A GENERALIZATION OF FIXED POINT THEOREMS ABOUT F-CONTRACTION IN PARTICULAR S-METRIC SPACES AND CHARACTERIZATION OF QUASI-CONTRACTION MAPS IN b-METRIC SPACES

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Abstract: In this paper, we define a new S-metric space and describe some fixed point theorems concerning F-contraction. In this way, we give some examples to illustrate our results. Also we give a partial answer to a question raised by Singh, Czerwik and Krol.

Keywords: Fixed point, F-contraction, S-metric space
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1. Introduction

Wardowski introduced a new contraction, the so-called F-contraction, and proved some fixed point results for such mappings on a complete metric space [9]. A mapping $T : X \to X$ is said to be an F-contraction if there exists $\tau > 0$ such that, if $d(Tx, Ty) > 0$, then $\tau + F(d(Tx, Ty)) \leq F(d(x, y))$, for all $x, y \in X$ [11, 12]. Later, Wardowski and Dung defined the notion of F-weak contractions in metric spaces and generalized the theorem of Wardowski [10]. Dung and Hang studied the notion of a generalized F-contraction and extended a fixed point theorem for such mappings [3]. Also, many author's studied F-contraction mappings and present some application of the map in b-metric space [13, 14, 1]. Let $F : \mathbb{R}^+ \to \mathbb{R}$ be a mapping satisfying: (F1) F is strictly increasing; (F2) For each sequence $\{a_n\}_{n\in\mathbb{N}}$ of positive numbers $\lim_{n\to\infty} a_n = 0$ if and only if $\lim_{n\to\infty} F(a_n) = -\infty$; (F3) There exists $k \in (0, 1)$ such that $\lim_{a\to 0^+} a^k F(a) = 0$. Piri and Kumam further described a large class of functions by replacing condition, (F3$'$) F is continuous on $(0, \infty)$ instead of the condition (F3) in the definition of F-contraction [6].

Motivated by these researches, in this paper we introduce new S-metric spaces and prove some fixed point theorems. Throughout this paper, $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{N}$ denote the set of all real numbers, the set of all nonnegative real numbers and the set of all positive integers, respectively.

Definition 1.1. Let X be a nonempty set. An S-metric on X is a continuous function $S : X^3 \to \mathbb{R}^+$ such that satisfies the following conditions for each x, y, z,and $a \in X$:

(S1) $S(x, y, z) > 0$ for all $x, y, z, \in X$ with $x \neq y \neq z$ or $x \neq y$ or $x \neq z$ or $y \neq z$;
(S2) $S(x, y, z) = 0$ if and only if $x = y = z$;
(S3) $S(x, y, z) \leq S(x, y, a) + S(a, y, z) + S(x, a, z)$;
(S4) $S(x, y, z) = S(x, z, y)$, $S(x, y, z) = S(y, x, z)$ and $S(x, y, z) = S(z, y, x)$. (symmetric)

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Example 1.1. Suppose that $X = \mathbb{R}$. Define $S : X^3 \to \mathbb{R}^+$ by

$$S(x, y, z) = |x - y| + |y - z| + |z - x|.$$

(X, S) is an S-metric space.

Definition 1.2. Suppose that (X, S) be an S-metric space, $\{x_n\}_{n=1}^{\infty}$ be a sequaence in X and $x \in X$ arbitrary. Then:

(i) The sequeance $\{x_n\}_{n=1}^{\infty}$ is said to be a Cauchy sequence, if for each $\varepsilon > 0$, exists $n_0 \in \mathbb{N}$ such that $S(x_n, x_m, x_m) < \varepsilon$ for each $m, n \geq n_0$;

(ii) The Sequence $\{x_n\}_{n=1}^{\infty}$ is said to be convergent to point $x \in X$, if for each $\varepsilon > 0$, exists a positive integer number n_0, such that for all $n \geq n_0$, $S(x_, x, x_n) < \varepsilon$;

(iii) (X, S) is said to be complete if every cauchy sequence is convergent.

Remark 1.1. Note that, if (X, d) be a metric space, then by definition

$$S(x, y, z) = d(x, y) + d(y, z) + d(z, x),$$

it can be easily shown that (X, d) is complete if and only if (X, S) is complete.

Definition 1.3. *Let $\mathcal{F}$ be the family of all functions $F : \mathbb{R}^+ \to \mathbb{R}^+$ such that:*

(F1) *F is strictly increasing that is for all $\alpha, \beta \in \mathbb{R}_+$ such that $\alpha < \beta$, $F(\alpha) < F(\beta)$;*

(F2) *For each sequence $\{x_n\}_{n=1}^{\infty}$ of positive numbers, $\lim_{n\to\infty} x_n = 0$ if and only if $\lim_{n\to\infty} F(x_n) = -\infty$;*

(F3) *$t \in (0, 1)$ exists such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.*

Example 1.2. $F_1(\alpha) = -\dfrac{1}{\alpha^2}$, $F_2(\alpha) = \ln \alpha$ *and* $F_3(\alpha) = \dfrac{1}{1 - e^{\alpha}}$.

Definition 1.4. [9] *Let (X, d) be a metric space. Self map T on X is said to be an F-contraction on (X, d), firstly, $F \in \mathcal{F}$ and secondly $k > 0$ exists such that for all $x, y \in X$, if $d(Tx, Ty) > 0$, then $k + F(d(Tx, Ty)) \leq F(d(x, y))$.*

Definition 1.5. [10] *Let (X, d) be a metric space. Self map T on X is said to be an F-weak contraction on (X, d), firstly, $F \in \mathcal{F}$ and secondly $k > 0$ exists such that for all $x, y \in X$, if $d(Tx, Ty) > 0$, then $k + F(d(Tx, Ty)) \leq F(M(x, y))$ where*

$$M(x, y) = \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2} \right\}.$$

2. Main Results

In the following, we are going to state and prove our main results.

Definition 2.1. *Let (X, S) be a S-metric space. Self map T on X is said to be an F-contaction on (X, S), firstly, $F \in \mathcal{F}$ and secondly $k > 0$ exists such that for all $x, y \in X$, if $S(Tx, Tx, Ty) > 0$, then*

$$k + F(S(Tx, Tx, Ty)) \leq F(S(x, x, y)). \tag{1}$$

Theorem 2.1. *Let (X, S) be a complete S-metric space and consider self map T on X is an F-contaction on (X, S). Then T has a unique fixed point in X.*

Proof. Let $x_0 \in X$ be arbitrary and fixed. Let $\{x_n\}_{n=1}^{\infty}$ be the picard sequence of T based on x_0, which is $Tx_n = x_{n+1}$ for $n = 0, 1, 2, \ldots,$. If $n_0 \in \mathbb{N}$ exists such that $S(x_{n_0}, x_{n_0}, x_{n_0+1}) = 0$ then x_{n_0} is a fixed point of T, and the existence part of the proof is finished. On the contrary case, assume that $S(x_n, x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N} \cup \{0\}$. Applying the contractivity condition (1), we get

$$k + F\left(S(Tx_n, Tx_n, Tx_{n+1})\right) \leq F\left(S(x_n, x_n, Tx_n)\right). \tag{2}$$

We will show that

$$S(x_{n+1}, x_{n+1}, Tx_{n+1}) \leq S(x_n, x_n, Tx_n), \tag{3}$$

for all $n \in \mathbb{N}$. On the contrary, Suppose that

$$S(x_{n_0+1}, x_{n_0+1}, Tx_{n_0+1}) \geq S(x_{n_0}, x_{n_0}, Tx_{n_0}),$$

for some $n_0 \in \mathbb{N}$. From (2), we have

$$F\left(S(x_{n_0+1}, x_{n_0+1}, Tx_{n_0+1})\right) \leq F\left(S(x_{n_0}, x_{n_0}, Tx_{n_0})\right) - k,$$

which together condition (F1) implies that

$$S(x_{n_0+1}, x_{n_0+1}, Tx_{n_0+1}) \leq S(x_{n_0}, x_{n_0}, Tx_{n_0}).$$

It gives us a contradiction. Therefore, (3) holds, so $\{S(x_n, x_n, Tx_n)\}$ is a decreasing positive sequence in $\mathbb{R}_+$ and it converges to some $R \geq 0$. We claim that $R = 0$. To support the claim, let be untrue and $R > 0$. Then for any $\varepsilon > 0$, it is possible to find a positive integer m such that $S(x_m, x_m, Tx_m) < R + \varepsilon$. By (F1) we get

$$F\left(S(x_m, x_m, Tx_m)\right) < F(R + \varepsilon). \tag{4}$$

Since $S(x_n, x_n, x_{n+1}) > 0$ for all n, then by repeatedly using (1) and taking (4) into account, we obtain

$$\begin{aligned} F\left(S(T^n x_m, T^n x_m, T^{n+1} x_m)\right) &\leq F\left(S(T^{n-1} x_m, T^{n-1} x_m, T^n x_m)\right) - k \\ &\leq F\left(S(T^{n-2} x_m, T^{n-2} x_m, T^{n-1} x_m)\right) - 2k \\ &\leq \cdots \\ &\leq F\left(S(x_m, x_m, Tx_m)\right) - nk \\ &\leq F(R + \varepsilon) - nk. \end{aligned}$$

Letting $n \to \infty$ in the above inequality, we get

$$\lim_{n \to \infty} F\left(S(T^n x_m, T^n x_m, T^{n+1} x_m)\right) = -\infty.$$

Hence from condition (F2), we obtain $\lim_{n \to \infty} S(T^n x_m, T^n x_m, T^{n+1} x_m) = 0$. Then

$$S(x_{m+n}, x_{m+n}, Tx_{m+n}) < R,$$

for n which large enough. It is a contradiction with the definition of R. Therefore

$$\lim_{n \to \infty} S(x_n, x_n, Tx_n) = 0. \tag{5}$$

Now, we prove that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in (X, S). Suppose that the contrary. Then $\varepsilon > 0$ exists for which we can find monotonically increasing sequences $\{p(n)\}_{n=1}^{\infty}$ and $\{q(n)\}_{n=1}^{\infty}$ of natural number such that

$$\begin{aligned} &p(n) > q(n) > n, \\ &S\left(x_{q(n)}, x_{q(n)}, x_{p(n)}\right) \geq \varepsilon, \\ &S\left(x_{q(n)}, x_{q(n)}, x_{p(n)-1}\right) < \varepsilon. \end{aligned} \tag{6}$$

Regarding (1) and (6), we can write

$$\begin{aligned} F\left(S\left(x_{q(n)}, x_{q(n)}, x_{p(n)}\right)\right) &\leq F\left(S\left(x_{q(n)-1}, x_{q(n)-1}, x_{p(n)-1}\right)\right) - k \\ &< F\left(S\left(x_{q(n)-1}, x_{q(n)-1}, x_{p(n)-1}\right)\right), \end{aligned} \tag{7}$$

which together (F1) implies

$$S\left(x_{q(n)}, x_{q(n)}, x_{p(n)}\right) < S\left(x_{q(n)-1}, x_{q(n)-1}, x_{p(n)-1}\right).$$

Using this together with property (S3) we get

$$\begin{aligned}\varepsilon &\le S\left(x_{q(n)}, x_{q(n)}, x_{p(n)}\right)\\ &< S\left(x_{q(n)-1}, x_{q(n)-1}, x_{p(n)-1}\right)\\ &\le S\left(x_{q(n)-1}, x_{q(n)-1}, x_{q(n)}\right) + 2S\left(x_{q(n)}, x_{q(n)-1}, x_{p(n)-1}\right)\\ &\le \cdots\\ &\le 2S\left(x_{q(n)-1}, x_{q(n)-1}, x_{q(n)}\right) + 6S\left(x_{p(n)}, x_{p(n)-1}, x_{p(n)}\right)\\ &\quad + S\left(x_{q(n)}, x_{q(n)}, x_{p(n)-1}\right).\end{aligned}$$

By virtue of thes fact and in view if (5) and (6), we have

$$\begin{aligned}\varepsilon &\le \lim_{n\to\infty} \sup S\left(x_{q(n)}, x_{q(n)}, x_{p(n)}\right)\\ &\le \lim_{n\to\infty} \sup S\left(x_{q(n)-1}, x_{q(n)-1}, x_{p(n)-1}\right)\\ &\le \varepsilon.\end{aligned} \tag{8}$$

Again, by using (7), (8), conditions (F1) and (F3) in Definition 1.3, we find that

$$\begin{aligned}F(\varepsilon) &\le F\left(\lim_{n\to\infty} \sup S\left(x_{q(n)}, x_{q(n)}, x_{p(n)}\right)\right)\\ &\le F\left(\lim_{n\to\infty} \sup S\left(x_{q(n)-1}, x_{q(n)-1}, x_{p(n)-1}\right)\right) - k\\ &\le F(\varepsilon) - k,\end{aligned}$$

which holds to a contradiction. Therefore, $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in the complete S-metric space X. Then $v \in X$ exists such that $x_n \to v$ as $n \to \infty$. That is for any $\varepsilon > 0$, $n_1 \in \mathbb{N}$ exists such that $S(v, v, x_n) < \varepsilon$ for all $n \ge n_1$. We are going to show that v is a fixed point of T. First note that $S(Tv, Tv, Tx_n) = 0$. Since, if for all n, $S(Tv, Tv, Tx_n) > 0$, then $F\left(S(Tv, Tv, Tx_n)\right) < F(S(v, v, x_n)) - k$. It inforce that $S(Tv, Tv, Tx_n) < S(v, v, x_n) < \varepsilon$. That is a contradiction. So, We have

$$S(Tv, Tv, v) = \lim_{n\to\infty} \sup S\left(Tv, Tv, Tx_{n+1}\right) = \lim_{n\to\infty} \sup S(Tv, Tv, Tx_n) < \varepsilon.$$

Thus $S(Tv, Tv, v) = 0$. Therefore $Tv = v$. Hence, v is a fixed point of T. Next we study the uniqueness of the fixed point of T. Assume that T has two different fixed points v_1 and v_2. Then $S(v_1, v_1, v_2) > 0$ and from condition (1), we get

$$\begin{aligned}0 < r &\le F\left(S(v_1, v_1, v_2)\right) - F\left(S(Tv_1, Tv_1, Tv_2)\right)\\ &= F\left(S(v_1, v_1, v_2)\right) - F\left(S(v_1, v_1, v_2)\right)\\ &= 0,\end{aligned}$$

which is a contradiction. Then $S(v_1, v_1, v_2) = 0$, and so $v_1 = v_2$. Therefore the fixed point is unique. □

Example 2.1. *Let $X = \mathbb{R}$ and $S(x, y, z) = |x - y| + |y - z| + |z - x|$. Define the mapping $T : X \to X$ by $T(x) = \frac{x}{3}$ and take $F(\alpha) = \ln \alpha$. We obtain the result that T is an F-contraction with $0 \le k \le \ln 3$. To see this, let us consider the following calculations. First, observe that*

$$k + F\left(S(Tx, Tx, Ty)\right) \le F\left(S(x, x, y)\right).$$

On the other hand, $S(Tx, Ty, Ty) = |Tx - Ty| + |Tx - Ty| = 2|Tx - Ty|$. So

$$F\left(S(Tx, Tx, Ty)\right) = F(2|Tx - Ty|) = \ln 2|Tx - Ty|,$$

and

$$F\left(S(x, x, y)\right) = F(2|x - y|) = \ln 2|x - y|.$$

Indeed, $k+\ln 2|Tx-Ty|\leq \ln 2|x-y|$. *Therefore,*

$$k\leq \ln\frac{2|x-y|}{2|Tx-Ty|}=\ln\frac{|x-y|}{|Tx-Ty|}=\ln\frac{|x-y|}{\frac{|x-y|}{3}}=\ln 3.$$

Definition 2.2. *Let* (X,S) *be a* S*-metric space. A self map* T *on* X *is said to be an* F*-weak contraction on* (X,S)*, if* $F\in\mathfrak{F}$ *and* $k>0$ *exists such that for all* $x,y\in X$*, which* $S(Tx,Tx,Ty)>0$*, we have*

$$k+F\left(S(Tx,Tx,Ty)\right)\leq F\left(M(x,y)\right), \tag{9}$$

where

$$M(x,y)=\max\left\{S(x,x,y),S(Tx,Tx,Ty),\frac{S(y,y,Tx)}{10},\frac{S(y,y,Ty)}{10}\right\}.$$

Theorem 2.2. *Let* (X,S) *be a complete* S*-metric space and* $T:X\to X$ *be an* F*-weak contraction satisfying the following condition:*

$$\max\left\{\frac{S(y,y,Ty)}{10},\frac{S(y,y,Ty)}{5}+\frac{S(Tx,Tx,Ty)}{10}\right\}\leq S(Tx,Tx,Ty).$$

Then T *has a unique fixed point in* X*.*

Proof. Let $x_0\in X$ be arbitrary and fixed. Define a sequence $\{x_n\}_{n=1}^{\infty}$ in X such that $x_1=Tx_0$, $x_{n+1}=Tx_n$ for all $n\in\mathbb{N}$. We may suppose that $x_{n+1}\neq x_n$ for all n, otherwise, T has obviously a fixed point. Then $S(x_n,x_n,x_{n+1})>0$ for all $n\in\mathbb{N}\cup\{0\}$ and hence (9) implies that

$$F\left(S(Tx_{n-1},Tx_{n-1},Tx_n)\right)\leq F\left(M(x_{n-1},x_n)\right)-k. \tag{10}$$

We obtain

$$\begin{aligned}\max\{S(x_{n-1},x_{n-1},x_n),S(Tx_{n-1},Tx_{n-1},Tx_n)\}&\leq M(x_{n-1},x_n)\\&\leq\max\Big\{S(x_{n-1},x_{n-1},x_n),S(Tx_{n-1},Tx_{n-1},Tx_n),\\&\quad\frac{S(x_n,x_n,Tx_{n-1})}{10},\frac{S(x_n,x_n,Tx_n)}{10}\Big\}\\&\leq\max\{S(x_{n-1},x_{n-1},x_n),S(x_n,x_n,x_{n+1})\}.\end{aligned}$$

Then (10) becomes that

$$F\left(S(Tx_{n-1},Tx_{n-1},Tx_n)\right)\leq F\left(\max\{S(x_{n-1},x_{n-1},x_n),S(x_n,x_n,x_{n+1})\}\right)-k.$$

If we assume that

$$\max\{S(x_{n-1},x_{n-1},x_n),S(x_n,x_n,x_{n+1})\}=S(x_n,x_n,x_{n+1}),$$

for some n then from (10), we have

$$\begin{aligned}F\left(S(Tx_{n-1},Tx_{n-1},Tx_n)\right)&\leq F\left(S(Tx_{n-1},Tx_{n-1},Tx_n)-k\right)\\&\leq F\left(S(Tx_{n-1},Tx_{n-1},Tx_n)\right),\end{aligned}$$

and using condition (F1) we conclude that

$$S(x_n,x_n,x_{n+1})<S(x_n,x_n,x_{n+1}),$$

which is a contradiction. Therefore,

$$\max\{S(x_{n-1},x_{n-1},x_n),S(x_n,x_n,x_{n+1})\}=S(x_{n-1},x_{n-1},x_n),$$

for each n. Applying again (10) and condition (F1), we deduce that

$$S(x_n,x_n,x_{n+1})<S(x_{n-1},x_{n-1},x_n).$$

That is $\{S(x_n, x_n, x_{n+1})\}_{n=1}^{\infty}$ is a stricly decreasing positive sequence in $\mathbb{R}^+$ and it converges to some $R \geq 0$. We declare that $R = 0$. Suppose it is not true. Then $R > 0$. For each $\varepsilon > 0$, let us choose $m \in \mathbb{N}$ such that $S(x_m, x_m, Tx_m) < R + \varepsilon$. From condition (F1), we have

$$F\left(S(x_m, x_m, Tx_m)\right) < F\left(R + \varepsilon\right). \tag{11}$$

Since T is an F-weak contraction and taking into account $S(Tx_m, Tx_m, T^2x_m) > 0$, we get

$$k + F\left(S(Tx_m, Tx_m, T^2x_m)\right) \leq F\left(M(x_m, Tx_m)\right). \tag{12}$$

Since,

$$M(x_m, Tx_m) = \max\left\{S(x_m, x_m, Tx_m), S(Tx_m, Tx_m, T^2x_m)\right\},$$

then from (9) and (F1), we get

$$\max\left\{S(Tx_m, Tx_m, T^2x_m), S(x_m, x_m, Tx_m)\right\} = S(x_m, x_m, Tx_m).$$

Hence (12) becaomes $F\left(S(Tx_m, Tx_m, T^2x_m)\right) \leq F\left(S(x_m, x_m, Tx_m)\right) - k$. This yields

$$\begin{aligned} F\left(S(T^2x_m, T^2x_m, T^3x_m)\right) &\leq F\left(S(Tx_m, Tx_m, T^2x_m)\right) - k \\ &\leq F\left(S(x_m, x_m, Tx_m)\right) - 2k. \end{aligned}$$

Continuing the above process and using (11), we observe that

$$\begin{aligned} F\left(S(T^nx_m, T^nx_m, T^{n+1}x_m)\right) &\leq \cdots \leq F\left(S(x_m, x_m, Tx_m)\right) - nk \\ &\leq F\left(R + \varepsilon\right) - nk. \end{aligned}$$

Passing to the limit $n \to \infty$ in the above relation, we obtain

$$\lim_{n\to\infty} F\left(S(T^nx_m, T^nx_m, T^{n+1}x_m)\right) = -\infty.$$

It follows from condition (F2) that $\lim_{n\to\infty} S(T^nx_m, T^nx_m, T^{n+1}x_m) = 0$. So

$$S(T^nx_m, T^nx_m, T^{n+1}x_m) = S(x_{m+n}, x_{m+n}, Tx_{m+n}) < R,$$

for n sufficiently large which is a contraction with the definition of R. Therefore

$$\lim_{n\to\infty} S(x_n, x_n, x_{n+1}) = 0. \tag{13}$$

Next, we intend to show that the sequence $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in (X, S). Arguing by contradiction, we assume that $\varepsilon > 0$ and the sequences $\{p(n)\}$ and $\{q(n)\}$ of natural numbers exists such that for all $n \in \mathbb{N}$,

$$\begin{aligned} &p(n) > q(n) > n, \\ &S\left(x_{q(n)}, x_{q(n)}, x_{p(n)}\right) \geq \varepsilon, \\ &S\left(x_{q(n)}, x_{q(n)}, x_{p(n)-1}\right) < \varepsilon. \end{aligned} \tag{14}$$

In the hight of (14) and condition (9), we find that

$$F\left(S\left(x_{q(n)-1}, x_{q(n)-1}, x_{p(n)-1}\right)\right) \leq F\left(M(x_{q(n)-1}, x_{p(n)-1})\right) - k. \tag{15}$$

Applying (S3) and our hypothesis, we get

$$\begin{aligned}
&\max\left\{ S\left(x_{q(n)-1},x_{q(n)-1},x_{p(n)-1}\right), S\left(Tx_{q(n)-1},Tx_{q(n)-1},Tx_{p(n)-1}\right) \right\} \\
&\le M\left(x_{q(n)-1},x_{p(n)-1}\right) \\
&= \max\left\{ S\left(x_{q(n)-1},x_{q(n)-1},x_{p(n)-1}\right), S\left(Tx_{q(n)-1},Tx_{q(n)-1},Tx_{p(n)-1}\right), \right. \\
&\quad \left. \frac{S\left(x_{p(n)-1},x_{p(n)-1},Tx_{q(n)-1}\right)}{10}, \frac{S\left(x_{q(n)-1},x_{q(n)-1},x_{p(n)-1}\right)}{10} \right\} \\
&\le \max\left\{ S\left(x_{q(n)-1},x_{q(n)-1},x_{p(n)-1}\right), S\left(Tx_{q(n)-1},Tx_{q(n)-1},Tx_{p(n)-1}\right), \right. \\
&\quad \frac{S\left(x_{p(n)-1},x_{p(n)-1},Tx_{p(n)-1}\right)}{5} + \frac{S\left(Tx_{q(n)-1},Tx_{q(n)-1},Tx_{p(n)-1}\right)}{10}, \\
&\quad \left. \frac{S\left(x_{p(n)-1},x_{p(n)-1},Tx_{p(n)-1}\right)}{10} \right\} \\
&\le \max\left\{ S\left(x_{q(n)-1},x_{q(n)-1},x_{p(n)-1}\right), \right. \\
&\quad \left. S\left(Tx_{q(n)-1},Tx_{q(n)-1},Tx_{p(n)-1}\right) \right\}.
\end{aligned}$$

a consequence of (15) and (F1), we have

$$\begin{aligned}
&\max\left\{ S\left(x_{q(n)-1},x_{q(n)-1},x_{p(n)-1}\right), S\left(x_{q(n)},x_{q(n)},x_{p(n)}\right) \right\} \\
&\quad = S\left(x_{q(n)-1},x_{q(n)-1},x_{p(n)-1}\right).
\end{aligned}$$

Again, according (15) becomes

$$F\left(S\left(x_{q(n)},x_{q(n)},x_{p(n)}\right)\right) \le F\left(S\left(x_{q(n)-1},x_{q(n)-1},x_{p(n)-1}\right)\right) - k,$$

and so by using (F1), we get

$$S\left(x_{q(n)},x_{q(n)},x_{p(n)}\right) < S\left(x_{q(n)-1},x_{q(n)-1},x_{p(n)-1}\right). \tag{16}$$

By (14), (16) and using (S3), we obtain

$$\begin{aligned}
\varepsilon &\le S\left(x_{q(n)},x_{q(n)},x_{p(n)}\right) \\
&\le S\left(x_{q(n)-1},x_{q(n)-1},x_{p(n)-1}\right) \\
&\le \cdots \\
&\le 2S\left(x_{q(n)-1},x_{q(n)-1},x_{q(n)}\right) \\
&\quad + 6S\left(x_{p(n)-1},x_{p(n)-1},x_{p(n)}\right) + S\left(x_{q(n)},x_{q(n)},x_{p(n)-1}\right).
\end{aligned}$$

Regarding to (13) and (14), we have

$$\begin{aligned}
\varepsilon &\le \lim_{n\to\infty} \sup S\left(x_{q(n)},x_{q(n)},x_{p(n)}\right) \\
&\le \lim_{n\to\infty} \sup S\left(x_{q(n)-1},x_{q(n)-1},x_{p(n)-1}\right) \\
&\le \varepsilon.
\end{aligned} \tag{17}$$

In view of (16) and (17), together with (F1), (F3)$'$, we have

$$\begin{aligned} F(\varepsilon) &\le F\left(\lim_{n\to\infty}\sup S\left(x_{q(n)}, x_{q(n)}, x_{p(n)}\right)\right) \\ &\le F\left(\lim_{n\to\infty}\sup S\left(x_{q(n)-1}, x_{q(n)-1}, x_{p(n)-1}\right)\right) - k \\ &\le F(\varepsilon) - k. \end{aligned}$$

It is a contradiction with $k > 0$ and it follows that $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence in X. By completeness (X, S), $\{x_n\}_{n=1}^{\infty}$ converges to some point $v \in X$. Therefore, for each $\varepsilon > 0$, $n_1 \in \mathbb{N}$ exists, such that $S(v, v, x_n) < \varepsilon$, for all $n \ge n_1$. We claim that v is a fixed point of T and $S(Tv, Tv, Tx_n) = 0$, for some $n \ge n_1$. If $S(Tv, Tv, Tx_n) > 0$ for all $n \ge n_1$, then from (9), we have $F(S(Tv, Tv, Tx_n)) \le F(M(v, x_n)) - k$. In view of (S3) and our assumptions, we obtain

$$\begin{aligned} \max\{S(v, v, x_n), S(Tv, Tv, Tx_n)\} &\le M(v, x_n) \\ &\le \max\Bigg\{S(v, v, x_n), S(Tv, Tv, Tx_n), \\ &\quad \frac{S(x_n, x_n, Tx_n)}{5} + \frac{S(Tv, Tv, Tx_n)}{10}, \\ &\quad \frac{S(x_n, x_n, Tx_n)}{10}\Bigg\} \\ &\le \max\{S(v, v, x_n), S(Tv, Tv, Tx_n)\}. \end{aligned}$$

Then (18) turns into

$$F\left(S(Tv, Tv, Tx_n)\right) \le F\left(\max\{S(v, v, x_n), S(Tv, Tv, Tx_n)\}\right) - k. \tag{18}$$

If $\max\{S(v, v, x_n), S(Tv, Tv, Tx_n)\} = S(Tv, Tv, Tx_n)$, then from (18) and condition (F1), we lead to a contradiction, and so consequently

$$\max\{S(v, v, x_n), S(Tv, Tv, Tx_n)\} = S(v, v, x_n),$$

we have $F\left(S(Tv, Tv, Tx_n)\right) \le F\left(S(v, v, x_n)\right) - k$ and we get

$$S(Tv, Tv, Tx_n) < S(v, v, x_n) < \varepsilon,$$

$$S(Tv, Tv, v) = \lim_{n\to\infty}\sup S(Tv, Tv, x_n) = \lim_{n\to\infty}\sup S(Tv, Tv, Tx_{n-1}) = 0,$$

which implies that $Tv = v$. Hence, v is a fixed point of T. Finally, we show that T has at nost one fixed point. Indeed, if v_1, $v_2 \in X$ are two fixed points of T such that $v_1 \neq v_2$, then from (9), we obtain

$$F\left(S(Tv_1, Tv_1, Tv_2)\right) \le F\left(M(v_1, v_2)\right) - k. \tag{19}$$

Applying (S3) and the a ssumption of the theorem, it follows that

$$\begin{aligned} S(v_1, v_1, v_2) \le M(v_1, v_2) &\le \max\Bigg\{S(v_1, v_1, v_2), S(Tv_1, Tv_1, Tv_2), \\ &\quad \frac{S(v_2, v_2, Tv_2)}{5} + \frac{S(Tv_1, Tv_1, Tv_2)}{10}, \frac{S(v_2, v_2, Tv_2)}{10}\Bigg\} \\ &\le \max\{S(v_1, v_1, v_2), S(Tv_1, Tv_1, Tv_2)\} \\ &= S(v_1, v_1, v_2), \end{aligned}$$

applying (18), we yield $S(v_1, v_1, v_2) < S(v_1, v_1, v_2)$, which is contradiction. Hence $v_1 = v_2$. This complete the proof. □

Example 2.2. *Let $X = \mathbb{R}$ and define $S : \mathbb{R} \to \mathbb{R}$ by*

$$S(x, , y, z) = |x - y| + |y - z| + |z - x|.$$

Then (X, S) is a complete S-metric space. Let $T : X \to X$ be defined by $T(x) = \frac{x}{3}$. Also, take $F(\alpha) = \ln \alpha$ for $\alpha > 0$. Note that,

$$M(x, y) = \max\left\{s(x, x, y), s(Tx, Tx, Ty), \frac{s(y, y, Tx)}{10}, \frac{s(y, y, Ty)}{10}\right\},$$

$s(x, x, y) = 2|x - y|$, $s(Tx, Tx, Ty) = 2|Tx - Ty|$,

$$\frac{s(y, y, Tx)}{10} = \frac{1}{10}(2|y - Tx|) = \frac{1}{5}|y - Tx|,$$

$$\frac{s(y, y, Ty)}{10} = \frac{1}{10}(2|y - Ty|),$$

and

$$M(x, y) = \max\left\{2|x - y|, 2|y|, \frac{1}{5}\left|y - \frac{x}{3}\right|\right\}.$$

We have $k + F(s(Tx, Tx, Ty)) \le F(M(x, y))$. So $k + \ln 2|\frac{x}{3} - \frac{y}{3}| \le F(M(x, y))$. If $M(x, y) = 2|x - y|$, then $k + \ln 2|\frac{x}{3} - \frac{y}{3}| < \ln 2|x - y|$ and so $k < \ln 3$. If $M(x, y) = \frac{2}{15}|y|$ then

$$k + \ln 2|\frac{x}{3} - \frac{y}{3}| < \ln \frac{2}{15}|y|,$$

or $k < \ln \dfrac{\frac{|y|}{5}}{|x - y|}$. Thus, $k \le \ln 3$. If $M(x, y) = \frac{1}{5}|y - \frac{x}{3}|$, then

$$k + \ln 2|\frac{x}{3} - \frac{y}{3}| \le \ln \frac{1}{5}|y - \frac{x}{3}|.$$

Finally $k \le \ln 3$.

3. Characterization of quasi-contraction maps

Now, we first introduce the concept of a quasi-contraction map in b-metric spaces.

Definition 3.1. *Let X be a nonempty set. Suppose that $D : X \times X \to [0, \infty)$ be a function satisfies the following conditions:*

1) *$D(x, y) = 0$ if and only if $x = y$,*
2) *$D(x, y) = D(y, x)$ for each $x, y, \in X$,*
3) *$D(x, y) \le k(D(x, z) + D(z, y))$ for each $x, y, z \in X$, where $k > 1$ is a constant.*

Then the pair (X, D) is called a b-metric space or a metric type space.

Definition 3.2. *Let (X, D) be a b-metric space. The self-map $T : X \to X$ is said to be quasi-contraction, if there exists a $0 \le c \le 1$ such that*

$$D(Tx, Ty) \le c \max\left\{D(x, y), \frac{D(x, Tx) + D(y, Ty)}{2}, \frac{D(x, Ty) + D(x, Ty)}{2}\right\},$$

for all $x, y, \in X$.

Definition 3.3. *Let (X, D) be a b-metric space and $\{x_n\}$ be a sequence in X.*

(i) *We say sequence $\{x_n\}$ is converges to x, if $D(x_n, x) \to 0$ as $n \to \infty$,*
(ii) *we say sequence $\{x_n\}$ is Cauchy sequence if for $\varepsilon > 0$, exists $n_0 \in \mathbb{N}$ such that for all natural number m, n, which $m, n \ge n_0$, then $D(x_n, x_m) < \varepsilon$.*

Theorem 3.1. *Let (X, D) be a complete b-metric space and $T : X \to X$ be a quasi-contraction map with $0 < kc < k^2c < 1$ $(0 < c < 1,\ k > 1)$. Then T has a unique fixed point $x^\circ \in X$ and for each $x \in X$, $\lim_{n\to\infty} T^n x = x^\circ$.*

Proof. Since T is quasi-contraction, for each $x, y, \in X$, we have

$$D(Tx,Ty) \le c \max\left\{D(x,y), \frac{D(x,Tx)+D(y,Ty)}{2}, \frac{D(x,Ty)+D(Tx,y)}{2}\right\}.$$

Let $x \in X$ be arbitrary. If for some $n_0 \in \mathbb{N}$, $T^{n_0-1}x = T^{n_0}x = T(T^{n_0-1})$, then $T^n x = T^{n_0-1}x$ for $n \ge n_0$. Thus $T^{n_0-1}x$ is a fixed point of T, the sequence $\{T^n x\}$ is convergent to $T^{n_0-1}x$ and the proof is finished. So we may assume that $T^{n-1}x \ne T^n x$ for each $n \in \mathbb{N}$. Now we show that $\{T^n x\}$ is a Cauchy sequence. To prove the claim, we first show that by induction for each $n \ge 2$, we have

$$D\left(T^{n-1}x, T^n x\right) \le \left(\frac{kc}{1-kc}^{n-1}\right) D(x,Tx).$$

If $n = 2$, we get

$$\begin{aligned} D(Tx,T^2x) &\le c \max\left\{D(x,Tx), \frac{D(x,Tx)+D(Tx,T^2x)}{2},\right. \\ &\qquad \left.\frac{D(x,T^2x)+D(Tx,Tx)}{2}\right\} \\ &= c \max\left\{D(x,Tx), \frac{D(x,Tx)+D(Tx,T^2x)}{2}, \frac{D(x,T^2x)}{2}\right\} \\ &\le c \max\left\{D(x,Tx), D(x,Tx)+D(Tx,T^2x), D(x,T^2x)\right\} \\ &\le c \max\left\{D(x,Tx)+D(Tx,T^2x), k[D(x,Tx)+D(Tx,T^2x)]\right\}. \end{aligned}$$

If

$$\begin{aligned} &\max\left\{D(x,Tx)+D(Tx,T^2x), k[D(x,Tx)+D(Tx,T^2x)]\right\} \\ &\quad = D(x,Tx)+D(Tx,T^2x), \end{aligned}$$

we have $D(Tx,T^2x) \le c\left(D(x,Tx)+D(Tx,T^2x)\right)$. It is easy to check that

$$D(Tx,T^2x) \le \frac{kc}{1-kc} D(x,Tx).$$

If

$$\begin{aligned} &\max\left\{D(x,Tx)+D(Tx,T^2x), k[D(x,Tx)+D(Tx,T^2x)]\right\} \\ &\quad = k[D(x,Tx)+D(Tx,T^2x)], \end{aligned}$$

we have $D(Tx,T^2x) \le ck[D(x,Tx)+D(Tx,T^2x)]$. Thus,

$$D(Tx,T^2x) \le \frac{kc}{1-kc} D(x,Tx).$$

If $n = 3$, then we get

$$\begin{aligned} D(T^2x,T^3x) &\le c \max\left\{D(Tx,T^2x), \frac{D(Tx,T^2x)+D(T^2x,T^3x)}{2},\right. \\ &\qquad \left.\frac{D(Tx,T^3x)+D(T^2x,T^2x)}{2}\right\} \\ &\le c \max\left\{D(Tx,T^2x)+D(T^2x,T^3x), D(x,T^3x)\right\} \\ &\le c \max\left\{D(Tx,T^2x)+D(T^2x,T^3x),\right. \\ &\qquad \left. k[D(Tx,T^2x)+D(T^2x,T^3x)]\right\}. \end{aligned} \tag{20}$$

If right-hand inequality (20) equal to $D(Tx, T^2x) + D(T^2x, T^3x)$, we get

$$\begin{aligned} D(T^2x, T^3x) &\le cD(Tx, T^2x) + D(T^2x, T^3x) \\ &\le kcD(Tx, T^2x) + kcD(T^2x, T^3x). \end{aligned}$$

So, $D(T^2x, T^3x) \le \dfrac{(kc)^2}{(1-kc)^2} D(x, Tx)$. By induction, we have

$$D(T^{n-1}x, T^n x) \le \left(\frac{kc}{1-kc}\right)^{n-1} D(x, Tx).$$

Now with $m = n + s$, we obtain

$$\begin{aligned} D\left(T^n x, T^m x\right) &= D\left(T^n x, T^{n+s}x\right) \\ &= k\left[D\left(T^n x, T^{n+1}x\right) + D\left(T^{n+1}x, T^{n+s}x\right)\right] \\ &\le k\left[D\left(T^n x, T^{n+1}x\right) + k\left[D\left(T^{n+1}x, T^{n+2}x\right) + D\left(T^{n+2}x, T^{n+s}x\right)\right]\right] \\ &\le kD\left(T^n x, T^{n+1}x\right) + k^2 D\left(T^{n+1}x, T^{n+2}x\right) + k^3 D\left(T^{n+2}x, T^{n+3}x\right) \\ &\quad + k^3 D\left(T^{n+3}x, T^{n+4}x\right) \\ &\le kD\left(T^n x, T^{n+1}x\right) + k^2 D\left(T^{n+1}x, T^{n+2}x\right) \\ &\quad + \cdots + k^s D\left(T^{n+s-1}x, T^{n+s}x\right) \\ &\le k\left(\frac{kc}{1-kc}\right)^n D\left(x, Tx\right)\left[1 + k\left(\frac{kc}{1-kc}\right)\right. \\ &\quad \left. + \cdots + k^{s-1}\left(\frac{kc}{1-kc}\right)^{s-1}\right] \\ &= k\left(\frac{kc}{1-kc}\right)^n D\left(x, Tx\right)\left[\frac{\left(\frac{k^2c}{1-kc}\right)^s - 1}{\frac{k^2c}{1-kc} - 1}\right]. \end{aligned}$$

If $n \to \infty$ and $s \to \infty$, we obtain $D(T^n x, T^m x) \to 0$ and a sequence $\{T^n x\}$ is Cauchy. Since (X, D) is a complete b-metric space, there exists a $x^\circ \in X$ such that $\lim\limits_{n\to\infty} T^n x = x^\circ$,

$$D(Tx^\circ, T^n x) \le kD(Tx^\circ, x^\circ) + kD(x^\circ, T^n x),$$

then

$$\lim_{n\to\infty} \sup D(Tx^\circ, T^n x) \le kD(Tx^\circ, x^\circ) + k \lim_{n\to\infty} \sup D(x^\circ, T^n x).$$

So $\lim\limits_{n\to\infty} \sup D(Tx^\circ, T^n x) \le kD(Tx^\circ, x^\circ)$,

$$\begin{aligned} D(Tx^\circ, T^{n+1}x) \le c \max \Bigg\{ &D(x^\circ, T^n x), \frac{D(x^\circ, Tx^\circ) + D(T^n x, T^{n+1}x)}{2}, \\ &\frac{D(x^\circ, T^{n+1}x) + D(Tx^\circ, T^n x)}{2} \Bigg\}, \end{aligned}$$

so

$$\lim_{n\to\infty} \sup D(Tx^\circ, T^{n+1}x) \le cD(x^\circ, Tx^\circ).$$

We obtain $D(Tx^\circ, x^\circ) \leq cD(x^\circ, Tx^\circ)$ which yields $D(Tx^\circ, x^\circ) = 0$ and $Tx^\circ = x^\circ$. To prove the uniqueness, suppose that $x^\circ \neq x^{\circ\circ}$ such that $Tx^\circ = x^\circ$ and $Tx^{\circ\circ} = x^{\circ\circ}$. We have

$$\begin{aligned} D(x^\circ, x^{\circ\circ}) &= D\left(Tx^\circ, Tx^{\circ\circ}\right) \\ &\leq c \max \left\{ D(x^\circ, x^{\circ\circ}x), \frac{D(x^\circ, Tx^\circ) + D(x^{\circ\circ}, Tx^{\circ\circ})}{2}, \right. \\ &\left. \frac{D(x^\circ, Tx^{\circ\circ}) + D(x^{\circ\circ}, Tx^\circ)}{2} \right\} \\ &= cD\left(x^\circ, x^{\circ\circ}\right). \end{aligned}$$

Therefore $D(x^\circ, x^{\circ\circ}) = 0$ and $x^\circ = x^{\circ\circ}$. □

Example 3.1. *Let $X = [0, 1]$, $D(x, y) = |x - y|(2 + |x - y|)$ for each $x, y, \in X$ and (X, D) is a complete b-metric space with $k = 2$. Let $T : X \to X$ be defined by $Tx = \frac{1}{2}$, whenever $x = 1$ and $Tx = \frac{1}{4}$, whenevere $x \neq 1$. It is strightforward to see that for each $x, y \in X$ and following inequality*

$$D(Tx, Ty) \leq \frac{1}{2} \max \left\{ D(x, y), \frac{D(x, Tx) + D(y, Ty}{2}, \frac{D(x, Ty) + D(y, Tx}{2} \right\},$$

satisfies for all cases.

1) *If $x = y = 1$, then $Tx = Ty = \frac{1}{2}$, hence $D(Tx, Ty) = D(\frac{1}{2}, \frac{1}{2}) = 0$.*
2) *If $x = 1$ and $y \neq 1$, then $Tx = \frac{1}{2}$ and $Ty = \frac{1}{4}$, hence*

$$D(Tx, Ty) = D(1, y) = |1 - y|(2 + |1 - y|).$$

We Have

$$\begin{aligned} D(Tx, Ty) &= D(\frac{1}{4}, \frac{1}{2}) = \frac{9}{16}, D(x, Tx) = D(1, \frac{1}{2}) = \frac{5}{4}, \\ D(y, Ty) &= |\frac{1}{4} - y|(2 + |y - \frac{1}{4}|), \\ D(x, Ty) &= D(1, \frac{1}{4}) = \frac{33}{16}, \\ D(y, Tx) &= D(y, \frac{1}{2}) = |y - \frac{1}{2}(2 + |y - \frac{1}{2}). \end{aligned}$$

3) *If $x \neq 1$ and $y \neq 1$, then $Tx = Ty = \frac{1}{4}$, hence $D(Tx, Ty) = D(\frac{1}{4}, \frac{1}{4}) = 0$.*
4) *If $x \neq 1$ and $y = 1$, then $Tx = \frac{1}{4}$ and $Ty = \frac{1}{2}$, hence $D(Tx, Ty) = \frac{9}{16}$.*

$$\begin{aligned} D(x, y) &= D(x, 1) = |x - 1|(2 + |x - 1|), \\ D(x, Tx) &= D(x, \frac{1}{4}) = |x - \frac{1}{4}|(2 + |x - \frac{1}{4}|), \\ D(y, Ty) &= D(1, \frac{1}{2}) = \frac{5}{4}, \\ D(x, Ty) &= D(x, \frac{1}{2}) = |x - \frac{1}{2}|(2 + |x - \frac{1}{2}|), \\ D(y, Tx) &= D(1, \frac{1}{4}) = \frac{33}{16}. \end{aligned}$$

Definition 3.4. *Let $T : X \to X$ be a map, T is called Lipschitzian, if there exists a constant $\lambda \geq 0$ such that $D(Tx, Ty) \leq \lambda D(x, y)$, for each $x, y \in X$. The smallest constant λ will be denoted $Lip(T)$.*

In the following, we give a fixed point theorem for Lipschitzian mappings in b-metric spaces.

Corollary 3.1. *Let (X, D) be a b-metric space and $T : X \to X$ be a map satisfies*

$$D(Tx, Ty) \leq cD(x, y),$$

or

$$D(Tx, Ty) \leq c\left[\frac{D(x, Tx) + D(y, Ty)}{2}\right],$$

or

$$D(Tx, Ty) \leq c\left[\frac{D(x, Ty) + D(y, Tx)}{2}\right].$$

Then T has a unique fixed point x° and for each $x \in X$, $\lim\limits_{n\to\infty} T^n x = x^\circ$.

Proof. Since T is a quasi-contraction with $kc < k^2c < 1$, then by the proof of Theorem 3.1, we have $\lim_{n\to\infty} T^n x = x^\circ$, for each $x \in X$. Now we consider some cases. **Case I.** If $D(T^{n+1}x, Tx^\circ) \leq cD(T^n x, x^\circ)$, and $\lim\limits_{n\to\infty} D(T^n x, x^\circ) = 0$, then

$$\lim_{n\to\infty} T^{n+1}x = Tx^\circ = x^\circ.$$

Case II. If $D(T^{n+1}x, Tx^\circ) \leq c\left[\frac{D(T^n x, T^{n+1}x) + D(x^\circ, Tx^\circ)}{2}\right]$, but $\lim\limits_{n\to\infty} T^{n+1}x = x^\circ$. Therefore

$$\begin{aligned}\lim_{n\to\infty} D(T^{n+1}x, Tx^\circ) &\leq \frac{c}{2}\left[\lim_{n\to\infty} D(T^n x, T^{n+1}x) + \lim_{n\to\infty} \frac{D(x^\circ, Tx^\circ)}{2}\right]\\ &= \frac{c}{2} D(x^\circ, Tx^\circ).\end{aligned}$$

Thus $D(x^\circ, Tx^\circ) \leq \frac{c}{2} D(x^\circ, Tx^\circ)$. We conclude that $D(x^\circ, Tx^\circ) = 0$ and $Tx^\circ = x^\circ$.
Case III. If $D(Tx, Ty) \leq \frac{c}{2}\left[D(x, Ty) + D(y, Tx)\right]$, then

$$\begin{aligned}D(T^{n+1}x, Tx^\circ) &\leq \frac{c}{2}\left[D(T^n x, x^\circ) + D(x^\circ, T^{n+1}x)\right]\\ \lim_{n\to\infty} D(T^{n+1}x, Tx^\circ) &\leq \frac{c}{2}\left[\lim_{n\to\infty} D(T^n x, x^\circ) + \lim_{n\to\infty} D(x^\circ, T^{n+1}x)\right].\end{aligned}$$

So $D(x^\circ, Tx^\circ) \leq \frac{c}{2} D(x^\circ, Tx^\circ)$. Hence $D(x^\circ, Tx^\circ) = 0$ and $Tx^\circ = x^\circ$. □

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