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PERIODICAL SOLUTIONS OF POISSON MULTI-TIME LINEAR EQUATIONS

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In this paper, we study the differential equation

$$\Delta x - q(t)\cdot x = 0,\quad t\in[0,T^1]\times\ldots\times[0,T^p]\subset R^p,$$

Where $p=1$ *is Hill's equation.*

We will consider the periodic solutions of this equation. We will study the minimum of the action that produces Poisson multi-time linear equation. Using minimizing sequences, we show that the action has a minimum periodical point x *which is solution for Poisson multi-time linear equation.*

Keywords: Poisson multi-time equation, periodical extremals, Euler-Lagrange equations

1. Introduction

The differential equations $x''-q(t)x=0$, with periodic boundary conditions $x(0)=x(T)$, $x'(0)=x'(T)$, was first examined by G.W. Hill in [22].

This differential equation of this work extends the case of Hill's equation. In the paper we will note by $W_T^{1,2}$ the Sobolev spaces of the functions $x\in L^2[T_0,R]$ which have the weak derivative $\frac{\partial x}{\partial t}\in L^2[T_0,R]$ [5], where $T_0=[0,T^1]\times\ldots\times[0,T^p]\subset R^p$ [17,12].

The weak derivatives are defined using the space C_T^∞ of all indefinitely differentiable multiple T-periodic functions from R^p into R.

We denote by H_T^1 the Hilbert space $W_T^{1,2}$. The geometry on H_T^1 is realized by the scalar product

$$\langle x,y\rangle=\int_{T_0}\left(x(t)y(t)+\delta^{\alpha\beta}\frac{\partial x}{\partial t^\alpha}(t)\frac{\partial y}{\partial t^\beta}(t)\right)dt^1\wedge\ldots\wedge dt^p,$$

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and the associated Euclidian norm $\|.\|$. These are induced by the scalar product (Riemannian metric)

$$G = \begin{pmatrix} 1 & 0 \\ 0 & \delta^{\alpha\beta} \end{pmatrix} \text{ on } R^{1+p}.$$

Let $t = \left(t^1,...,t^p\right)$ be a generic point in R^p.

The opposite faces of the parallelepiped T_0 can be described by the equations $S_\alpha^- : t^\alpha = 0$, $S_\alpha^+ : t^\alpha = T^\alpha$ for each $\alpha = 1,...p$.

We denote

$$\|x\|_{L^2} = \int_{T_0} x^2(t) dt^1 \wedge ... \wedge dt^p,$$

$$\left\|\frac{\partial x}{\partial t}\right\|_{L^2} = \int_{T_0} \left(\delta^{\alpha\beta} \frac{\partial x}{\partial t^\alpha}(t) \frac{\partial x}{\partial t^\beta}(t)\right) dt^1 \wedge ... \wedge dt^p,$$

The Poisson multi-time linear equation with periodic boundary conditions is

$$\Delta x - q(t) \cdot x = 0 \qquad (1)$$

$$x\big|_{S_\alpha^-} = x\big|_{S_\alpha^+}, \quad \frac{\partial x}{\partial t}\bigg|_{S_\alpha^-} = \frac{\partial x}{\partial t}\bigg|_{S_\alpha^+}, \quad \alpha = 1,...p.$$

2. Action that produces Poisson multi -time linear equation

We consider the multi-time variable $t = \left(t^1,...,t^p\right) \in R^p$, the functions

$$x : R^p \to R, \quad \left(t^1,...,t^p\right) \to x\left(t^1,...,t^p\right),$$

and we denote $x_\alpha = \dfrac{\partial x}{\partial t^\alpha}$, $\alpha = 1,2,..., p$.

The Lagrange functions

$$L : R^{p+1+p} \to R, \quad \left(t, x, \frac{\partial x}{\partial t}\right) \to L\left(t, x, \frac{\partial x}{\partial t}\right)$$

give the Euler-Lagrange equations

$$\frac{\partial}{\partial t^{\alpha}}\frac{\partial L}{\partial x_{\alpha}}=\frac{\partial L}{\partial x},\quad \alpha=1,2,...,p\ \ [17]$$

(second order PDE_s equation on the *n*-dimensional space).

We consider the Lagrangian

$$L:T_0\times R\times R^p\to R,\ \left(t,x,\frac{\partial x}{\partial t}\right)\to L\left(t,x,\frac{\partial x}{\partial t}\right)$$

$$L\left(t,x(t),\frac{\partial x}{\partial t}(t)\right)=\left|\frac{\partial x}{\partial t}(t)\right|^2+q(t)x^2(t)$$

where $q:T_0\to R$.

The function that realizes the minimum of the action

$$\varphi(x)=\int_{T_0}L\left(t,x(t),\frac{\partial x}{\partial t}(t)\right)dt^1\wedge...\wedge dt^p$$

verifies a PDE_s (1).

3. Continuously differentiable action

The next theorem establishes some conditions in which the action

$$\varphi:W_T^{1,2}\to R,\ \ \varphi(x)=\int_{T_0}\left(\left|\frac{\partial x}{\partial t}(t)\right|^2+q(t)x^2(t)\right)dt^1\wedge...\wedge dt^p.$$

is continuously differentiable.

Theorem 1.

Let

$$L:T_0\times R\times R^p\to R,\ (t,x,y)\to L(t,x,y),$$

$$L\left(t,x(t),\frac{\partial x}{\partial t}(t)\right)=\left|\frac{\partial x}{\partial t}(t)\right|^2+q(t)x^2(t)$$

where $q:T_0\to R$, is integrable function.

If there is $M\in R^+$, such that $|q(t)|\le M$, for any $t\in T_0$, then functional φ defined by $\varphi(x)=\int_{T_0}L\left(t,x(t),\frac{\partial x}{\partial t}(t)\right)dt^1\wedge...\wedge dt^p$

is continuously differentiable on $W_T^{1,2}$, and his gradient derives from the formula

$$(\nabla\varphi(x), z) = \int_{T_0}\left[\left(\nabla_x L\left(t, x(t), \frac{\partial x}{\partial t}(t)\right), z(t)\right) + \left(\nabla_y L\left(t, x(t), \frac{\partial x}{\partial t}(t)\right), \frac{\partial z}{\partial t}(t)\right)\right] dt^1 \wedge \ldots \wedge dt^p \quad (2)$$

Proof. It is sufficient to prove that φ has the derivative $\varphi'(x) \in \left(W_T^{1,p}\right)^*$ given by the relation (2) and that the function

$$\varphi'(x) \in W_T^{1,p} \to \left(W_T^{1,p}\right)^*,\ x \to \varphi'(x)$$

is continuous.

For x and y fixed in $W_T^{1,2}$, $t \in T_0$, $\lambda \in [-1,1]$, let

$$F(\lambda, t) = L\left(t, x(t) + \lambda z(t), \frac{\partial x}{\partial t}(t) + \lambda \frac{\partial z}{\partial t}(t)\right)$$

and

$$\psi(\lambda) = \int_{T_0} F(\lambda, t) dt^1 \wedge \ldots \wedge dt^p = \varphi(x + \lambda y).$$

Because

$$\left|\nabla_x L\left(t, x(t), \frac{\partial x}{\partial t}(t)\right)\right| = |2q(t) \cdot x(t)| \le 2M \cdot |x(t)| \in L^2\left(T_0, R^+\right)$$

and

$$\left|\nabla_y L\left(t, x(t), \frac{\partial x}{\partial t}(t)\right)\right| = \left|\frac{\partial x}{\partial t}\right| \in L^2\left(T_0, R^+\right)$$

we have

$$|\nabla_\lambda F(\lambda, t)| \le d(t) \in L^1\left(T_0, R^+\right).$$

Thus, Leibniz formula of differentiation under integral is applicable and

$$\psi'(0) = \int_{T_0} D_\lambda F(0, t) dt$$

$$= \int_{T_0}\left[\left(\nabla_x L\left(t, x(t), \frac{\partial x}{\partial t}(t)\right), z(t)\right) + \left(\nabla_y L\left(t, x(t), \frac{\partial x}{\partial t}(t)\right), \frac{\partial z}{\partial t}(t)\right)\right] dt^1 \wedge \ldots \wedge dt^p$$

$$= \int_{T_0}\left(2q(t)x(t)z(t) + 2\delta_{\alpha\beta}\frac{\partial x}{\partial t^\alpha}(t)\frac{\partial z}{\partial t^\beta}(t)\right) dt^1 \wedge \ldots \wedge dt^p$$

We have

$$\psi'(0)=\int_{T_0}\left(2q(t)x(t)z(t)+2\frac{\partial x}{\partial t}(t)2\frac{\partial z}{\partial t}(t)\right)dt^1\wedge\ldots\wedge dt^p$$

and with Cauchy-Schwartz inequality we find

$$\varphi'(0)\le 2\left(\int_{T_0}\left|\left(q(t)x(t),\frac{\partial x}{\partial t}(t)\right)\right|^2 dt^1\wedge\ldots\wedge dt^p\right)^{\frac{1}{2}}\cdot\left(\int_{T_0}\left|\left(z(t),\frac{\partial z}{\partial t}(t)\right)\right|^2 dt^1\wedge\ldots\wedge dt^p\right)^{\frac{1}{2}}$$
$$\le c\cdot\|z\|.$$

The Krasnoselski theorem implies that the application

$$x\to\left(\nabla_x L\left(\cdot,x,\frac{\partial x}{\partial t}\right)\right)$$

from $W_T^{1,2}$ to $L^1\times L^2$ is continuous, so φ' is continuous from $W_T^{1,2}$ into $\left(W_T^{1,2}\right)^*$ and the proof is complete.

4. Poisson multi-time linear equation

Theorem 2.

Let be the Lagrangian

$$L:T_0\times R\times R^p\to R,\ \left(t,x,\frac{\partial x}{\partial t}\right)\to L\left(t,x,\frac{\partial x}{\partial t}\right),\ \ \alpha=1,2,\ldots,p.$$

$$L\left(t,x(t),\frac{\partial x}{\partial t}(t)\right)=\left|\frac{\partial x}{\partial t}(t)\right|^2+q(t)x^2(t)$$

where $q:T_0\to R$.

If q is integrable and bounded real function, then exist x that minimizes the action

$$\varphi(x)=\int_{T_0}L\left(t,x(t),\frac{\partial x}{\partial t}(t)\right)dt^1\wedge\ldots\wedge dt^p\ \text{ in }\ H_T^1$$

and the Dirichlet problem

$$\Delta x - q(t)x(t) = 0\,,\ \ x\big|_{S_\alpha^+} = x\big|_{S_\alpha^-}\,,\ \ \frac{\partial x}{\partial t}\bigg|_{S_\alpha^+} = \frac{\partial x}{\partial t}\bigg|_{S_\alpha^-}$$ has at least one solution.

Proof. For $u \in H_T^1$ we use the decomposition $u = \overline{u} + \widetilde{u}$,
where

$$\overline{u} = \frac{1}{T_1 ... T_p} \int_{T_0} u(t) dt^1 \wedge ... \wedge dt^p\ .$$

The convexity of function

$$P(t, x(t)) = q(t) x^2(t)$$

implies

$$P(t, x(t)) \geq P(t, \overline{x}) + (\nabla P(t, \overline{x}), x(t) - \overline{x}).$$

It follows

$$\varphi(x) = \int_{T_0} \left|\frac{\partial x}{\partial t}\right|^2 dt^1 \wedge ... \wedge dt^p + \int_{T_0} P(t, \overline{x}) dt^1 \wedge ... \wedge dt^p +$$
$$+ \int_{T_0} (\nabla P(t, \overline{x}), x(t) - \overline{x}) dt^1 \wedge ... \wedge dt^p$$
$$= \int_{T_0} \left|\frac{\partial x}{\partial t}\right|^2 dt^1 \wedge ... \wedge dt^p + \int_{T_0} P(t, \overline{x}) dt^1 \wedge ... \wedge dt^p +$$
$$+ \int_{T_0} (\nabla P(t, \overline{x}), \widetilde{x}(t)) dt^1 \wedge ... \wedge dt^p$$

On the other hand, by Schwartz inequality we can write

$$(\nabla P(y, \overline{x}), \widetilde{x}(t)) \leq |\nabla P(y, \overline{x})| \cdot |\widetilde{x}(t)| \leq C_1 |\widetilde{x}(t)|\,,\ \ C_1 > 0$$

The function $q(t)$ is integrable. It follows

$$\varphi(x) \geq \int_{T_0} \left|\frac{\partial x}{\partial t}\right|^2 dt^1 \wedge ... \wedge dt^p + C_2 - C_1 \int_{T_0} |\widetilde{x}(t)| dt^1 \wedge ... \wedge dt^p$$

Because the function $\widetilde{x}(t)$ has the mean zero, we can use the Wirtinger inequality [1] and we find

$$\varphi(x) \geq \int_{T_0} \left|\frac{\partial x}{\partial t}\right|^2 dt^1 \wedge ... \wedge dt^p + C_2 - C_3 \left(\int_{T_0} \left|\frac{\partial x}{\partial t}\right|^2 dt^1 \wedge ... \wedge dt^p \right)^{\frac{1}{2}}.$$

The function of degree two from the right side of the inequality is bounded below since the action $\varphi(x)$ is bounded below. Let (u_k) be a minimizing sequence for the action φ.
We have

$$\varphi(x_k) \geq \int_{T_0} \left| \frac{\partial x_k}{\partial t} \right|^2 dt^1 \wedge \ldots \wedge dt^p + C_2 - C_3 \left(\int_{T_0} \left| \frac{\partial x_k}{\partial t} \right|^2 dt^1 \wedge \ldots \wedge dt^p \right)^{\frac{1}{2}}.$$

This means that the limit of $\|x_k\|$ is not ∞ because the sequence $\varphi(x_k)$ is bounded below. So, the sequence (x_k) is bounded in H_T^1. The Hilbert space H_T^1 is reflexive and φ is a continuously, convex function bounded below.

By consequence the action φ has a minimum point $x \in H_T^1$ that verifies Euler-Lagrange equations (1) and periodic boundary conditions because x has the weak derivatives and weak divergence Δx [1].

5. Conclusions

This work generalizes the case of Hill's equation [22]. We establish the conditions under which the action that produces the Poisson multi-time linear equation has a minimum on H_T^1. The function x that realizes the minimum of action, verifies Euler-Lagrange equation (the differential equation $\Delta x - q(t) \cdot x = 0$) and respects the boundary conditions

$$x\big|_{S_\alpha^+} = x\big|_{S_\alpha^-}, \quad \frac{\partial x}{\partial t}\bigg|_{S_\alpha^+} = \frac{\partial x}{\partial t}\bigg|_{S_\alpha^-}, \quad \alpha = 1,2,\ldots, p.$$

R E F E R E N C E S

[1] *I. Duca, A-M. Teleman, C. Udrişte,* Poisson-Gradient Dynamical Systems with Convex Potential, Proceedings of the 3-rd International Colloquium " Mathematics in Engineering and Numerical Physics ", 7-9 October 2004, Bucharest.

[2] *M. Forger, C. Paufler, H. Romer*, The Poisson bracket for Poisson forms in multisymplectic field theory, Reviews in Mathematical Physics, 15, 7 (2003), 705-743.

[3] *I. V. Kanatchikov*, Geometric (pre)quantization in the polysymplectic approach to field theory, arXiv: hep-th/0112263 v3, 3 Jun 2002, 1-12; Differential Geometry and Its Applications, Proc. Conf., Opava (Czech Republic), August 27-31, 2001, Silesian University Opava, 2002.

[4] *L. V. Kantorovici, G. P. Akilov,* Analiză funcţională (Functional analysis), Editura ştiinţifică şi enciclopedică, Bucureşti, 1980.

[5] *J. Mawhin, M. Willem*, Critical Point Theory and Hamiltonian Systems, Springer-Verlag, 1989.
[6] *S. G. Mihlin*, Ecuaţii liniare cu derivate parţiale (Linear equations with partial derivatives), Editura ştiinţifică şi enciclopedică, Bucureşti,
1983.
[7] *M. Neagu, C. Udrişte,* From PDE Systems and Metrics to Geometric Multi-time Field Theories, Seminarul de Mecanică, Sisteme Dinamice Diferenţiale, 79, Universitatea de Vest din Timişoara, 2001.
[8] *P. J. Olver*, The Equivalence Problem and Canonical Forms for Quadratic Lagrangians, Advances in Applied Mathematics, 9, (1998), 226-227.
[9] *C. Paufler, H. Romer*, De Donder-Weyl equations and multisymplectic geometry, arXiv: math-ph/0506022 v2, 20 Jun 2005.
[10] *N. Roman-Roy*, Multisymplectic Lagrangian and Hamiltonian formalism of first -order classical field theories, arXiv: math-ph/0107019 v1, 20 Jul 2001, vol XX (XXXX), No. X, 1-9.
[11] *R. E. Showalter*, Hilbert Space Methods for Partial Differential Equations, Electronic Journal of Differential Equations Monograph 01, 1994.
[12] *A.-M. Teleman*, *C. Udrişte,* On Hamiltonian Formalisms in Mathematical Physics, 4-th International Conference of Balkan Society of Geometers, Aristotle University of Thessaloniki, 26-30 June 2002.
[13] *C. Udrişte*, Nonclassical Lagrangian Dynamics and Potential Maps, Conference in Mathematics in Honour of Professor Radu Roşca on the occasion of his Ninetieth Birthday, Katholieke University Brussel, Katholieke University Leuwen, Belgium, December 11-16, 1999; http://xxx.lanl.gov/math.DS/0007060, (2000).
[14] *C. Udrişte*, Geometric Dynamics, Kluwer Academic Publishers, Dodrecht /Boston /London, Mathematics and Its Applications, 513, (2000); Southeast Asian Bulletin of Mathematics, Springer-Verlag 24 (2000).
[15] *C. Udrişte*, Solutions of DEs and PDEs as Potential Maps Using First Order Lagrangians, Centenial Vrânceanu, Romanian Academy, University of Bucharest, June 30-July 4, (2000); http://xxx.lanl.gov/math.DS/ 0007061, (2000); Balkan Journal of Geometry and Its Applications 6, 1, 93-108, 2001.
[16] *C. Udrişte*, From integral manifolds and metrics to potential maps, Atti dell'Accademia Peloritana dei Pericolanti, Classe 1 di Scienze Fis. Mat. e Nat., 81-82, A 01006 (2003-2004), 1-14.
[17] *C. Udrişte, I. Duca*, Periodical Solutions of Multi-Time Hamilton Equations, Analele Universităţii Bucureşti, 55, 1 (2005), 179-188.
[18] *C. Udrişte, M. Neagu*, From PDE Systems and Metrics to Generalized Field Theories, http://xxx.lanl.gov/abs/math.DG/0101207.
[19] *C. Udrişte, M. Postolache*, Atlas of Magnetic Geometric Dynamics, Geometry Balkan Press, Bucharest, 2001.
[20] *C. Udrişte, A.-M. Teleman*, Hamilton Approaches of Fields Theory, IJMMS, 57 (2004), 3045-3056; ICM Satelite Conference in Algebra and Related Topics, University of Hong-Kong, 13-18.08.02.
[21] *C. Udrişte, M. Ferrara, D. Opriş*, Economic Geometric Dynamics, Geometry Balkan Press, Bucharest, 2004.
[22] *G.W. Hill*, On the part of the motion of the lunar perigee which is a function of the mean motions of the sun and the moon, Acta Math. 8 (1886), 1-3.