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DIFFERENTIATION AND WEAK INTEGRATION FOR CONE-VALUED CURVES

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Locally convex cones are a generalization of locally convex topological vector spaces which are not necessarily embedded in vector spaces. We define the concepts of differentiation and weak integration for cone-valued curves. Also, we prove the Fundamental Theorems of Calculus for this type of functions. We find some conditions for the existence of weak integral in locally convex cones.

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1. Introduction

The theory of locally convex cones as developed in [4] and [5], uses an order theoretical concept or convex quasiuniform structure to introduce a topological structure on a cone. Examples of locally convex cones contain classes of functions that take infinite values and families of convex subsets of vector spaces. These types of structures are not vector space and also may not even be embedded into a larger vector spaces in order to apply technics from topological vector spaces. These structures are studied in the general theory of locally convex cones.

A *cone* is a set $\mathcal{P}$ endowed with an addition and a scalar multiplication for nonnegative real numbers. The addition is supposed to be associative and commutative, and there is a neutral element $0 \in \mathcal{P}$. For the scalar multiplication we must have $\alpha(\beta a) = (\alpha\beta)a$, $(\alpha + \beta)a = \alpha a + \beta a$, $\alpha(a + b) = \alpha a + \alpha b$, $1a = a$ and $0a = 0$ for all $a, b \in \mathcal{P}$ and $\alpha, \beta \geq 0$.

Let $\mathcal{P}$ be a cone. A collection $\mathcal{U}$ of convex subsets $U \subseteq \mathcal{P}^2 = \mathcal{P} \times \mathcal{P}$ is called a convex quasiuniform structure on$\mathcal{P}$, if the following properties hold:

($U1$) $\Delta \subseteq U$ for every $U \in \mathcal{U}$ ($\Delta = \{(a, a) : a \in \mathcal{P}\}$);

($U2$) for all $U, V \in \mathcal{U}$ there is a $W \in \mathcal{U}$ such that $W \subseteq U \cap V$;

($U3$) $\lambda U o \mu U \subseteq (\lambda + \mu)U$ for all $U \in \mathcal{U}$ and $\lambda, \mu > 0$;

($U4$) $\alpha U \in \mathcal{U}$ for all $U \in \mathcal{U}$ and $\alpha > 0$.

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Here, for $\boldsymbol{U}, \boldsymbol{V} \subseteq \boldsymbol{\mathcal{P}}^2$, by $\boldsymbol{UoV}$ we mean the set of all $(a, b) \in \boldsymbol{\mathcal{P}}^2$ such that there is some $\boldsymbol{c} \in \boldsymbol{\mathcal{P}}$ with $(\boldsymbol{a}, \boldsymbol{c}) \in \boldsymbol{U}$ and $(\boldsymbol{c}, \boldsymbol{b}) \in \boldsymbol{V}$.
Let $\boldsymbol{\mathcal{P}}$ be a cone and $\boldsymbol{\mathcal{U}}$ be a convex quasiuniform structure on $\boldsymbol{\mathcal{P}}$. We shall say $(\boldsymbol{\mathcal{P}}, \boldsymbol{\mathcal{U}})$ is a locally convex cone if
($U5$) for each $\boldsymbol{a} \in \boldsymbol{\mathcal{P}}$ and $\boldsymbol{U} \in \boldsymbol{\mathcal{U}}$ there is some $\rho > 0$ such that $(\boldsymbol{0}, \boldsymbol{a}) \in \boldsymbol{\rho U}$.
With every convex quasiuniform structure $\boldsymbol{\mathcal{U}}$ on $\boldsymbol{\mathcal{P}}$ we associate two topologies: The neighborhood bases for an element a in the upper and lower topologies are given by the sets
$\boldsymbol{U}(\boldsymbol{a}) = \{\boldsymbol{b} \in \boldsymbol{\mathcal{P}} : (\boldsymbol{b}, \boldsymbol{a}) \in \boldsymbol{U}\}$, ***resp.*** $(\boldsymbol{a})\boldsymbol{U} = \{\boldsymbol{b} \in \boldsymbol{\mathcal{P}} : (\boldsymbol{a}, \boldsymbol{b}) \in \boldsymbol{U}\}$, $\boldsymbol{U} \in \boldsymbol{\mathcal{U}}$.
The common refinement of the upper and lower topologies is called symmetric topology. A neighborhood base for $a \in \boldsymbol{\mathcal{P}}$ in this topology is given by the sets

$$\boldsymbol{U}(\boldsymbol{a})\boldsymbol{U} = \boldsymbol{U}(\boldsymbol{a}) \cap (\boldsymbol{a})\boldsymbol{U}, \qquad \boldsymbol{U} \in \boldsymbol{\mathcal{U}}.$$

Let $\boldsymbol{\mathcal{U}}$ and $\boldsymbol{\mathcal{W}}$ be convex quasiuniform structures on $\boldsymbol{\mathcal{P}}$. We say that $\boldsymbol{\mathcal{U}}$ is finer than $\boldsymbol{\mathcal{W}}$ if for every $\boldsymbol{W} \in \boldsymbol{\mathcal{W}}$ there is $\boldsymbol{U} \in \boldsymbol{\mathcal{U}}$ such that $\boldsymbol{U} \subseteq \boldsymbol{W}$.

The extended real number system $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is a cone endowed with the usual algebraic operations, in particular $\boldsymbol{a} + \infty = +\infty$ for all $a \in \overline{\mathbb{R}}$, $\boldsymbol{\alpha} \cdot (+\infty) = +\infty$ for all $\boldsymbol{\alpha} > \boldsymbol{0}$ and $\boldsymbol{0} \,.\, (+\infty) = \boldsymbol{0}$. We set $\widetilde{\boldsymbol{\mathcal{V}}} = \{\widetilde{\boldsymbol{\varepsilon}} : \boldsymbol{\varepsilon} > \boldsymbol{0}\}$, where $\widetilde{\boldsymbol{\varepsilon}} = \{(\boldsymbol{a}, \boldsymbol{b}) \in \overline{\mathbb{R}}^2 : \boldsymbol{a} \leq \boldsymbol{b} + \boldsymbol{\varepsilon}\}$. Then $\widetilde{\boldsymbol{\mathcal{V}}}$ is a convex quasiuniform structure on $\overline{\mathbb{R}}$ and $(\overline{\mathbb{R}}, \widetilde{\boldsymbol{\mathcal{V}}})$ is a locally convex cone. For $\boldsymbol{a} \in \mathbb{R}$ the intervals $(-\infty, \boldsymbol{a} + \boldsymbol{\varepsilon}]$ are the upper and the intervals $[\boldsymbol{a} - \boldsymbol{\varepsilon}, +\infty]$ are the lower neighborhoods, while for $\boldsymbol{a} = +\infty$ the entire cone $\overline{\mathbb{R}}$ is the only upper neighborhood, and $\{+\infty\}$ is open in the lower topology. The symmetric topology is the usual topology on $\mathbb{R}$ with as an isolated point $+\infty$.

For cones $\boldsymbol{\mathcal{P}}$ and $\boldsymbol{\mathcal{Q}}$, a mapping $\boldsymbol{T} : \boldsymbol{\mathcal{P}} \to \boldsymbol{\mathcal{Q}}$ is called a *linear operator* if $\boldsymbol{T}(\boldsymbol{a} + \boldsymbol{b}) = \boldsymbol{T}(\boldsymbol{a}) + \boldsymbol{T}(\boldsymbol{b})$ and $\boldsymbol{T}(\boldsymbol{\alpha a}) = \boldsymbol{\alpha T}(\boldsymbol{a})$ hold for all $\boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{\mathcal{P}}$ and $\boldsymbol{\alpha} \geq \boldsymbol{0}$. If both $(\boldsymbol{\mathcal{P}}, \boldsymbol{\mathcal{U}})$ and $(\boldsymbol{\mathcal{Q}}, \boldsymbol{\mathcal{W}})$ are locally convex cones, the operator $\boldsymbol{T}$ is called *(uniformly) continuous* if for every $\boldsymbol{W} \in \boldsymbol{\mathcal{W}}$ one can find $\boldsymbol{U} \in \boldsymbol{\mathcal{U}}$ such that $\boldsymbol{T} \times \boldsymbol{T}(\boldsymbol{U}) \subseteq \boldsymbol{W}$. A (uniformly) continuous linear operator is continuous with respect to upper, lower and symmetric topologies.

A *linear functional* on $\boldsymbol{\mathcal{P}}$ is a linear operator $\mu : \boldsymbol{\mathcal{P}} \to \overline{\mathbb{R}}$. We denote the set of all linear functional on $\boldsymbol{\mathcal{P}}$ by $\boldsymbol{L}(\boldsymbol{\mathcal{P}})$ (the algebraic dual of $\boldsymbol{\mathcal{P}}$). For a subset F of $\boldsymbol{\mathcal{P}}^2$ we define *polar* $\boldsymbol{F}^{\circ}$ as below

$$\boldsymbol{F}^{\circ} = \{\mu \in \boldsymbol{L}(\boldsymbol{\mathcal{P}}) : \mu(\boldsymbol{a}) \leq \mu(\boldsymbol{b}) + \boldsymbol{1}, \forall (\boldsymbol{a}, \boldsymbol{b}) \in \boldsymbol{F}\}.$$

Clearly $(\{(\boldsymbol{0}, \boldsymbol{0})\}^{\circ}) = \boldsymbol{L}(\boldsymbol{\mathcal{P}})$. A linear functional μ on $(\boldsymbol{\mathcal{P}}, \boldsymbol{\mathcal{U}})$ is (uniformly) continuous if there is $\boldsymbol{U} \in \boldsymbol{\mathcal{U}}$ such that $\mu \in \boldsymbol{U}^{\circ}$. The *dual cone* $\boldsymbol{\mathcal{P}}^*$ of a locally convex cone $(\boldsymbol{\mathcal{P}}, \boldsymbol{\mathcal{U}})$ consists of all continuous linear functionals on $\boldsymbol{\mathcal{P}}$ and is the union of all polars $\boldsymbol{U}^{\circ}$ of neighborhoods $\boldsymbol{U} \in \boldsymbol{\mathcal{U}}$. Foe example, the dual cone $\overline{\mathbb{R}}^*$ of $\overline{\mathbb{R}}$ consists of all nonnegative reals and the singular functional $\overline{\boldsymbol{0}}$ such that $\overline{\boldsymbol{0}}\,(x) = 0$ for all $\boldsymbol{x} \in \mathbb{R}$ and $\overline{\boldsymbol{0}}(+\infty) = +\infty$.

We shall say that a locally convex cone $(\mathcal{P},\mathcal{U})$ has the strict separation property if the following holds:
(*SP*) For all $a,b \in \mathcal{P}$ and $U \in \mathcal{U}$ such that $(a,b) \notin \rho U$ for some $\rho > 1$, there is a linear functional $\mu \in U^{\circ}$ such that $\mu(a) > \mu(b) + 1$ ([4], II, 2.12).

The locally convex cone $(\mathcal{P},\mathcal{U})$ is called separated whenever its symmetric topology is Hausdorff. Also, we say that $\mathcal{P}^*$ separates the points of $\mathcal{P}$, whenever for $a,b \in \mathcal{P}$ we have $a = b$ if and only if $\mu(a) = \mu(b)$ for all $\mu \in \mathcal{P}^*$. If $\mathcal{P}^*$ separates the points of $\mathcal{P}$, then $(\mathcal{P},\mathcal{U})$ is a separated locally convex cone. Indeed, if $a,b \in \mathcal{P}$ and $a \neq b$, then there is $\mu \in \mathcal{P}^*$ such that $\mu(a) \neq \mu(b)$. Since the symmetric topology of $(\overline{\mathbb{R}},\widetilde{\mathcal{V}})$ is Hausdorff, there is $\varepsilon > 0$ such that $\tilde{\varepsilon}(a)\tilde{\varepsilon} \cap \tilde{\varepsilon}(b)\tilde{\varepsilon} = \emptyset$. Now, by the continuity of μ, there is $U \in \mathcal{U}$ such that $(\mu \times \mu)(U) \subseteq \tilde{\varepsilon}$ and we have $U(a)U \cap U(b)U = \emptyset$.
The convex subset $U \subseteq \mathcal{P}$ is called uniformly convex if
(1) for each $a \in \mathcal{P}$, $(a,a) \in U$,
(2) for $\alpha,\beta > 0$, $(\alpha U) \circ (\beta U) \subseteq (\alpha + \beta)U$.
Also, we shall say that the subset F of $\mathcal{P}^2$ has the property (*CP*) if the following holds:
(*CP*) if $(a,b) \notin F$, then there is $\mu \in \mathcal{P}^*$ such that $\mu(a) > \mu(b) + 1$ and $\mu(c) \leq \mu(d) + 1$ for all $(c,d) \in F$.

In [4], a dual pair is defined as follows: A *dual pair* $(\mathcal{P},\mathcal{Q})$ consists of two cones $\mathcal{P}$ and $\mathcal{Q}$ with a bilinear mapping $(a,x) \to \langle a,x \rangle : \mathcal{P} \times \mathcal{Q} \to \overline{\mathbb{R}}$. Suppose that $(\mathcal{P},\mathcal{U})$ is a locally convex cone. We shall say that $F \subseteq \mathcal{P}^2$ is *u-bounded (uniformly-bounded)* if it is absorbed by each $U \in \mathcal{U}$. A subset A of $\mathcal{P}$ is called *bounded above(below)* whenever $A \times \{0\}(res. \{0\} \times A)$ is u-bounded (see [2]).

If $(\mathcal{P},\mathcal{Q})$ is a dual pair, then every $x \in \mathcal{Q}$ is a linear mapping on $\mathcal{P}$. We denote the coarsest convex quasiuniform structure on $\mathcal{P}$ that makes all $x \in \mathcal{Q}$ continuous by $\mathcal{U}_\sigma(\mathcal{P},\mathcal{Q})$. In fact, $(\mathcal{P},\mathcal{U}_\sigma(\mathcal{P},\mathcal{Q}))$ is the projective limit of $(\overline{\mathbb{R}},\widetilde{\mathcal{V}})$ by $x \in \mathcal{Q}$ as linear mappings on $\mathcal{P}$ (projective limits of locally convex cones were defined in [3]).
Let $(\mathcal{P},\mathcal{Q})$ be a dual pair. We shall say that a subset B of P is $\mathcal{U}_\sigma(\mathcal{P},\mathcal{Q})$-bounded below whenever it is bounded below in locally convex cone $(\mathcal{P},\mathcal{U}_\sigma(\mathcal{P},\mathcal{Q}))$.
Let $\mathcal{B}$ be a collection of $(\mathcal{P},\mathcal{U}_\sigma(\mathcal{P},\mathcal{Q}))$-bounded below subsets of $\mathcal{P}$ such that
(*a*) $\alpha B \in \mathcal{B}$ for all $B \in \mathcal{B}$ and $\alpha > 0$,
(*b*) For all X, $Y \in \mathcal{B}$ there is $Z \in \mathcal{B}$ such that $X \cup Y \subset Z$.
(*c*) $\mathcal{P}$ is spanned by $\bigcup_{B\in\mathcal{B}} B$.
For $B \in \mathcal{B}$ we set

$$U_B = \{(x,y) \in \mathcal{Q}^2 : \langle b,x \rangle \leq \langle b,y \rangle + 1, \text{for all } b \in B\}$$

and

$$\mathcal{U}_{\mathcal{B}}(\mathcal{P},\mathcal{Q}) = \{U_B : B \in \mathcal{B}\}.$$

It is proved in [4], page 37, that $\mathcal{U}_{\mathcal{B}}(\mathcal{P},\mathcal{Q})$ is a convex quasiuniform structure on $\mathcal{Q}$ and $(\mathcal{Q},\mathcal{U}_{\mathcal{B}}(\mathcal{P},\mathcal{Q}))$ is a locally convex cone. If $b \in B$ for $B \in \mathcal{B}$, then $b \in B \subseteq U_B^{\circ}$. Now $\mathcal{P} \subseteq (\mathcal{Q},(\mathcal{Q},\mathcal{U}_{\mathcal{B}}(\mathcal{P},\mathcal{Q}))^*$ by (*c*). This shows that $\mathcal{U}_{\mathcal{B}}(\mathcal{P},\mathcal{Q})$ is finer than $\mathcal{U}_{\sigma}(\mathcal{P},\mathcal{Q})$.

2. Differentiation and weak integration for cone-valued curves

We define the concept of differentiation for cone-valued curves. **Definition 2.1**.*Let $(\mathcal{P},\mathcal{U})$ be a separated locally convex cone. We consider on $\mathcal{P}$ and $[0,+\infty)$, the symmetric topology and the usual Euclidean topology, respectively. A $\mathcal{P}$-valued continuous map γ, defined on an interval $I = (a,b) \subseteq [0,\infty)$ for some $a,b \in [0,+\infty)$ and $a < b$, is called a C^0-curve. A C^0-curve $\gamma : I \to \mathcal{P}$ is called a C^1-curve, whenever*

(1) for every $t \in I$ there is $\gamma'(t) \in \mathcal{P}$ such that for each $U \in \mathcal{U}$ there is $\delta > 0$ such that $s < \delta$ implies that $(\gamma(t+s),\gamma(t)+s\gamma'(t)) \in sU$ and $(\gamma(t)+s\gamma'(t),\gamma(t+s)) \in sU$,

(2) the map $\gamma': I \to \mathcal{P} : t \to \gamma'(t)$ is continuous.

We set $\gamma^{(1)} = \gamma'$ and $\gamma^{(k)} = (\gamma^{(k-1)})'$. Recursively, for $k \in \mathbb{N}$, we call γ, C^k-curve if γ is a C^{k-1}-curve and $\gamma^{(k-1)}$ is C^1-curve. We note that for each $t \in I$, if $\gamma'(t)$ exists, then it is unique, since $(\mathcal{P},\mathcal{U})$ is a separated locally convex cone.

It is easy to see that if $\gamma : I \to \mathcal{P}$ and $\varphi : I \to \mathcal{P}$ are C^k-curve, then $\gamma + \varphi$ and $\alpha\gamma$, are *Ck*-curve for $\alpha \geq 0$. Therefore the collection of all $\mathcal{P}$-valued C^k-curves on the interval *I* is a cone denoted by $C^k(I,\mathcal{P})$. Obviously, we have $C^{k+1}(I,\mathcal{P}) \subseteq C^k(I,\mathcal{P})$ for all $k \in \mathbb{N}$.

The cone $C^\infty(I,\mathcal{P})$: Let $(\mathcal{P},\mathcal{U})$ be a separated locally convex cone. Then the collection of $\mathcal{P}$-valued infinitely differentiable functions on *I* is a cone denoted by $C^\infty(I,\mathcal{P})$. For $\gamma,\eta \in C^\infty(I,\mathcal{P})$, $U \in \mathcal{U}$ and $n \in \mathbb{N}$, we set $(\gamma,\eta) \in \vartheta_U^n$ if and only if $(\gamma^{(n)}(x),\eta^{(n)}(x)) \in U$ for all $x \in I$. The collection of all neighborhoods ϑ_U^n is a base for a convex quasiuniform structure on $C^\infty(I,\mathcal{P})$, denoted by $\mathcal{U}_\infty$. The collection of all bounded below functions in $C^\infty(I,\mathcal{P})$, with respect to $\mathcal{U}_\infty$ is a cone, denoted by $C_b^\infty(I,\mathcal{P})$. Therefore $(C_b^\infty(I,\mathcal{P}),\mathcal{U}_\infty)$ is a locally convex cone. For each $x \in I$, $n \in \mathbb{N}$ and $\mu \in \mathcal{P}^*$, we claim that the linear functional $\mu_x^n : C_b^\infty(I,\mathcal{P}) \to \overline{\mathbb{R}}$, $\mu_x^n(\gamma) = \mu(\gamma^{(n)}(x))$ is an element of $(C_b^\infty(I,\mathcal{P}),\mathcal{U}_\infty)^*$. Indeed, there is $U \in \mathcal{U}$ such that $\mu \in U^\circ$. Now, if $(\gamma,\eta) \in \vartheta_U^n$, then $(\gamma^{(n)}(x),\eta^{(n)}(x)) \in U$. Therefore $\mu(\gamma^{(n)}(x)) \leq \mu(\eta^{(n)}(x)) + 1$. This shows that $\mu_x^n \in (\vartheta_U^n)^\circ$. In the special case, for the linear functional $\mu_x^0 : C_b^\infty(I,\mathcal{P}) \to \overline{\mathbb{R}}$, $\mu_x^0(\gamma) = \mu(\gamma(x))$, we have $\mu_x^0 \in (\vartheta_U^0)^\circ$.

Example 2.1. *We consider the locally convex cone* $(\overline{\mathbb{R}}, \widetilde{\mathcal{V}})$. *The constant curve* $\boldsymbol{\gamma} : (\mathbf{0}, +\infty) \to \overline{\mathbb{R}}$, $\boldsymbol{\gamma}(\boldsymbol{t}) = +\infty$ *is a* $\boldsymbol{C}^{\infty}$*-curve. We have* $\boldsymbol{\gamma}^{(\boldsymbol{k})}(\boldsymbol{t}) = +\infty$ *for each* $\boldsymbol{k} \in \mathbb{N}$. *This function acts similar to the exponential function* $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{e}^{\boldsymbol{x}}$ *in the classical calculus. If we consider constant curve* $\boldsymbol{\gamma} : (\mathbf{0}, +\infty) \to \overline{\mathbb{R}}$, $\boldsymbol{\gamma}(\boldsymbol{s}) = \boldsymbol{b}$, *where* $\boldsymbol{b} \in \mathbb{R}$, *then we have* $\boldsymbol{\gamma}'(\boldsymbol{t}) = \mathbf{0}$.

Theorem 2.2. *Let* $(\boldsymbol{\mathcal{P}}, \boldsymbol{\mathcal{U}})$ *and* $(\boldsymbol{\mathcal{Q}}, \boldsymbol{\mathcal{W}})$ *be locally convex cones and* $\boldsymbol{T}: \boldsymbol{\mathcal{P}} \to \boldsymbol{\mathcal{Q}}$ *be a continuous linear mapping. If* $\boldsymbol{\gamma} : \boldsymbol{I} \to \boldsymbol{\mathcal{P}}$ *is a* $\boldsymbol{C}^{\mathbf{1}}$*-curve, then* $\boldsymbol{To\gamma}$ *is also a* $\boldsymbol{C}^{\mathbf{1}}$*-curve and* $(\boldsymbol{To\gamma})' = \boldsymbol{To\gamma}'$.

Proof. Let $\boldsymbol{W} \in \boldsymbol{\mathcal{W}}$ be arbitrary. There is $\boldsymbol{U} \in \boldsymbol{\mathcal{U}}$ such that $(\boldsymbol{a}, \boldsymbol{b}) \in \boldsymbol{U}$ implies that $(\boldsymbol{T}(\boldsymbol{a}), \boldsymbol{T}(\boldsymbol{b})) \in \boldsymbol{W}$ for each $\boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{\mathcal{P}}$. Now, since γ is $\boldsymbol{C}^{\mathbf{1}}$-curve, there is $\delta > 0$ such that for $s < \delta$, we have $(\boldsymbol{\gamma}(\boldsymbol{t} + \boldsymbol{s}), \boldsymbol{\gamma}(\boldsymbol{t}) + \boldsymbol{s}\boldsymbol{\gamma}'(\boldsymbol{t}) \in \boldsymbol{sU}$ and $(\boldsymbol{\gamma}(\boldsymbol{t}) + \boldsymbol{s}\boldsymbol{\gamma}'(\boldsymbol{t}), \boldsymbol{\gamma}(\boldsymbol{t} + \boldsymbol{s})) \in \boldsymbol{sU}$. Then $(\boldsymbol{T}(\boldsymbol{\gamma}(\boldsymbol{t} + \boldsymbol{s})), \boldsymbol{T}(\boldsymbol{\gamma}(\boldsymbol{t}) + \boldsymbol{s}\boldsymbol{\gamma}'(\boldsymbol{t}))) \in \boldsymbol{sW}$ and $(\boldsymbol{T}(\boldsymbol{\gamma}(\boldsymbol{t}) + \boldsymbol{s}\boldsymbol{\gamma}'(\boldsymbol{t})), \boldsymbol{T}(\boldsymbol{\gamma}(\boldsymbol{t} + \boldsymbol{s}))) \in \boldsymbol{sW}$. The linearity of T implies that $(\boldsymbol{T}(\boldsymbol{\gamma}(\boldsymbol{s} + \boldsymbol{t})), \boldsymbol{T}(\boldsymbol{\gamma}(\boldsymbol{t})) + \boldsymbol{sT}(\boldsymbol{\gamma}'(\boldsymbol{t})) \in \boldsymbol{sW}$ and $(\boldsymbol{T}(\boldsymbol{\gamma}(\boldsymbol{t})) + \boldsymbol{sT}(\boldsymbol{\gamma}'(\boldsymbol{t})), \boldsymbol{T}(\boldsymbol{\gamma}(\boldsymbol{s} + \boldsymbol{t}))) \in \boldsymbol{sW}$. Then we have $(\boldsymbol{To\gamma})' = \boldsymbol{To\gamma}'$.

Lemma 2.3. *Consider the locally convex cone* $(\overline{\mathbb{R}}, \widetilde{\mathcal{V}})$ *endowed with its symmetric topology. For every continuous curve* $\boldsymbol{\gamma}: \boldsymbol{I} \to \overline{\mathbb{R}}$, *we have* $\boldsymbol{\gamma}(\boldsymbol{x}) = \infty$ *for all* $\boldsymbol{x} \in \boldsymbol{I}$ *or* $\gamma(x) < \infty$ *for all* $x \in I$.

Proof. Let the assertion is false. Then there is $x \in I$ such that $\gamma(x) = \infty$ and $\gamma(y) < \infty$ for all $\boldsymbol{y} \in \boldsymbol{I} \backslash \{x\}$. We choose the sequence $(\boldsymbol{x}_{\boldsymbol{n}})_{\boldsymbol{n} \in \mathbb{N}}$ in $\boldsymbol{I} \backslash \{\boldsymbol{x}\}$ such that $\boldsymbol{x}_{\boldsymbol{n}} \to \boldsymbol{x}$. Now the continuity of $\boldsymbol{\gamma}$ yields that $(\boldsymbol{\gamma}(\boldsymbol{x}_{\boldsymbol{n}}))_{\boldsymbol{n} \in \mathbb{N}}$ is convergent to $\boldsymbol{\gamma}(\boldsymbol{x}) = +\infty$. Since $+\infty$ is an isolated point in the symmetric topology of $(\overline{\mathbb{R}}, \widetilde{\mathcal{V}})$, we realize that there is $\boldsymbol{m} \in \mathbb{N}$ such that $\boldsymbol{\gamma}(\boldsymbol{x}_{\boldsymbol{n}}) = +\infty$ for all $n \geq m$. This is a contradiction. Therefore, the assertion is true.

Now, by considering the Lemma 2.3, we introduce the integral $\int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{\gamma}(\boldsymbol{t})\boldsymbol{dt}$ for a continuous curve $\boldsymbol{\gamma} : \boldsymbol{I} \to \overline{\mathbb{R}}$: if $\gamma(x) < \infty$ for all $\boldsymbol{x} \in \boldsymbol{I}$, we mean $\int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{\gamma}(\boldsymbol{t})\boldsymbol{dt}$ the usual Riemann integral and when $\gamma(x) = +\infty$ for all $\boldsymbol{x} \in \boldsymbol{I}$, we set $\int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{\gamma}(\boldsymbol{t})\boldsymbol{dt} := +\infty$ for $\boldsymbol{a} \neq \boldsymbol{b}$ and $\int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{\gamma}(\boldsymbol{t})\boldsymbol{dt} = \mathbf{0}$ for $\boldsymbol{a} = \boldsymbol{b}$.

Definition 2.4. *Let* $(\boldsymbol{\mathcal{P}}, \boldsymbol{\mathcal{U}})$ *be a locally convex cone such that* $\boldsymbol{\mathcal{P}}^*$ *separates the points of* $\boldsymbol{\mathcal{P}}$, *and* $\boldsymbol{\gamma} : \boldsymbol{I} \to \boldsymbol{\mathcal{P}}$ *be a* $\boldsymbol{C}^{\mathbf{0}}$*-curve, and a, b* $\in I$ *with* $a \leq b$. *If there is* $\boldsymbol{p} \in \boldsymbol{\mathcal{P}}$ *such that for each* $\mu \in \boldsymbol{\mathcal{P}}^*$,

$$\mu(\boldsymbol{p}) = \int_{\boldsymbol{a}}^{\boldsymbol{b}} \mu(\boldsymbol{\gamma}(\boldsymbol{t}))\boldsymbol{dt},$$

$\boldsymbol{p} \in \boldsymbol{\mathcal{P}}$ *is called the weak integral of the* $\boldsymbol{\gamma}$, *from a to b, and denoted by*

$$\boldsymbol{p} := \int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{\gamma}(\boldsymbol{t})\boldsymbol{dt}.$$

In Definition 2.4, if $a = b$, then $\int_a^b \mu(\boldsymbol{\gamma(t)})\boldsymbol{dt}$=0=$\mu(0)$ for all $\mu \in \boldsymbol{\mathcal{P}}^*$. This shows that $\int_a^b \boldsymbol{\gamma(t)dt} = \mathbf{0}$ for $\boldsymbol{a, b} \in \boldsymbol{I}$, with $a = b$. Also, we note that the element $\boldsymbol{p} \in \boldsymbol{\mathcal{P}}$ in Definition 2.4 is uniquely determined if it exists, since $\boldsymbol{\mathcal{P}}^*$ separates the points of $\boldsymbol{\mathcal{P}}$.

Lemma 2.5. *Let $(\boldsymbol{\mathcal{P}}, \boldsymbol{\mathcal{U}})$ be a locally convex cone such that $\boldsymbol{\mathcal{P}}^*$ separates the points of $\boldsymbol{\mathcal{P}}$. Suppose $\gamma: I \to \boldsymbol{\mathcal{P}}$ and $\varphi: I \to \boldsymbol{\mathcal{P}}$ are continuous curves such that the weak integrals $\int_a^b \boldsymbol{\gamma(t)dt}$ and $\int_a^b \boldsymbol{\varphi(t)dt}$ exist for $\boldsymbol{a, b} \in I$. Then the followings hold.*

(a) $\int_a^b \big(\boldsymbol{\gamma(t)} + \boldsymbol{\varphi(t)}\big)\boldsymbol{dt} = \int_a^b \boldsymbol{\gamma(t)dt} + \int_a^b \boldsymbol{\varphi(t)dt}$,

(b) for $\boldsymbol{\alpha} \geq \mathbf{0}$, $\int_a^b \boldsymbol{\alpha\gamma(t)dt} = \boldsymbol{\alpha} \int_a^b \boldsymbol{\gamma(t)dt}$,

(c) if for $a, b, c \in I$, with $a < c < b$, the integrals $\int_a^c \boldsymbol{\gamma(t)dt}$ and $\int_c^b \boldsymbol{\gamma(t)dt}$ exist, then $\int_a^c \boldsymbol{\gamma(t)dt} + \int_c^b \boldsymbol{\gamma(t)dt} = \int_a^b \boldsymbol{\gamma(t)dt}$.

Proof. For (*a*), let $\boldsymbol{p} = \int_a^b \boldsymbol{\gamma(t)dt}$ and $\boldsymbol{q} = \int_a^b \boldsymbol{\varphi(t)dt}$. Then for each $\mu \in \boldsymbol{\mathcal{P}}^*$, we have $\mu(\boldsymbol{p}) = \int_a^b \mu\big(\boldsymbol{\gamma(t)}\big)\boldsymbol{dt}$ and $\mu(\boldsymbol{q}) = \int_a^b \mu(\boldsymbol{\varphi(t)})\boldsymbol{dt}$. This yields that for every $\mu \in \boldsymbol{\mathcal{P}}^*$,

$\int_a^b \mu(\boldsymbol{\gamma(t)} + \boldsymbol{\varphi(t)})\boldsymbol{dt} = \int_a^b \mu(\boldsymbol{\gamma(t)})\boldsymbol{dt} + \int_a^b \mu(\boldsymbol{\varphi(t)})\boldsymbol{dt} = \mu(\boldsymbol{p}) + \mu(\boldsymbol{q}) = \mu(\boldsymbol{p} + \boldsymbol{q})$.

Therefore $\int_a^b \big(\boldsymbol{\gamma(t)} + \boldsymbol{\varphi(t)}\big)\boldsymbol{dt} = \boldsymbol{p} + \boldsymbol{q}$, by definition of weak integral. For (*b*), let $\boldsymbol{p} = \int_a^b \boldsymbol{\gamma(t)dt}$ and $\boldsymbol{\alpha} \geq \mathbf{0}$. Then for each $\mu \in \boldsymbol{\mathcal{P}}^*$, $\mu(\boldsymbol{p}) = \int_a^b \mu(\boldsymbol{\gamma(t)})\boldsymbol{dt}$. Now, we have

$$\mu(\boldsymbol{\alpha p}) = \boldsymbol{\alpha}\mu(\boldsymbol{p}) = (\boldsymbol{\alpha}\mu)(\boldsymbol{p}) = \int_a^b (\boldsymbol{\alpha}\mu)\big(\boldsymbol{\gamma(t)}\big)\boldsymbol{dt} = \int_a^b \mu\big(\boldsymbol{\alpha\gamma(t)}\big)\boldsymbol{dt}.$$

Now, by the definition of weak integral, we have $\int_a^b \boldsymbol{\alpha\gamma(t)dt} = \boldsymbol{\alpha} \int_a^b \boldsymbol{\gamma(t)dt}$.

For (*c*), let $\boldsymbol{m} = \int_a^c \boldsymbol{\gamma(t)dt}, \boldsymbol{n} = \int_c^b \boldsymbol{\gamma(t)dt}$ and $\boldsymbol{p} = \int_a^b \boldsymbol{\gamma(t)dt}$. Then for $\mu \in \boldsymbol{\mathcal{P}}^*$,

$$\mu(\boldsymbol{m} + \boldsymbol{n}) = \mu(\boldsymbol{m}) + \mu(\boldsymbol{n}) = \int_a^c \mu(\boldsymbol{\gamma(t)})\boldsymbol{dt} + \int_c^b \mu(\boldsymbol{\gamma(t)})\ \boldsymbol{dt} = \int_a^b \mu(\boldsymbol{\gamma(t)})\boldsymbol{dt} = \mu(\boldsymbol{p}).$$

Since $\boldsymbol{\mathcal{P}}^*$ separates the points of $\boldsymbol{\mathcal{P}}$, we have $m + n = p$.

Theorem 2.6 (The First Fundamental Theorem of Calculus).*Let $(\boldsymbol{\mathcal{P}}, \boldsymbol{\mathcal{U}})$ be a locally convex cone such that $\boldsymbol{\mathcal{P}}^*$ separates the points of $\boldsymbol{\mathcal{P}}$, $\gamma: I \to \boldsymbol{\mathcal{P}}$ be a continuous curve and $\boldsymbol{a} \in \boldsymbol{I}$. Also assume that the weak integral $\boldsymbol{\varphi(t)} = \int_a^t \boldsymbol{\gamma(m)dm}$ exists for each $\boldsymbol{t} \in \boldsymbol{I}$. Then $\boldsymbol{\varphi} : \boldsymbol{I} \to \boldsymbol{\mathcal{P}}$ is a $\boldsymbol{C^1}$-curve and $\varphi' = \gamma$.*

Proof. Since the weak integral $\boldsymbol{\varphi(t)} = \int_{\boldsymbol{a}}^{\boldsymbol{t}} \boldsymbol{\gamma(t)dt}$exists, we have $\mu(\boldsymbol{\varphi(t)}) = \int_{\boldsymbol{a}}^{\boldsymbol{t}} \mu(\boldsymbol{\gamma(m)})\boldsymbol{dm}$, for all $\mu \in \mathcal{P}^*$. For $\mu \in \mathcal{P}^*$, if $\mu(\boldsymbol{\gamma(m)}) < \infty$ for all $\boldsymbol{m} \in \boldsymbol{I}$, the usual First Fundamental Theorem of Calculus yields that $(\mu(\boldsymbol{\varphi(t)})' = \mu(\boldsymbol{\gamma(t)})$. Then we have $\mu(\boldsymbol{\varphi'(t)} = \mu(\boldsymbol{\gamma(t)})$ by Lemma 2.2. Now, since $\mathcal{P}^*$separates the points of $\mathcal{P}$, we have $\boldsymbol{\varphi'(t) = \gamma(t)}$. If $\mu(\gamma(t)) = \infty$ for all $\boldsymbol{t} \in \boldsymbol{I}$, Example 2.1 shows that the relation $(\mu(\boldsymbol{\varphi(t)})' = \mu(\boldsymbol{\gamma(t)})$ is true. Then the assertion holds.

Theorem 2.7 (The Second Fundamental Theorem of Calculus). *Let*$(\mathcal{P}, \mathcal{U})$ *be a separated locally convex cone such that* $\mathcal{P}^*$ *separates the points of* $\mathcal{P}$, *and* $\boldsymbol{\gamma}: \boldsymbol{I} \to \mathcal{P}$ *be a* $\boldsymbol{C^1}$*-curve, and* $\boldsymbol{a, b} \in \boldsymbol{I}$. *Then* $\boldsymbol{\gamma(b) = \gamma(a) + p}$, *where* $\boldsymbol{p} = \int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{\gamma'(t)dt}$.
Proof. Let $\mu \in \mathcal{P}^*$. By Lemma 2.2, $\mu\boldsymbol{o\gamma} : \boldsymbol{I} \to \overline{\mathbb{R}}$ is a $\boldsymbol{C^1}$-curve and we have $(\mu o\gamma)' = \mu \boldsymbol{o}\gamma'$. By Lemma 2.3, $(\mu\boldsymbol{o\gamma})'(\boldsymbol{x}) = \infty$ for all $\boldsymbol{x} \in \boldsymbol{I}$ or $(\mu\boldsymbol{o}\gamma)'(x) < \infty$ for all $\boldsymbol{x} \in \boldsymbol{I}$. Let $(\mu\boldsymbol{o}\gamma)'(x) < \infty$ for all $\boldsymbol{x} \in \boldsymbol{I}$. Now the Classical Fundamental Theorem of Calculus yields that

$$\int_{\boldsymbol{a}}^{\boldsymbol{b}} \mu\boldsymbol{o\gamma'(t)dt} = \int_{\boldsymbol{a}}^{\boldsymbol{b}} (\mu\boldsymbol{o\gamma})'\boldsymbol{(t)dt} = \mu(\boldsymbol{\gamma(b)}) - \mu(\boldsymbol{\gamma(a)}).$$

This shows that

$$\mu(\boldsymbol{p} + \boldsymbol{\gamma(a)}) = \mu(\boldsymbol{p}) + \mu(\boldsymbol{\gamma(a)}) = \int_{\boldsymbol{a}}^{\boldsymbol{b}} \mu(\boldsymbol{\gamma'(t)})\boldsymbol{dt} + \mu(\boldsymbol{\gamma(a)}) = \mu(\boldsymbol{\gamma(b)}).$$

Now, since $\mathcal{P}^*$ separates the points of $\mathcal{P}$, we conclude that $p + \gamma(a) = \gamma(b)$. If $(\mu\boldsymbol{o\gamma})'(\boldsymbol{x}) = \infty$ for all $\boldsymbol{x} \in \boldsymbol{I}$, then $(\mu\boldsymbol{o\gamma})(\boldsymbol{x}) = \infty$ for all $\boldsymbol{x} \in \boldsymbol{I}$ by Example 2.1. Then the relation $\mu(\boldsymbol{p}) + \mu(\boldsymbol{\gamma(a)}) = \mu(\boldsymbol{\gamma(b)})$ holds too in this case. Therefore $p + \gamma(a) = \gamma(b)$, since $\mathcal{P}^*$ separates the points of $\mathcal{P}$.

Proposition 2.8.*Let* $(\mathcal{P}, \mathcal{U})$*be a locally convex cone such that* $\mathcal{P}^*$ *separates the points of* $\mathcal{P}$, $\gamma : I \to \mathcal{P}$ *be a continuous curve and a,* $\boldsymbol{b} \in \boldsymbol{I}$ *with* $a \leq b$. *Suppose that the weak integral* $\int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{\gamma(t)dt}$ *exists, and V is a uniformly convex subset of* $\mathcal{P}^2$ *with* (*CP*) *such that* $\gamma(I) \subseteq V(0)V = \{m \in \mathcal{P} : (m,0) \in V, (0, m) \in V\}$. *Then*

$$\int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{\gamma(t)dt} \in (\boldsymbol{b} - \boldsymbol{a})\boldsymbol{V}(\boldsymbol{0})\boldsymbol{V}.$$

Proof. We remember that $\boldsymbol{V(0)} = \{\boldsymbol{m} : (\boldsymbol{m}, \boldsymbol{0}) \in \boldsymbol{V}\}$, $(0)V = \{m : (0, m) \in V\}$ and $\boldsymbol{V(0)V} = \boldsymbol{V(0)} \cap \boldsymbol{V(0)}$. If $\boldsymbol{a = b}$, then $\int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{\gamma(t)dt} \in (\boldsymbol{b} - \boldsymbol{a})\boldsymbol{V(0)V}$. Suppose $\boldsymbol{a} \neq \boldsymbol{b}$ and $\boldsymbol{p} = \int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{\gamma(t)dt} \notin (\boldsymbol{b} - \boldsymbol{a})\boldsymbol{V(0)V}$. Then $\boldsymbol{p} \notin (\boldsymbol{b} - \boldsymbol{a})\boldsymbol{V(0)}$ or $\boldsymbol{p} \notin (\boldsymbol{b} - \boldsymbol{a})(\boldsymbol{0})\boldsymbol{V}$. Let $\boldsymbol{p} \notin (\boldsymbol{b} - \boldsymbol{a})\boldsymbol{V(0)}$. Then $(\boldsymbol{p}, \boldsymbol{0}) \notin (\boldsymbol{b} - \boldsymbol{a})\boldsymbol{V}$. Since $\boldsymbol{V}$ has (*CP*), there is $\mu \in \mathcal{P}^*$such that $\mu(\boldsymbol{p}) > \boldsymbol{b} - \boldsymbol{a}$and $\mu(m) \leq \mu(n) + 1$ for all $(m, n) \in V$. This shows that $\mu(\boldsymbol{\gamma(t)}) \leq \mu(\boldsymbol{0}) + \boldsymbol{1}$ for all $\boldsymbol{t} \in \boldsymbol{I}$. Then for each $\boldsymbol{t} \in \boldsymbol{I}$, $\mu(\boldsymbol{\gamma(t)}) \leq \boldsymbol{1}$. Therefore, we have

$$\mu(\boldsymbol{p}) = \int_a^b \mu(\boldsymbol{\gamma(t)})\boldsymbol{dt} \;\le\; \int_a^b \boldsymbol{dt} \;= \boldsymbol{b} - \boldsymbol{a}.$$

This contradiction yields that

$$\int_a^b \boldsymbol{\gamma(t)dt} \in (\boldsymbol{b}-\boldsymbol{a})\boldsymbol{V}\,(\boldsymbol{0}).$$

We can prove that $\int_a^b \boldsymbol{\gamma(t)dt} \in (\boldsymbol{b}-\boldsymbol{a})(\boldsymbol{0})\boldsymbol{V}$in a similar way. Therefore $\int_a^b \boldsymbol{\gamma(t)dt} \in (\boldsymbol{b}-\boldsymbol{a})\boldsymbol{V}(\boldsymbol{0})\boldsymbol{V}$.

Example 2.2.*Let $\boldsymbol{X}$ be a topological space, and let $\boldsymbol{\mathcal{F}}(\boldsymbol{X},\overline{\mathbb{R}})$be the cone of all $\overline{\mathbb{R}}$-valued lower semicontinuous functions on X, where $\overline{\mathbb{R}}$is endowed with the usual, that is, the one-point compactification topology. The cone $\boldsymbol{\mathcal{F}}(\boldsymbol{X},\overline{\mathbb{R}})$ is considered endowed with the pointwise operations. For $\rho>0$, we set $\widetilde{\boldsymbol{\rho}} = \{(\boldsymbol{f},\boldsymbol{g}) \in \boldsymbol{\mathcal{P}}^2 : \boldsymbol{f}(\boldsymbol{x}) \le \boldsymbol{g}(\boldsymbol{x}) + \boldsymbol{\rho}\}$. The collection $\boldsymbol{\mathcal{W}} = \{\widetilde{\boldsymbol{\rho}} : \boldsymbol{\rho} > \boldsymbol{0}\}$ is a convex quasiuniform structure on $\boldsymbol{\mathcal{F}}(\boldsymbol{X},\overline{\mathbb{R}})$. We denote the subcone of all bound below elements of $\boldsymbol{\mathcal{F}}(\boldsymbol{X},\overline{\mathbb{R}})$ with respect to $\boldsymbol{\mathcal{W}}$ by $\boldsymbol{\mathcal{P}}$. Then $(\boldsymbol{\mathcal{P}},\boldsymbol{\mathcal{W}})$is a locally convex cone. If $\boldsymbol{x} \in \boldsymbol{X}$ and $\mu \in \overline{\mathbb{R}}^*$, then $\mu_{\boldsymbol{x}} : \boldsymbol{\mathcal{P}} \to \overline{\mathbb{R}}$, defined by $\mu_{\boldsymbol{x}}(\boldsymbol{f}) = \mu\big(\boldsymbol{f}(\boldsymbol{x})\big)$ is a continuous functional on $\boldsymbol{\mathcal{P}}$. The dual cone $\boldsymbol{\mathcal{P}}^*$separates the points of $\boldsymbol{\mathcal{P}}$. Consider the continuous curve$\boldsymbol{\gamma} : [\boldsymbol{0},+\infty) \to \boldsymbol{\mathcal{P}}$, $\boldsymbol{\gamma}(\boldsymbol{t}) = \boldsymbol{f_t}$, where $\boldsymbol{f_t} : \boldsymbol{X} \to \overline{\mathbb{R}}$, $\boldsymbol{f_t}(\boldsymbol{x}) = \boldsymbol{t}$ for each $x \in X$. We claim that $\boldsymbol{\gamma}'(\boldsymbol{t}) = \boldsymbol{f_1}$. Let $\rho>0$. Then we have $\boldsymbol{t}+\boldsymbol{s} \le \boldsymbol{t}+\boldsymbol{s}+\boldsymbol{s\rho}$ for each $\boldsymbol{s} \in \boldsymbol{I}$. This shows that $\boldsymbol{f_{t+s}}(\boldsymbol{x}) \le \boldsymbol{f_t}(\boldsymbol{x}) + \boldsymbol{s}\boldsymbol{f_1}(\boldsymbol{x}) + \boldsymbol{s\rho}$ and $\boldsymbol{f_t}(\boldsymbol{x}) + \boldsymbol{s}\boldsymbol{f_1}(\boldsymbol{x}) \le \boldsymbol{f_{t+s}}(\boldsymbol{x}) + \boldsymbol{s\rho}$. This yields that $(\boldsymbol{f_{t+s}},\boldsymbol{f_t} + \boldsymbol{s}\boldsymbol{f_1}) \in \boldsymbol{s}\widetilde{\boldsymbol{\rho}}$ and $(\boldsymbol{f_t} + \boldsymbol{s}\boldsymbol{f_1},\boldsymbol{f_{t+s}}) \in \boldsymbol{s}\widetilde{\boldsymbol{\rho}}$.*
*Therefore $\boldsymbol{\gamma}'(\boldsymbol{t}) = \boldsymbol{f_1}$. If $\boldsymbol{\varphi}(\boldsymbol{t}) = \boldsymbol{f_{t^2}}$, we can prove that $\boldsymbol{\varphi}'(\boldsymbol{t}) = \boldsymbol{f_{2t}}$ and by theinduction **if** $\boldsymbol{\varphi}(\boldsymbol{t}) = \boldsymbol{f_{t^n}}$, for $\boldsymbol{n} \in \mathbb{N}$, then $\boldsymbol{\varphi}'(\boldsymbol{t}) = \boldsymbol{f_{nt^{n-1}}}$. Now, we conclude that$\int_0^t \boldsymbol{\varphi}'(\boldsymbol{s})\boldsymbol{ds} = \boldsymbol{\varphi}(\boldsymbol{t})$.*

Remark 2.9. Let $\widehat{\boldsymbol{\mathcal{P}}}$ be a locally convex cone such that $\boldsymbol{\mathcal{P}}^*$ separates the points of $\boldsymbol{\mathcal{P}}$. We claim that for the constant curve $\boldsymbol{\gamma}(\mathbf{t}) = \mathbf{c}$, where $\mathbf{c} \in \boldsymbol{\mathcal{P}}$, we have $\mathbf{ac} + \int_{\mathbf{a}}^{\mathbf{b}} \boldsymbol{\gamma}(\mathbf{t})\mathbf{dt} = \mathbf{bc}$ for a, b $\in[0,\infty)$ with $\mathbf{a} \neq \mathbf{b}$ (in the case a = b, the relation is obvious). Indeed, if for $\mu \in \boldsymbol{\mathcal{P}}^*$, $\mu(\boldsymbol{\gamma}(\mathbf{t}) = \mu(\mathbf{c}) = \infty$, then $\int_{\mathbf{a}}^{\mathbf{b}} \mu(\boldsymbol{\gamma}(\mathbf{t}))\mathbf{dt} = \infty$ and $\mathbf{b}\mu(\mathbf{c}) = \infty$. Therefore $\mathbf{ac} + \int_{\mathbf{a}}^{\mathbf{b}} \boldsymbol{\gamma}(\mathbf{t})\mathbf{dt} = \mathbf{bc}$ is true. If for $\mu \in \boldsymbol{\mathcal{P}}^*$, $\mu(\boldsymbol{\gamma}(\mathbf{t}) = \mu(\mathbf{c}) < \infty$, then by the classical integration, we have $\int_{\mathbf{a}}^{\mathbf{b}} \mu(\mathbf{c})\mathbf{dt} = (\mathbf{b} - \mathbf{a})\mu(\mathbf{c}) = \mu((\mathbf{b}-\mathbf{a})\mathbf{c})$. Therefore for each $\mu \in \boldsymbol{\mathcal{P}}^*$, $\mu(\mathbf{ac}) + \int_{\mathbf{a}}^{\mathbf{b}} \mu(\boldsymbol{\gamma}(\mathbf{t}))\mathbf{dt} = \mu(\mathbf{bc})$. Since $\boldsymbol{\mathcal{P}}^*$ separates the points of $\boldsymbol{\mathcal{P}}$, we conclude that $\mathbf{ac} + \int_{\mathbf{a}}^{\mathbf{b}} \boldsymbol{\gamma}(\mathbf{t})\mathbf{dt} = \mathbf{bc}$.

The concept of completion for locally convex cones has been established in [1]. It is proved in [4] that if $(\boldsymbol{\mathcal{P}},\boldsymbol{\mathcal{U}})$ is a locally convex cone with (*SP*), then for $\boldsymbol{\mathcal{B}} = \{\boldsymbol{U}^{\circ} : \boldsymbol{U} \in \boldsymbol{\mathcal{U}}\}$, the convex quasiuniform structure $\boldsymbol{\mathcal{U}_{\mathcal{B}}}(\boldsymbol{\mathcal{P}},\boldsymbol{\mathcal{P}}^*)$ and $\boldsymbol{\mathcal{U}}$ are equivalent. For a locally convex cone $(\mathcal{P},\mathcal{U})$ with (*SP*), the completion $\widehat{\mathcal{P}}$ of $\mathcal{P}$, is the subcone $\bigcap_{U\in\mathcal{U}}(\mathcal{P} + (\{0\} \times U^{\circ})^{\circ})$ of $L(\mathcal{P}^*)$ endowed with the convex convex quasiuniform structure $\widehat{\mathcal{U}} = \mathcal{U}_{\mathcal{B}}(\widehat{\mathcal{P}},\mathcal{P}^*)$, where $\mathcal{B} = \{U^{\circ}: U \in \mathcal{U}\}$. For details see [1].

Theorem 2.10.*Let* $(\mathcal{P},\mathcal{U})$*be a locally convex cone such that* $\mathcal{P}^*$ *separates the points of* $\mathcal{P}$*, and* $\gamma : [0,+\infty) \to \mathcal{P}$*be a continuous curve. Then there is a unique differentiable curve* $\varphi : [0,+\infty) \to \widehat{\mathcal{P}}$*, where* $\widehat{\mathcal{P}}$ *is the completion of* $\widehat{\mathcal{P}}$*such that* $\varphi'(t) = \gamma(t)$ *for each* $t \in [0,+\infty)$*and* $\varphi(0) = 0$.

Proof. Firstly, we show uniqueness. Let $\varphi : [0,+\infty) \to \widehat{\mathcal{P}}$ be a curve with derivative γ and $\varphi(0) = 0$. For every $\mu \in \mathcal{P}^*$ the composition $\mu o \varphi$ is an antiderivative of $\mu o \gamma$ with initial value 0, so it is uniquely determined, and since $\mathcal{P}^*$separates the points of $\mathcal{P}$, φ is also uniquely determined. Now, we show the existence of φ. For each $t \in [0,+\infty)$, we define $\varphi_t : (\mathcal{P}^*,\mathcal{U}_{\mathcal{B}}(\mathcal{P}^*,\mathcal{P})) \to (\overline{\mathbb{R}},\widetilde{\mathcal{V}})$, $\varphi_t(\mu) = \int_0^t (\mu o \gamma)(s)ds$. For each $t \in [0,+\infty)$, φ_t is a linear functional on $\mathcal{P}^*$. Then $\varphi_t(\in L(\mathcal{P}^*)$ for all $t \in [0,+\infty)$. Now, we define $\varphi: [0,+\infty) \to L(\mathcal{P}^*)$, $\varphi(t) = \varphi_t$. We claim that $\varphi' = \gamma$. Let $\mu \in \mathcal{P}^*$. Then we have $\mu o\gamma(t) = \infty$ or $\mu o\gamma(t) < \infty$ for each $t \in [0,\infty)$ by the Example 2.1. If for each $t \in [0,\infty)$, $\mu o\gamma(t) = \infty$, then for $t > 0$, we have $\varphi_t(\mu) = \int_0^t \mu o\gamma(s)ds = \int_0^t \infty ds = \infty$. This shows that $\varphi(t) = \infty$. Therefore the assertion holds in this case by the Example 2.1. Now, let for each $t \in [0,\infty)$, $\mu o\gamma(t) < \infty$. We identify $a \in \mathcal{P}$ with the linear mapping $\psi_a : \mathcal{P}^* \to \overline{\mathbb{R}}$, $\psi_a(\mu) = \mu(a)$. Let $U \in \mathcal{U}$, $t \in [0,\infty)$ and $s > 0$. There is $\delta_t > 0$ such that $\gamma(m) \in (\delta_t U)(\gamma(t))(\delta_t U)$, for each $m \in [t,t+s]$ by the continuity of γ. We set $V = \delta_t U$. Clearly, we have $U_{V^{\circ}} \in \mathcal{U}_{\mathcal{B}}(\mathcal{P}^*,\mathcal{P})$.Now, for $t \in [0,\infty)$, $s \leq \delta$and $\mu \in V^{\circ}$we have

$$\mu(\gamma(m)) \leq \mu(\gamma(t)) + 1,$$

for all $m \in [t,t + s]$. Therefore

$$\int_t^{t+s} \mu(\gamma(m))dm \leq \int_t^{t+s} (\mu(\gamma(t)) + 1)dm.$$

This shows that

$$\int_t^{t+s} (\mu o\gamma)(m)dm \leq s\mu(\gamma(t)) + s.$$

Then

$$\int_0^{t+s} (\mu o\gamma)(m)dm - \int_0^{t} (\mu o\gamma)(m)dm \leq s\mu(\gamma(t)) + s.$$

This shows that

$$\int_0^{t+s} (\mu o\gamma)(m)dm \leq \int_0^{t} (\mu o\gamma)(m)dm + s\mu\big(\gamma(t)\big) + s.$$

Therefore

$$\varphi_{t+s}(\mu) + s(\mu) \leq \varphi_t(\mu) + s\mu(\gamma(t)) + s.$$

This yields that

$$\big(\boldsymbol{\varphi}(\boldsymbol{t}+\boldsymbol{s}), \boldsymbol{\varphi}(\boldsymbol{t})+\boldsymbol{s}\boldsymbol{\gamma}(\boldsymbol{t})\big) \in \boldsymbol{s}\boldsymbol{U}_{\boldsymbol{V}^{\circ}}\ \circ.$$

In a similar way we can prove that

$$\big(\boldsymbol{\varphi}(\boldsymbol{t})+\boldsymbol{s}\boldsymbol{\gamma}(\boldsymbol{t}), \boldsymbol{\varphi}(\boldsymbol{t}+\boldsymbol{s})\big) \in \boldsymbol{s}\boldsymbol{U}_{\boldsymbol{V}^{\circ}}.$$

Then we have $\varphi' = \gamma$ by the definition of differentiation. It remains to show that $\boldsymbol{\varphi_t} \in \widehat{\boldsymbol{\mathcal{P}}}$ for all $\boldsymbol{t} \in [\boldsymbol{0},\infty)$. If $\boldsymbol{t}=\boldsymbol{0}$, it is clear. For each $\boldsymbol{t} \in (\boldsymbol{0},\infty)$ there is $\boldsymbol{\delta_t}$ such that $\boldsymbol{\gamma}([\boldsymbol{0},\boldsymbol{t}]) \subseteq (\boldsymbol{\delta_t U})(\boldsymbol{0})(\boldsymbol{\delta_t U})$ by the continuity of γ. We claim that $\boldsymbol{\varphi_t} \in (\{\boldsymbol{0}\} \times (\boldsymbol{t\delta_t U})^{\circ})^{\circ}$. Let $\mu \in (\boldsymbol{t\delta_t U})^{\circ}$. Then $\mu \in \frac{\boldsymbol{1}}{\boldsymbol{t\delta_t}}\boldsymbol{U}^{\circ}$. For $\boldsymbol{s} \in [\boldsymbol{0},\boldsymbol{t}]$, since, $(\boldsymbol{0}, \frac{\boldsymbol{1}}{\boldsymbol{\delta_t}}\boldsymbol{\gamma}(\boldsymbol{s})) \in \boldsymbol{U}$, we have $(\boldsymbol{t\delta_t}\mu)(\frac{\boldsymbol{1}}{\boldsymbol{\delta_t}}\boldsymbol{\gamma}(\boldsymbol{s})) \geq -\boldsymbol{1}$. Then for $\boldsymbol{s} \in [\boldsymbol{0},\boldsymbol{t}]$, $\mu(\boldsymbol{\gamma}(\boldsymbol{s})) \geq -\frac{\boldsymbol{1}}{\boldsymbol{t}}$. This yields that

$$\boldsymbol{\varphi_t}(\mu) = \int_{\boldsymbol{0}}^{\boldsymbol{t}} \mu \boldsymbol{o}\boldsymbol{\gamma}(\boldsymbol{s})\boldsymbol{ds} \geq \int_{\boldsymbol{0}}^{\boldsymbol{t}} -\frac{\boldsymbol{1}}{\boldsymbol{t}}\,\boldsymbol{ds} = -\boldsymbol{1}.$$

Then $\boldsymbol{0} \leq \boldsymbol{\varphi_t}(\mu)+\boldsymbol{1}$. In fact, for $(\{\boldsymbol{0}\},\mu) \in \boldsymbol{0} \times (\boldsymbol{t\delta_t U})$ we have $\boldsymbol{\varphi_t}(\boldsymbol{0}) = \boldsymbol{0} \leq \boldsymbol{\varphi_t}(\mu)+\boldsymbol{1}$. This shows that $\boldsymbol{\varphi_t} \in (\{\boldsymbol{0}\} \times (\boldsymbol{t\delta_t U})^{\circ})^{\circ}$. Then $\boldsymbol{\varphi_t} \in \widehat{\boldsymbol{\mathcal{P}}}$ for all $\boldsymbol{t} \in [\boldsymbol{0},\infty)$.

Corollary 2.11.*Let $(\boldsymbol{\mathcal{P}},\boldsymbol{\mathcal{U}})$be an upper complete locally convex cone such that $\boldsymbol{\mathcal{P}}^*$separates the points of $\boldsymbol{\mathcal{P}}$, and $\boldsymbol{\gamma}:[\boldsymbol{0},\infty)\to\boldsymbol{\mathcal{P}}$ be a continuous curve. Then the weak integral $\int_{\boldsymbol{a}}^{\boldsymbol{b}}\boldsymbol{\gamma}(\boldsymbol{t})\boldsymbol{dt}$ exists. In fact, if $(\boldsymbol{\mathcal{P}},\boldsymbol{\mathcal{U}})$ is upper complete, then$\big(\widehat{\boldsymbol{\mathcal{P}}},\widehat{\boldsymbol{\mathcal{U}}}\big) = (\boldsymbol{\mathcal{P}},\boldsymbol{\mathcal{U}})$ and therefore $\boldsymbol{\varphi_t} \in \boldsymbol{\mathcal{P}}$ for each $\boldsymbol{t} \in [\boldsymbol{a},\boldsymbol{b}]$, by Theorem 2.10.Also, we have $\int_{\boldsymbol{a}}^{\boldsymbol{b}}\boldsymbol{\gamma}(\boldsymbol{t})\boldsymbol{dt}+\boldsymbol{\varphi}(\boldsymbol{a}) = \boldsymbol{\varphi}(\boldsymbol{b})$ by Theorem 2.7.*

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