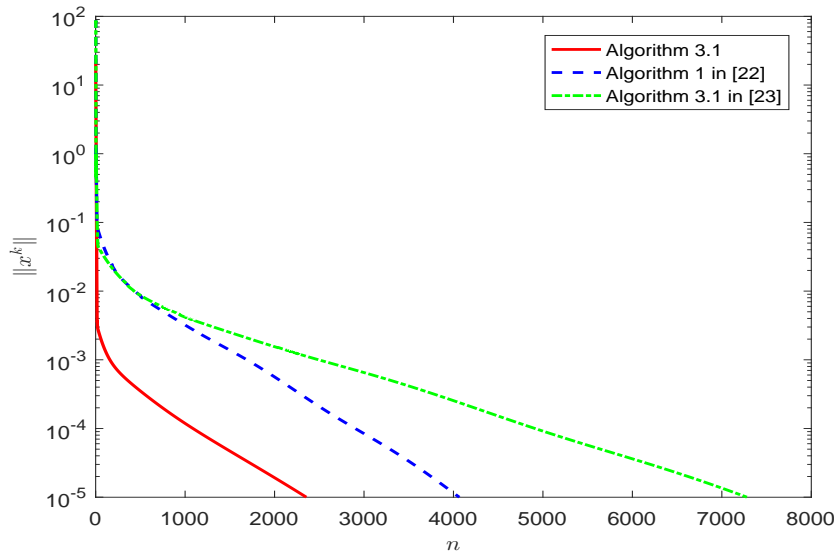

FIGURE 1. Comparison of Algorithm 3.1 and Algorithm 1 in [22] and Algorithm 3.1 in [23].

Take $\|x^k\| < 10^{-5}$ as the stopping criterion in Table 1. Figure 1 and Table 1 show that Algorithm 3.1 is better than Algorithm 1 in [22] and Algorithm 3.1 in [23] in the number of iterations and the cpu time.

Next we give an example in an infinite dimensional Hilbert space.

m	Algorithm 3.1		Algorithm 1 in [22]		Algorithm 3.1 in [23]	
	Iter.	Sec.	Iter.	Sec.	Iter.	Sec.
20	2545	0.304765	3517	0.453531	5025	1.016516
30	2520	0.378789	3743	1.895805	6839	3.411048
40	2523	0.436611	4621	2.439548	20891	12.074621
50	2219	1.366350	-	-	42706	38.102084

TABLE 1. Comparison of Algorithm 3.1 and Algorithm 1 in [22] and Algorithm 3.1 in [23] for different m.

Example 4.2. Suppose that $H = L^2([0,1])$ with norm $\|x\| := \left(\int_0^1 |x(t)|^2 dt \right)^{\frac{1}{2}}$ and inner product $\langle x,y \rangle := \int_0^1 x(t)y(t)dt, \;\; x,y \in H$. Let $C := \{x \in H : \|x\| \le 1\}$ be the unit ball. Define an operator $A : C \to H$ by

$$A(x)(t) = \int_0^1 (x(t) - F(t,s)f(x(s)))ds + g(t), \quad x \in C, t \in [0,1], \tag{40}$$

where

$$F(t,s) = \frac{2tse^{t+s}}{e\sqrt{e^2-1}}, \quad f(x) = \cos x, \quad g(t) = \frac{2te^t}{e\sqrt{e^2-1}}.$$

A is monotone and L-Lipschitz-continuous with $L = 2$ (hence uniformly continuous) and $\Gamma = \{0\}$.
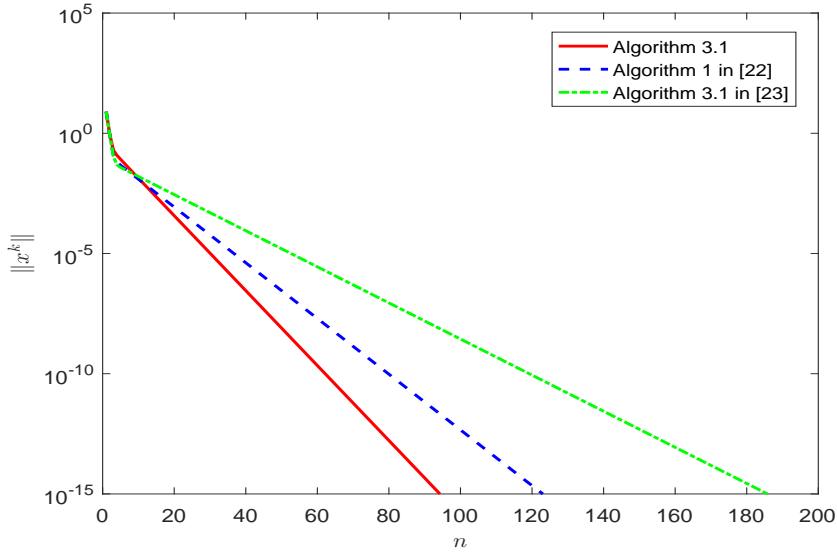

FIGURE 2. Comparison of Algorithm 3.1 and Algorithm 1 in [22] and Algorithm 3.1 in [23].

Take $\|x^k\| < 10^{-15}$ as the stopping criterion in Table 2. Figure 2 and Table 2 illustrate that Algorithm 3.1 is better than Algorithm 1 in [22] and Algorithm 3.1 in [23].

5. **Acknowledgments**

$x^0(t)$	Algorithm 3.1		Algorithm 1 in [22]		Algorithm 3.1 in [23]	
	Iter.	Sec.	Iter.	Sec.	Iter.	Sec.
1	84	0.159966	124	0.163477	186	0.246614
$\frac{1}{t}$	83	0.163891	124	0.171908	189	0.271120
t^2	83	0.167847	125	0.187206	193	0.266201
e^t	85	0.186619	127	0.189535	183	0.281754

TABLE 2. Comparison of Algorithm 3.1 and Algorithm 1 in [22] and Algorithm 3.1 in [23] in different initial point.

References

[1] F. Alvarez, *Weak convergence of a relaxed and inertial hybrid projection-proximal point algorithm for maximal monotone operators in Hilbert space*, SIAM J. Optim., **14**(2004), 773–782.

[2] H.H. Bauschke and P.L. Combettes, Convex Analysis and Monotone Operator Theory in Hilbert Spaces, 2nd ed. Springer, New York, 2017.

[3] L.C. Ceng, A. Petrusel, J.C. Yao and Y. Yao, *Systems of variational inequalities with hierarchical variational inequality constraints for Lipschitzian pseudocontractions*, Fixed Point Theory, **20**(2019), 113–133.

[4] Y. Censor, A. Gibali and S. Reich, *The subgradient extragradient method for solving variational inequalities in Hilbert space*, J. Optim. Theory Appl., **148**(2011), 318–335.

[5] V. Dadashi and M. Postolache, *Forward-backward splitting algorithm for fixed point problems and zeros of the sum of monotone operators*, Arab. J. Math., **9**(2020), 89–99.

[6] Q.L. Dong, Y.J. Cho and T.M. Rassias, *The projection and contraction methods for finding common solutions to variational inequality problems*, Optim. Lett., **12(8)**(2018), 1871–1896.

[7] Q.L. Dong, Y.J. Cho, L.L. Zhong and T.M. Rassias, *Inertial projection and contraction algorithms for variational inequalities*, J. Global Optim., **70(3)**(2018), 687–704.

[8] Q.L. Dong, D. Jiang and A. Gibali,*A modified subgradient extragradient method for solving the variational inequality*, Numer. Algor., **79(3)**(2018), 927–940.

[9] Q.L. Dong, Y.Y. Lu and J. Yang, *The extragradient algorithm with inertial effects for solving the variational inequality*, Optim., **65**(2016), 2217–2226.

[10] Q.L. Dong, J. Yang and H.B. Yuan, *The projection and contraction algorithm for solving variational inequality problems in Hilbert spaces*, J. Nonlinear Convex Anal., **20**(2019), 111–122.

[11] F. Facchinei and J.S. Pang, Finite-dimensional variational inequalities and complementarity problems, Vol. II, Springer Ser. Oper. Res., Springer-Verlag, New York, 2003.

[12] P.T. Harker and J.S. Pang, A damped-Newton method for the linear complementarity problem, in Computational Solution of Nonlinear Systems of Equations, Lectures in Appl. Math. 26, G. Allgower and K. Georg, eds., AMS, Providence, RI, 1990, pp. 265-284.

[13] S. He and H. Tian, *Selective projection methods for solving a class of variational inequalities*, Numer Algor., **80**(2019), 617–634.

[14] S. He and H.K. Xu, *The selective projection method for convex feasibility and split feasibility problems*, J. Nonlinear Sci. Appl., **19(7)**(2018), 1199–1215.

[15] D.V. Hieu, P.K. Anh and, L.D. Muu, *Modified hybrid projection methods for finding common solutions to variational inequality problems*, Comput. Optim. Appl., **66**(2017), 75–96.

[16] G.M. Korpelevich, *The extragradient method for finding saddle points and other problems*, Ekon. Mat. Metody, **12**(1976), 747–756.

[17] R. Kraikaew and S. Saejung, *Strong Convergence of the Halpern subgradient extragradient method for solving variational inequalities in Hilbert spaces*, J. Optim. Theory Appl., **163(2)**(2014), 399–412.

[18] Yu.V. Malitsky and V.V. Semenov, *A hybrid method without extrapolation step for solving variational inequality problems*, J. Global Optim., **61**(2015), 193–202.

[19] R.T. Rockafellar, *Monotone operators and the proximal point algorithm*, SIAM J. Control Optim., **14(5)**(1976), 877–898.

[20] S. Saejung and P. Yotkaew, *Approximation of zeros of inverse strongly monotone operators in Banach spaces*, Nonlinear Anal., **75**(2012), 742–750.

[21] D.R. Sahu, A. Pitea and M. Verma, *A new iteration technique for nonlinear operators as concerns convex programming and feasibility problems*, Numer. Algorithms, **83**(2020), 421–449.

[22] Y. Shehu, Q.L. Dong and D. Jiang, *Single projection method for pseudo-monotone variational inequality in Hilbert spaces*, Optim., **68**(2019), 385–409.

[23] M.V. Solodov and B.F. Svaiter, *A new projection method for variational inequality problems*, SIAM J. Control Optim., **37**(1999), 765–776.

[24] B.S. Thakur, D. Thakur and M. Postolache, *A new iterative scheme for numerical reckoning fixed points of Suzuki's generalized nonexpansive mappings*, Appl. Math. Comput., **275**(2016), 147–155.

[25] B.S. Thakur, D. Thakur and M. Postolache, *A new iteration scheme for approximating fixed points of nonexpansive mappings*, Filomat, **30**(2016), 2711–2720.

[26] D. Thakur, B.S. Thakur and M. Postolache, *New iteration scheme for numerical reckoning fixed points of nonexpansive mappings*, J. Inequal. Appl., **2014**(2014), Art. No. 328.

[27] D.V. Thong and D.V. Hieu, *Modified subgradient extragradient method for variational inequality problems*, Numer. Algor., **79**(2018), 597–610.

[28] D.V. Thong and D.V. Hieu, *Inertial subgradient extragradient algorithms with line-search process for solving variational inequality problems and fixed point problems*, Numer. Algor., **80**(2019), 1283–1307.

[29] G.I. Usurelu, *Split feasibility handled by a single-projection three-step iteration with comparative analysis*, J. Nonlinear Convex Anal., **22**(2021), 544–558.

[30] G.I. Usurelu and M. Postolache, *Algorithm for generalized hybrid operators with numerical analysis and applications*, J. Nonlinear Variational Anal., in printing.

[31] J. Yang, *Self-adaptive inertial subgradient extragradient algorithm for solving pseudomonotone variational inequalities*, Appl. Anal., **100(6)**(2019), 1–12.

[32] Y. Yao, L. Leng, M. Postolache and X. Zheng, *Mann-type iteration method for solving the split common fixed point problem*, J. Nonlinear Convex Anal., **18**(2017), 875–882.

[33] Y. Yao, H. Li and M. Postolache, *Iterative algorithms for split equilibrium problems of monotone operators and fixed point problems of pseudo-contractions*, Optim., in press, DOI: 10.1080/02331934.2020.1857757.

[34] Y. Yao, Y.C. Liou and M. Postolache, *Self-adaptive algorithms for the split problem of the demicontractive operators*, Optim., **67**(2018), 1309-1319.

[35] Y. Yao, Y.C. Liou and J.C. Yao, *Split common fixed point problem for two quasi-pseudo-contractive operators and its algorithm construction*, Fixed Point Theory Appl., **2015**(2015), Article ID 127.

[36] Y. Yao, Y.C. Liou and J.C. Yao, *Iterative algorithms for the split variational inequality and fixed point problems under nonlinear transformations*, J. Nonlinear Sci. Appl., **10**(2017), 843–854.

[37] Y. Yao, M. Postolache and Y.C. Liou, *Strong convergence of a self-adaptive method for the split feasibility problem*, Fixed Point Theory Appl., **2013**(2013), Art. No. 201.

[38] Y. Yao, M. Postolache, Y.C. Liou and Z. Yao, *Construction algorithms for a class of monotone variational inequalities*, Optim. Lett., **10**(2016), 1519–1528.

[39] Y. Yao, M. Postolache and J.C. Yao, *Iterative algorithms for the generalized variational inequalities*, U.P.B. Sci. Bull., Series A, **81**(2019), 3–16.

[40] Y. Yao, M. Postolache and J.C. Yao, *An iterative algorithm for solving the generalized variational inequalities and fixed points problems*, Mathematics, **7**(2019), Art. No. 61.

[41] Y. Yao, M. Postolache and J.C. Yao, *Strong convergence of an extragradient algorithm for variational inequality and fixed point problems*, U.P.B. Sci. Bull., Series A, **82(1)**(2020), 3–12.

[42] Y. Yao, M. Postolache and Z. Zhu, *Gradient methods with selection technique for the multiple-sets split feasibility problem*, Optim., **69**(2020), 269–281.

[43] Y. Yao, X. Qin and J.C. Yao, *Projection methods for firmly type nonexpansive operators*, J. Nonlinear Convex Anal., **19**(2018), 407–415.

[44] Y. Yao, J.C. Yao, Y.C. Liou and M. Postolache, *Iterative algorithms for split common fixed points of demicontractive operators without priori knowledge of operator norms*, Carpathian J. Math., **34**(2018), 459–466.

[45] H. Zegeye, N. Shahzad and Y. Yao, *Minimum-norm solution of variational inequality and fixed point problem in Banach spaces*, Optim., **64** (2015), 453–471.

[46] C. Zhang, Z. Zhu, Y. Yao and Q. Liu, *Homotopy method for solving mathematical programs with bounded box-constrained variational inequalities*, Optim., **68**(2019), 2293–2312.

[47] X. Zhao, M.A. Kobis, Y. Yao and J.C. Yao, *A projected subgradient method for nondifferentiable quasiconvex multiobjective optimization problem*, J. Optim. Theory Appl., **190**(2021), 82–107.

[48] X. Zhao, J.C. Yao and Y. Yao, *A proximal algorithm for solving split monotone variational inclusions*, U.P.B. Sci. Bull., Series A, **82(3)**(2020), 43–52.

[49] X. Zhao and Y. Yao, *Modified extragradient algorithms for solving monotone variational inequalities and fixed point problems*, Optim., **69**(2020), 1987–2002.

[50] L.J. Zhu, Y. Yao and M. Postolache, *Projection methods with linesearch technique for pseudomonotone equilibrium problems and fixed point problems*, U.P.B. Sci. Bull., Series A, **83(1)**(2021), 3–14.