# MINIMAL MEAN KINETIC ENERGY MODIFICATION OF THE DESIGN DEFECT FOR PITCH CURVE WITH CONCAVE CUSPS OF *N*-LOBE NONCIRCULAR GEAR

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*A modification model with minimal mean kinetic energy characteristic is established of the design defect of pitch curve with concave cusps. The modification model can be solved via the Euler-Lagrange Equation, and the unified algorithm of minimal mean kinetic energy modification for the pitch curve with concave cusps can be developed. Meanwhile, the changing situation of modified transmission ratio can be investigated, and it shows that the design defect can be modified within the smaller transmission error range. Numerical example illustrates the validity and feasibility of the proposed modification model and algorithm.*

**Keywords**: Noncircular gear, pitch curve, concave cusps, design defect, modification model.

## 1. Introduction

Due to the complexity of manufacturing and the stability of transmission for noncircular gears mainly depend on the pitch curve, therefore, the innovative design of pitch curve with noncircular characteristic became the subject of study for many researchers. Elliptical gears, for example, were generated by conjugate ellipses as the pitch curves of gears, whose rotation centers are coinciding with their foci [1-3]. In particularly, an elliptic interpolation algorithm for elliptical gears on the basis of the principle of central angle division of an arc was developed by Hu [4]. In order to increase the number of speed cycles per revolution, a general generation method of *N*-lobe or high-order elliptical gears from a basic ellipse and their rack by means of a conjugate shaper cutter with an involute tooth profile were proposed in Refs. [5-9]. In addition, the transmission mathematical models among cutter, driving and driven elliptical bevel gears was established by Figliolini and Angeles [10], and the design of noncircular bevel gears with concave pitch curves was discussed by Shi and co-workers [11].

More recently, some novel pitch curves for noncircular gears were proposed to expand its applications in automatic machines and instruments. A general method for generating pitch curves of *N*-lobe noncircular gears based on Bézier and B-spline nonparametric curves was developed by Hector and co-workers [12]. The pitch curves of *N*-lobe noncircular gears can be generated by

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plane regular $N$-curved polygon or spiral of Archimedes [13]. Noncircular gears with steepest rotation pitch curves can be synthesized, and this research is applied to modify the pitch curve with discontinuity points [14-15]. The Archimedes spiral, quadratic curve and Pascal curve were also applied in pitch surface design for noncircular bevel gear [16]. Moreover, the design methods of pitch curves with minimal rotary inertia and minimal mean kinetic energy characteristics are proposed in Ref. [17] and Ref. [18], respectively.

Due to the presence of cusps, the stability of transmission is reduced and the universality of application is limited for $N$-lobe noncircular gears with discretionary pitch curves. In order to improve the stability of transmission and expand the application of engineering for $N$-lobe noncircular gears ($N$-LNGs), a minimal mean kinetic energy modification model of pitch curve with concave cusps for $N$-LNG is established according to the idea of the Ref. [18]. Meanwhile, the general formulation and modification algorithm of the pitch curve with concave cusps are obtained based on the model. Numerical example is shown to validate the proposed modification model and algorithm.

## 2. Minimal mean kinetic energy modification model of the pitch curve with concave cusps for *N*-LNG

Closed $N$-lobe noncircular gear pitch curve $r(\eta)$ with concave cusps is shown in Fig. 1(a), points $c_1$, $c_2$, ..., $c_N$ are concave cusps of the pitch curve $r(\eta)$, the $x$-axis of fixed coordinate system $\Gamma(o\text{-}xy)$ is coincide with the straight line $oc_N$, polar angle $\eta$ is measured counterclockwise from the positive direction of $x$-axis. From Fig. 1(a), we know that these concave cusps have a certain rotated period $P$ and $P$=2π/$N$, then the pitch curve $r(\eta)$ can be expressed as [15]

$$r(\eta)=\begin{cases} r_1(\eta),\ \eta\in[0,2\pi/N] \text{ and } r_1(0)=r_1(2\pi/N) \\ r_k(\eta)=r_1(\eta-\dfrac{2(k-1)\pi}{N}),\ \eta\in[\dfrac{2(k-1)\pi}{N},\dfrac{2k\pi}{N}] \end{cases} \tag{1}$$

where $N$ is the number of lobes for pitch curve r($\eta$) and $N$ is an integer, $k$=1,2,...,$N$.

Referring to Fig. 1(b), in order to improve the manufacturing and simplify the calculation, assuming the modification pitch curve $r_{m1}(\eta)$ can be used to replace two infinitesimal pitch curves $\overset{\frown}{a_1c_1}$ and $\overset{\frown}{c_1b_1}$ around the concave cusp $c_1$, points $a_1$ and $b_1$ are tangency points that the modification pitch curve $r_{m1}(\eta)$ makes with pitch curve $r_1(\eta)$ and $r_2(\eta)$, respectively. Therefore, the modification pitch curve $r_m(\eta)$ can be expressed as

$$r_m(\eta)=\begin{cases} r_{m1}(\eta),\ \eta\in[\eta_{a1},\eta_{b1}] \\ r_{mk}(\eta)=r_{m1}(\eta-\dfrac{2(k-1)\pi}{N}),\ \eta\in[\eta_{a1}+\dfrac{2(k-1)\pi}{N},\eta_{b1}+\dfrac{2k\pi}{N}] \end{cases} \tag{2}$$

where $\eta_{a1}$ and $\eta_{b1}$ are the polar angles of points $a_1$ and $b_1$, respectively.

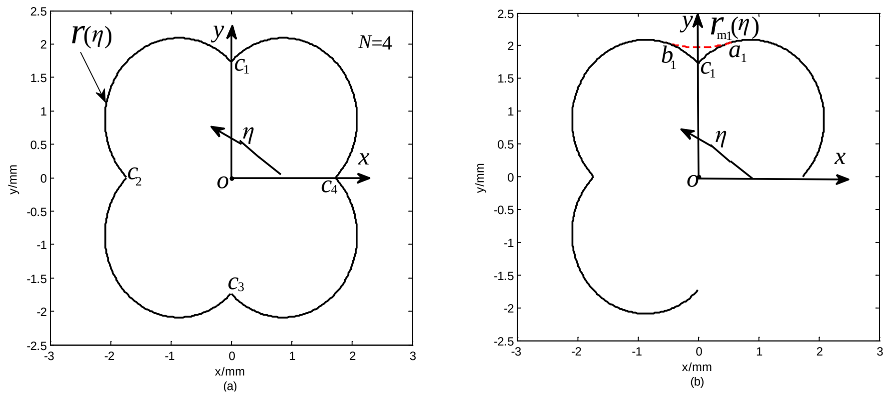


Fig. 1 Pitch curve with concave cusps and its modification model for *N*-lobe noncircular gear

Assuming the mass of noncircular gear is $Q$, then the minimum $E_{\min}$ of mean kinetic energy $E$ of noncircular gear through the modification pitch curve $r_{m1}(\eta)$ at the given angular velocity $\omega$ from $t_0=0$ to $t_1=T$ can be expressed as [18]

$$\begin{cases} E_{\min}=\min\int_{\eta_{a1}}^{\eta_{b1}} F d\eta=\min\int_{\eta_{a1}}^{\eta_{b1}} \dfrac{Q\omega r_{m1}(\eta)\sqrt{r_{m1}^2(\eta)+r_{m1}'^2(\eta)}}{2\int_{\eta_{a1}}^{\eta_{b1}}\dfrac{\sqrt{r_{m1}^2(\eta)+r_{m1}'^2(\eta)}}{\omega r_{m1}(\eta)}d\eta}d\eta \\ s.t.\ T=\int_{\eta_{a1}}^{\eta_{b1}} dt(\eta)=\int_{\eta_{a1}}^{\eta_{b1}}\dfrac{\sqrt{r_{m1}^2(\eta)+r_{m1}'^2(\eta)}}{\omega r_{m1}(\eta)}d\eta \end{cases} \tag{3}$$

where $T$ is rotation time that noncircular gear through the modification pitch curve $r_{m1}(\eta)$ at the given angular velocity $\omega$ from $\eta=\eta_{a1}$ to $\eta=\eta_{b1}$. Parameter $\omega$ is given angular velocity of the noncircular gear.

According to the solution method of the above variational problem [18-19], the minimal mean kinetic energy modification model for pitch curve with concave cusps of *N*-LNG can be expressed as

$$\begin{cases} r_{m1}(\eta)=\sqrt{\dfrac{h_1}{\cos(2\eta-h_2)}},\ \eta\in[\eta_{a1},\eta_{b1}] \\ s.t.\begin{cases} r_{m1}(\eta_{a1})=r_1(\eta_{a1}),\ r_{m1}(\eta_{b1})=r_2(\eta_{b1}) \\ r_{m1}'(\eta_{a1})=r_1'(\eta_{a1}),\ r_{m1}'(\eta_{b1})=r_2'(\eta_{b1}) \end{cases} \end{cases} \tag{4}$$

where $h_1$, $h_2$ are undetermined parameters.

### 3. Model analysis and algorithm design

According to the requirements of noncircular gear pitch curve, we know that the value of pitch curve $r_{m1}(\eta)$ is greater than zero. Therefore, Eq. (4) should satisfy the following constraint

$$\begin{cases} h_1 > 0 \\ 2\eta - \dfrac{\pi}{2} < h_2 < 2\eta + \dfrac{\pi}{2},\ \eta \in [\eta_{a1}, \eta_{b1}] \end{cases} \tag{5}$$

Furthermore, according to the characteristic of concave cusp, we know that the first-order derivatives $r'_1(\eta_{a1})$ and $r'_2(\eta_{b1})$ around the concave cusp $c_1$ should satisfy the following conditions [15, 17-18]

$$\tan(2\eta_{a1} - h_2) < 0,\ \tan(2\eta_{b1} - h_2) > 0 \tag{6}$$

Along with Eq. (5), the constraints of undetermined parameters $h_1$, $h_2$ can be obtained

$$\begin{cases} h_1 > 0 \\ 2\eta_{a1} < h_2 < 2\eta_{a1} + \dfrac{\pi}{2} \text{ and } 2\eta_{b1} - \dfrac{\pi}{2} < h_2 < 2\eta_{b1} \end{cases} \tag{7}$$

Along with Eq. (4), undetermined parameters $h_1$, $h_2$ and polar angles $\eta_{a1}$, $\eta_{b1}$ can be obtained by the following conditions

$$\begin{cases} \sqrt{\dfrac{h_1}{\cos(2\eta_{a1} - h_2)}} = r_1(\eta_{a1}),\ \sqrt{\dfrac{h_1}{\cos(2\eta_{b1} - h_2)}} = r_2(\eta_{b1}) \\ \tan(2\eta_{a1} - h_2)\sqrt{\dfrac{h_1}{\cos(2\eta_{a1} - h_2)}} = r'_1(\eta_{a1}),\ \tan(2\eta_{b1} - h_2)\sqrt{\dfrac{h_1}{\cos(2\eta_{b1} - h_2)}} = r'_2(\eta_{b1}) \end{cases} \tag{8}$$

Along with Eq. (1), the modified pitch curve $R(\eta)$ with minimal mean kinetic energy characteristic for the pitch curve with concave cusps is shown in Fig. 2, and its polar equation $R(\eta)$ can be expressed as

$$R(\eta) = \begin{cases} \begin{cases} r_1(\eta), \eta \in [\eta_{bN}, \eta_{a1}] \\ r_{m1}(\eta) = \sqrt{\dfrac{h_1}{\cos(2\eta - h_2)}}, \eta \in [\eta_{a1}, \eta_{b1}] \end{cases}, k = 1 \\ \begin{cases} r_k(\eta) = r_1(\eta - \dfrac{2(k-1)}{N}\pi), \eta \in [\eta_{b(k-1)}, \eta_{ak}] \\ r_{mk}(\eta) = r_{m1}(\eta - \dfrac{2(k-1)}{N}\pi), \eta \in [\eta_{ak}, \eta_{bk}] \end{cases}, \begin{cases} k = 2, 3, \ldots N \\ \eta_{ak} = \eta_{a1} + \dfrac{2(k-1)\pi}{N}, \\ \eta_{bk} = \eta_{b1} + \dfrac{2(k-1)\pi}{N} \end{cases} \end{cases} \tag{9}$$

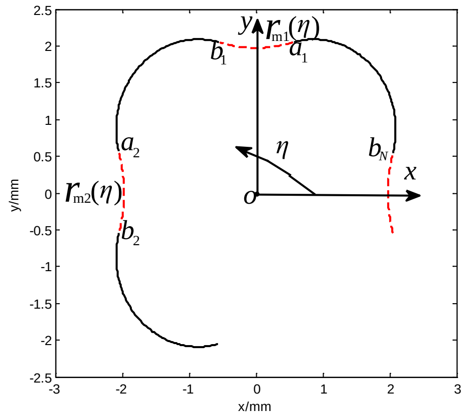


Fig. 2 Sketch of the modified pitch curve $R(\eta)$ for the pitch curve with concave cusps

Furthermore, the pitch curve $R_e(\eta_e)$ conjugated with the modified pitch curve $R(\eta)$ for $N$-LNGs can be expressed [13-18]

$$\begin{cases} R_e(\eta_e) = a_e \pm R(\eta) \\ \eta_e = \int_0^{\eta} \dfrac{R(\eta)}{a_e \pm R(\eta)} d\eta \end{cases} \tag{10}$$

where the '+' of symbol '±' in Eq. (10) is applies to the internal meshing noncircular gears, the '-' of symbol '±' in Eq. (10) is applies to the external meshing noncircular gears.

## 4. Modification examples

Pitch curve with concave cusps $r(\eta)$ of 2-lobed noncircular driving gear satisfied the transmission ratio $i(\eta)$ of Eq. (12) can be expressed as

$$r(\eta) = \begin{cases} r_1(\eta) = \sqrt{3}\sin(\eta) + 3, \eta \in [0, \pi] \\ r_2(\eta) = r_1(\eta - \pi), \eta \in [\pi, 2\pi] \end{cases} \tag{11}$$

$$i(\eta) = \frac{r_1(\eta)}{8.3330 - r_1(\eta)}, \eta \in [0, \pi] \tag{12}$$

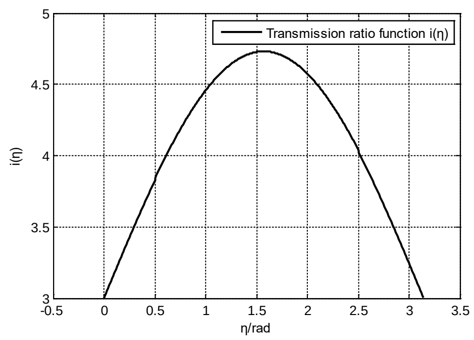


Fig. 3 Transmission ratio function $i(\eta)$ of Eq. (12)

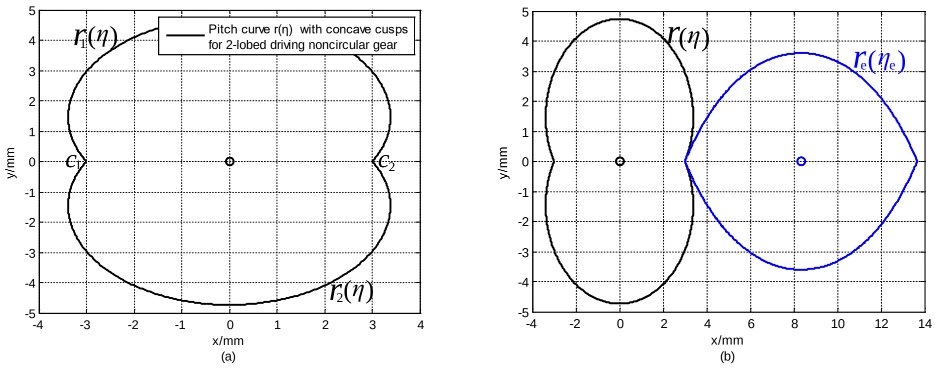


Fig. 4 Pitch curve $r(\eta)$ with concave cusps for 2-lobed driving noncircular gear

Fig. 3 and Fig. 4 are the transmission ratio function $i(\eta)$ of Eq. (12) and the pitch curve $r(\eta)$ with concave cusps for 2-lobed noncircular driving gear and its conjugate external meshing noncircular driven gear pitch curve $r_e(\eta_e)$. Fig. 5(a) and Fig. 5(b) are the modified pitch curve $R(\eta)$ of Eq. (13) and its conjugate external meshing noncircular gear pitch curve $R_e(\eta_e)$, and the corresponding modification parameters can be obtained by the proposed minimal mean kinetic energy modification algorithm and they are listed in Table 1.

$$R(\eta)=\begin{cases} r_1(\eta)=\sqrt{3}\sin(\eta)+3, \eta\in[3.3721-\pi, 2.9111] \\ r_{m1}(\eta)=\sqrt{\dfrac{10.3271}{\cos(2\eta)}}, \eta\in[2.9111, 3.3721] \\ r_2(\eta)=r_1(\eta-\pi), \eta\in[3.3721, 2.9111+\pi] \\ r_{m2}(\eta)=r_{m1}(\eta-\pi), \eta\in[2.9111+\pi, 3.3721+\pi] \end{cases} \tag{13}$$

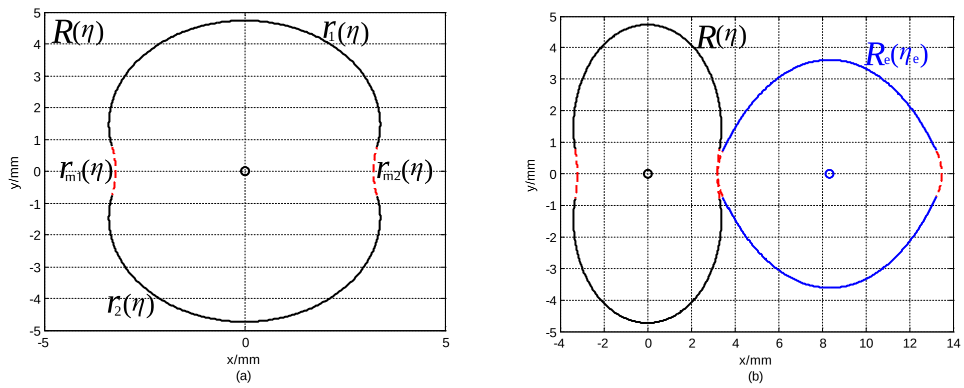


Fig. 5 Modified pitch curve $R(\eta)$ and its conjugate external meshing noncircular gear pitch curve $R_e(\eta_e)$

*Table 1*

**Modification parameters of 2-lobed driving noncircular gear pitch curve $r(\eta)$ with concave cusps depicted in Fig. 4(a)**

| Modified pith curve | Polar angles $\eta_{a1}$, $\eta_{b1}$ | Undetermined constants $h_1$, $h_2$ | External meshing |
|---|---|---|---|
| $R(\eta)$ (Referring to Fig. 5(a) and Eq. (13)) | $\begin{cases}\eta_{a1}=2.9111\\ \eta_{b1}=3.3721\end{cases}$ | $\begin{cases}h_1=10.3271\\ h_2=2\pi\end{cases}$ | $\begin{cases}R_e(\eta_e)=8.3476-R(\eta)\\ \eta_e=\int_0^{\eta}\frac{R(\eta)}{8.3476-R(\eta)}d\eta\end{cases}$ (Referring to Fig. 5(b)) |

$$i_m(\eta)=\begin{cases}\dfrac{r_{m2}(\eta)}{8.3476-r_{m2}(\eta)}, & \eta\in[0,3.3721-\pi]\\ \dfrac{r_1(\eta)}{8.3476-r_1(\eta)}, & \eta\in[3.3721-\pi,2.9111]\\ \dfrac{r_{m1}(\eta)}{8.3476-r_{m1}(\eta)}, & \eta\in[2.9111,\pi]\end{cases} \tag{14}$$

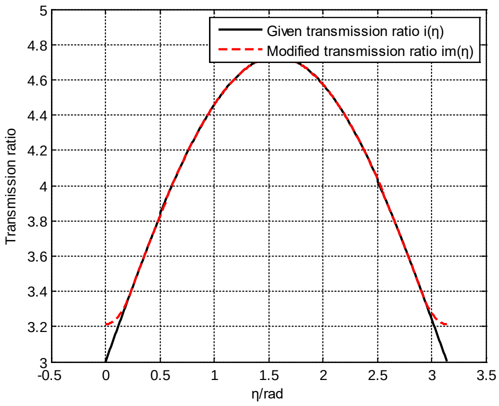


Fig. 6 The changing situation of modified transmission ratio $i_m(\eta)$ of Eq. (14)

Changing situation of the modified transmission ratio $i_m(\eta)$ ($\eta\in[0,\pi]$) of Eq. (14) is depicted in Fig. 6, and it shows that the modified transmission ratio $i_m(\eta)$ approximates to the given transmission ratio $i(\eta)$ and the design defect of pitch curve with concave cusps can be modified within the smaller transmission error range according to the proposed minimal mean kinetic energy modification model.

## 5. Conclusions

The mathematical model of the minimal mean kinetic energy modification for pitch curve with concave cusps of *N*-LNG has been proposed in this paper. The corresponding modification algorithm has been developed based on the mathematical model. Comparison of the modified and given transmission ratio shows that the design defect of pitch curve with concave cusps for *N*-LNG can be modified within the smaller transmission error range. Meanwhile, the feasibility and validity of proposed minimal mean kinetic energy modification model have been verified by the illustrated example. This research should be helpful in the advancement of design and manufacture of *N*-lobed noncircular gears.

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