WEAK AND STRONG CONVERGENCE THEOREMS OF ITERATIVE PROCESS FOR TWO FINITE FAMILIES OF MAPPINGS

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In this paper, weak and strong convergence of two-step iteration sequences to a common fixed point for a pair of a finite family of asymptotically nonexpansive mappings and a finite family of generalized nonexpansive multivalued mappings in a nonempty closed convex subset of uniformly convex Banach spaces are presented. The results obtained in this paper extend and improve some recent known results.

Keywords: Two-step iterative process, Common fixed point, Generalized nonexpansive multivalued mapping, Asymptotically nonexpansive mappings, Weak convergence.

MSC2010: 47H10, 47H09.

1. Introduction

Fixed point iteration process for nonexpansive and asymptotically nonexpansive singlevalued mappings in Hilbert spaces and Banach spaces has been studied extensively by many authors to solve nonlinear operator equation as well as variational inequalities, see e.g. [1-8] and the works referred there.

In recent years, approximation of fixed point of nonexpansive multivalued mappings by iteration has been studied by many authors, (see [9-19]). The theory of multivalued mappings has applications in control theory, convex optimization, differential equations and economics. Recently, Eslamian and Abkar [20] proved the existence of common fixed point for a pair consisting of a generalized nonexpansive multivalued mapping and an asymptotically nonexpansive mapping in uniformly convex Banach spaces. The purpose of this paper is to introduce a two-step iterative process for approximating the common fixed points of a pair consisting of a finite family of asymptotically nonexpansive mappings and a finite family of generalized nonexpansive multivalued mappings and then prove weak and strong convergence theorems for such iterative process in uniformly convex Banach spaces. The results obtained in this paper extend and improve some recent known results.

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2. **Preliminaries**

Recall that a Banach space X is said to be uniformly convex if for each $t \in [0,2]$, the modulus of convexity of X given by:

$$\delta(t) = inf\{1 - \frac{1}{2}\|x+y\| : \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq t\}$$

satisfies the inequality $\delta(t) > 0$ for all $t > 0$. A Banach space X is said to satisfy Opial's condition if $x_n \to z$ weakly as $n \to \infty$ and $z \neq y$ imply that

$$\lim sup_{n\to\infty}\|x_n - z\| < \lim sup_{n\to\infty}\|x_n - y\|.$$

All Hilbert spaces, all finite dimensional Banach spaces and $\ell^p (1 \leq p < \infty)$ have the Opial property.

A subset $D \subset X$ is called proximinal if for each $x \in X$, there exists an element $y \in D$ such that

$$\| x - y \| = dist(x, D) = inf\{\| x - z \| : z \in D\}.$$

We denote by $CB(D), K(D)$ and $P(D)$ the collection of all nonempty closed bounded subsets, nonempty compact subsets, and nonempty proximinal bounded subsets of D respectively. The Hausdorff metric H on $CB(X)$ is defined by

$$H(A,B) := \max\{\sup_{x\in A} dist(x,B), \sup_{y\in B} dist(y,A)\},$$

for all $A, B \in CB(X)$.
Let $T : X \to 2^X$ be a multivalued mapping. An element $x \in X$ is said to be a fixed point of T, if $x \in Tx$. The set of fixed points of T will be denoted by $F(T)$.

Definition 2.1. *A multivalued mapping $T : X \to CB(X)$ is called*

(i) *nonexpansive if*

$$H(Tx, Ty) \leq \|x - y\|, \quad x, y \in X.$$

(ii) *quasi nonexpansive if $F(T) \neq \emptyset$ and $H(Tx, Tp) \leq \| x - p \|$ for all $x \in X$ and all $p \in F(T)$.*

In [21], Garcia-Falset, Llorens-Fuster and Suzuki, introduced a new condition on singlevalued mappings, called condition (E), which is weaker than nonexpansiveness. Very recently, Abkar and Eslamian [22] used a modified condition for multivalued mappings as follows:

Definition 2.2. *A multivalued mapping $T : X \to CB(X)$ is said to satisfy condition (E_μ) provided that*

$$dist(x, Ty) \leq \mu\, dist(x, Tx) + \|x - y\|, \quad x, y \in X.$$

We say that T satisfies condition (E) whenever T satisfies (E_μ) for some $\mu \geq 1$.

Lemma 2.1. *Let $T : X \to CB(X)$ be a multivalued nonexpansive mapping, then T satisfies the condition (E_1).*

Lemma 2.2. *([19]) Let X be a uniformly convex Banach space and let $B_r(0) = \{x \in X : \| x \| \leq r\}$, for $r > 0$. Then there exists a continuous, strictly increasing convex function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(0) = 0$ such that*

$$\|a_1x_1 + a_2x_2 + \cdots + a_kx_k\|^2 \leq a_1\|x_1\|^2 + a_2\|x_2\|^2 + \cdots + a_k\|x_k\|^2 - a_ia_j\varphi(\|x_i - x_j\|)$$

for all $x_i, x_j \in B_r(0)$ *and* $a_i, a_j \in [0,1]$ *for* $i, j = 1, 2, \cdots, k$ *and* $\sum_{i=1}^{k} a_i = 1$.

Lemma 2.3. *([7]) Let* $\{a_n\}, \{b_n\}$ *and* $\{\delta_n\}$ *be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1+\delta_n)a_n + b_n.$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ *and* $\sum_{n=1}^{\infty} b_n < \infty$*, then* $\lim_{n\to\infty} a_n$ *exists. In particular , if* $\{a_n\}$ *has a subsequence converging to 0, then* $\lim_{n\to\infty} a_n = 0$.

Definition 2.3. *([23]) Let* D *be a nonempty convex subset of a Banach space* X*. A map* $f : D \to D$ *is called asymptotically nonexpansive if there exists a sequence* $\{k_n\}$ *of real numbers with* $\{k_n\} \geq 1$ *and* $lim_{n\to\infty} k_n = 1$ *such that*

$$\|f^n x - f^n y\| \leq k_n \|x - y\|$$

for all $x, y \in D$ *and all* $n \geq 1$.

Lemma 2.4. *([9]) Let* X *be a uniformly convex Banach space,* D *be a nonempty closed convex subset of* X*, and* $f : D \to D$ *be an asymptotically nonexpansive mapping. If* $x_n \to x$ *weakly and* $x_n - fx_n \to 0$ *strongly, then* $x = fx$.

3. Main results

We are interested in sequences in the following process.
(A): Let X be a Banach space, D be a nonempty convex subset of X and $T_i : D \to CB(D)$ and $f_i : D \to D$ $(i = 1, 2, ..., m)$ be finite given mappings. Then, for $x_0 \in D$, we consider the following iterative process:

$$y_n = b_{n,0}x_n + b_{n,1}z_{n,1} + b_{n,2}z_{n,2} + \cdots + b_{n,m}z_{n,m}, \qquad n \geq 0,$$

$$x_{n+1} = a_{n,0}x_n + a_{n,1}f_1^n y_n + a_{n,2}f_2^n y_n + \cdots + a_{n,m}f_m^n y_n, \qquad n \geq 0,$$

where $z_{n,i} \in T_i(x_n)$ and $\{a_{n,k}\}$, $\{b_{n,k}\}$ are sequences of numbers in [0,1] such that for every natural number n

$$\sum_{k=0}^{m} a_{n,k} = \sum_{k=0}^{m} b_{n,k} = 1.$$

Definition 3.1. *A mapping* $T : D \to CB(D)$ *is said to satisfy condition (I) if there is a non decreasing function* $g : [0,\infty) \to [0,\infty)$ *with* $g(0) = 0$*,* $g(r) > 0$ *for* $r \in (0,\infty)$ *such that*

$$dist(x, T(x)) \geq g(dist(x, F(T))).$$

A family $\{T_i : D \to CB(D), \quad i = 1, ..., m\}$ *is said to satisfy condition (II) if there is a non decreasing function* $g : [0,\infty) \to [0,\infty)$ *with* $g(0) = 0$*,* $g(r) > 0$ *for* $r \in (0,\infty)$ *such that for some* $i = 1, .., m$*,*

$$dist(x, T_i(x)) \geq g(dist(x, \bigcap_{i=1}^{m} F(T_i))).$$

In this sequel, $\mathcal{F} = \bigcap_{i=1}^{m}(F(T_i) \bigcap F(f_i))$ *is the set of all common fixed points of the mappings* f_i *and* T_i *for* $i = 1, 2, \cdots, m$.

Theorem 3.1. *Let D be a nonempty closed convex subset of a uniformly convex Banach space X. Let $T_i : D \to CB(D)$, $(i = 1, 2, \cdots, m)$ be a finite family of quasi-nonexpansive multivalued mappings, and $f_i : D \to D$, $(i = 1, 2, \cdots, m)$ be a finite family of asymptotically nonexpansive mappings with sequence $\{k_{n,i}\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, where $k_n = max\{k_{n,i}; 1 \le i \le m\}$. Assume that $\mathcal{F} \neq \emptyset$ and $T_i(p) = \{p\}, (i = 1, 2, \cdots, m)$ for each $p \in \mathcal{F}$. Let $\{x_n\}$ be the iterative process defined by (A), and $a_{n,k}, b_{n,k} \in [a, 1) \subset (0, 1)$ for $k = 0, 1, \cdots, m$. Then*

(i) $\lim_{n\to\infty} \|x_n - p\|$ *exists for all* $p \in \mathcal{F}$,

(ii) $\lim_{n\to\infty} dist(T_i(x_n), x_n) = \lim_{n\to\infty} \|f_i x_n - x_n\| = 0$, $(i = 1, ..., m.)$

Proof. Let $p \in \mathcal{F}$. Then, we have

$$\begin{aligned}
\|y_n - p\| &= \|b_{n,0}x_n + b_{n,1}z_{n,1} + b_{n,2}z_{n,2} + \cdots + b_{n,m}z_{n,m} - p\| \\
&\le b_{n,0}\|x_n - p\| + b_{n,1}\|z_{n,1} - p\| + b_{n,2}\|z_{n,2} - p\| + ... + b_{n,m}\|z_{n,m} - p\| \\
&= b_{n,0}\|x_n-p\|+b_{n,1}dist(z_{n,1}, T_1(p))+b_{n,2}dist(z_{n,2}, T_2(p))+...+b_{n,m}dist(z_{n,m}, T_m(p)) \\
&\le b_{n,0}\|x_n-p\|+b_{n,1}H(T_1(x_n), T_1(p))+b_{n,2}H(T_2(x_n), T_2(p))+...+b_{n,m}H(T_m(x_n), T_m(p)) \\
&\le b_{n,0}\|x_n - p\| + b_{n,1}\|x_n - p\| + b_{n,2}\|x_n - p\| + ... + b_{n,m}\|x_n - p\| = \|x_n - p\|.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|x_{n+1} - p\| &= \|a_{n,0}x_n + a_{n,1}f_1^n y_n + a_{n,2}f_2^n y_n + ... + a_{n,m}f_m^n y_n - p\| \\
&\le a_{n,0}\|x_n - p\| + a_{n,1}\|f_1^n y_n - p\| + a_{n,2}\|f_2^n y_n - p\| + ... + a_{n,m}\|f_m^n y_n - p\| \\
&\le a_{n,0}\|x_n - p\| + a_{n,1}k_{n,1}\|y_n - p\| + a_{n,2}k_{n,2}\|y_n - p\| + ... + a_{n,m}k_{n,m}\|y_n - p\| \\
&\le a_{n,0}\|x_n - p\| + a_{n,1}k_n\|x_n - p\| + a_{n,2}k_n\|x_n - p\| + ... + a_{n,m}k_n\|x_n - p\| \\
&\le (1 + (k_n - 1))\|x_n - p\|.
\end{aligned}$$

Since $\sum_{n=1}^{\infty}(k_n-1) < \infty$, by lemma 2.5 we have $\lim_{n\to\infty} \|x_n-p\|$ exists for all $p \in \mathcal{F}$. Since the sequences $\{x_n\}$ and $\{y_n\}$ are bounded, we can find $r > 0$ depending on p such that $x_n - p, y_n - p \in B_r(0)$ for all $n \ge 0$. From Lemma 2.4, for $i = 1, 2, ..., m$, we obtain that

$$\begin{aligned}
\|y_n - p\|^2 &= \|b_{n,0}x_n + b_{n,1}z_{n,1} + b_{n,2}z_{n,2} + ... + b_{n,m}z_{n,m} - p\|^2 \\
&\le b_{n,0}\|x_n - p\|^2 + b_{n,1}\|z_{n,1} - p\|^2 + b_{n,2}\|z_{n,2} - p\|^2 + ... \\
&\quad + b_{n,m}\|z_{n,m} - p\|^2 - b_{n,0}b_{n,i}\varphi(\|x_n - z_{n,i}\|) \\
&= b_{n,0}\|x_n - p\|^2 + b_{n,1}dist(z_{n,1}, T_1(p))^2 + b_{n,2}dist(z_{n,2}, T_2(p))^2 + ... \\
&\quad + b_{n,m}dist(z_{n,m}, T_m(p))^2 - b_{n,0}b_{n,i}\varphi(\|x_n - z_{n,i}\|) \\
&\le b_{n,0}\|x_n - p\| + b_{n,1}H(T_1(x_n), T_1(p))^2 + b_{n,2}H(T_2(x_n), T_2(p))^2 + ... \\
&\quad + b_{n,m}H(T_m(x_n), T_m(p))^2 - b_{n,0}b_{n,i}\varphi(\|x_n - z_{n,i}\|) \\
&\le b_{n,0}\|x_n - p\|^2 + b_{n,1}\|x_n - p\|^2 + b_{n,2}\|x_n - p\|^2 + ... \\
&\quad + b_{n,m}\|x_n - p\|^2 - b_{n,0}b_{n,i}\varphi(\|x_n - z_{n,i}\|) \\
&= \|x_n - p\|^2 - b_{n,0}b_{n,i}\varphi(\|x_n - z_{n,i}\|).
\end{aligned}$$

It follows from Lemma 2.4 that

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \|a_{n,0}x_n + a_{n,1}f_1^n y_n + a_{n,2}f_2^n y_n + ... + a_{n,m}f_m^n y_n - p\|^2 \\
&\leq a_{n,0}\|x_n - p\|^2 + a_{n,1}\|f_1^n y_n - p\|^2 + a_{n,2}\|f_2^n y_n - p\|^2 + ... \\
&\quad + a_{n,m}\|f_m^n y_n - p\|^2 - a_{n,0}a_{n,i}\varphi(\|x_n - f_i^n y_n\|) \\
&\leq a_{n,0}\|x_n - p\|^2 + a_{n,1}k_{n,1}^2\|y_n - p\|^2 + a_{n,2}k_{n,2}^2\|y_n - p\|^2 + ... \\
&\quad + a_{n,m}k_{n,m}^2\|y_n - p\|^2 - a_{n,0}a_{n,i}\varphi(\|x_n - f_i^n y_n\|) \\
&\leq a_{n,0}\|x_n - p\|^2 + a_{n,1}k_n^2\|x_n - p\|^2 + a_{n,2}k_n^2\|x_n - p\|^2 + ... + a_{n,m}k_n^2\|x_n - p\|^2 \\
&\quad - a_{n,0}a_{n,i}\varphi(\|x_n - f_i^n y_n\|) - a_{n,i}b_{n,0}b_{n,i}k_n^2\varphi(\|x_n - z_{n,i}\|) \\
&\leq (1 + (k_n^2 - 1))\|x_n - p\|^2 - a_{n,0}a_{n,i}\varphi(\|x_n - f_i^n y_n\|) - a_{n,i}b_{n,0}b_{n,i}k_n^2\varphi(\|x_n - z_{n,i}\|).
\end{aligned}$$

So we have

$$\begin{aligned}
a^3\varphi(\|x_n - z_{n,i}\|) &\leq a_{n,i}b_{n,0}b_{n,i}k_n^2\varphi(\|x_n - z_{n,i}\|) \\
&\leq \| x_n - p \|^2 - \| x_{n+1} - p \|^2 + (k_n^2 - 1))\|x_n - p\|^2,
\end{aligned}$$

which implies that

$$\sum_{n=1}^{\infty} a^3\varphi(\|x_n - z_{n,i}\|) \leq \|x_1 - p\|^2 + \sum_{n=1}^{\infty}(k_n^2 - 1))\|x_n - p\|^2 < \infty,$$

from which it follows that $\lim_{n\to\infty}\varphi(\|x_n - z_{n,i}\|) = 0$. Similarly it can be shown that $\lim_{n\to\infty}\varphi(\|x_n - f_i^n y_n)\| = 0$. From Lemma 2.4, since φ is continuous at 0 and is strictly increasing, we have

$$\lim_{n\to\infty}\|x_n - z_{n,i}\| = \lim_{n\to\infty}\|x_n - f_i^n y_n\| = 0.$$

Moreover for $i = 1, 2, ..., m$, we get $dist(x_n, T_i x_n) \leq \|x_n - z_{n,i}\| \to 0$ as $n \to \infty$. Using (A) we have

$$\lim_{n\to\infty}\|y_n - x_n\| = \lim_{n\to\infty}(b_{n,1}\|z_{n,1} - x_n\| + b_{n,2}\|z_{n,2} - x_n\| + ... + b_{n,m}\|z_{n,m} - x_n\|) = 0,$$

and

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}(a_{n,1}\|f_1^n y_n - x_n\| + a_{n,2}\|f_2^n y_n - x_n\| + ... + a_{n,m}\|f_m^n y_n - x_n\|) = 0.$$

Also we have

$$\begin{aligned}
\|x_n - f_i^n x_n\| &\leq \|x_n - f_i^n y_n\| + \|f_i^n y_n - f_i^n x_n\| \\
&\leq \|x_n - f_i^n y_n\| + k_n\|y_n - x_n\| \to 0 \quad n \to \infty,
\end{aligned}$$

and

$$\begin{aligned}
\|x_{n+1} - f_i^n x_{n+1}\| &\leq \|x_{n+1} - x_n\| + \|x_n - f_i^n x_n\| + \|f_i^n x_{n+1} - f_i^n x_n\| \\
&\leq \|x_{n+1} - x_n\| + \|x_n - f_i^n x_n\| + k_n\|x_{n+1} - x_n\| \to 0.
\end{aligned}$$

Hence

$$\begin{aligned}
\|x_{n+1} - f_i x_{n+1}\| &\leq \|x_{n+1} - f_i^{n+1}x_{n+1}\| + \|f_i^{n+1}x_{n+1} - f_i x_{n+1}\| \\
&\leq \|x_{n+1} - f_i^{n+1}x_{n+1}\| + k_1\|f_i^n x_{n+1} - x_{n+1}\| \to 0 \qquad as \qquad n \to \infty,
\end{aligned}$$

which implies

$$\lim_{n\to\infty} \|f_i x_n - x_n\| = 0 \qquad i = 1, 2, ..., m.$$

□

Theorem 3.2. *Suppose that $X, D, f_i, T_i (i = 1, 2, ..., m)$ are as in theorem 3.2 with $\mathcal{F} \neq \emptyset$ and $T_i(p) = \{p\}, (i = 1, 2, \cdots, m)$ for each $p \in \mathcal{F}$. Let $\{x_n\}$ be the iterative process defined by (A), and $a_{n,k}, b_{n,k} \in [a, 1) \subset (0, 1)$ for $k = 0, 1, \cdots, m$. Then $\{x_n\}$ converges strongly to a common fixed point of T_i and f_i ,$i = 1, 2, \cdots, m$ if and only if $\liminf_{n\to\infty} dist(x_n, \mathcal{F}) = 0$.*

Proof. Necessity it obvious. Conversely, suppose that $\lim_{n\to\infty} inf dist(x_n, \mathcal{F}) = 0$. As in Theorem 3.2, we have

$$\|x_{n+1} - p\| \leq (1 + (k_n - 1))\|x_n - p\|,$$

for all $p \in \mathcal{F}$. This implies that

$$dist(x_{n+1}, \mathcal{F}) \leq (1 + (k_n - 1))dist(x_{n+1}, \mathcal{F})$$

Since $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, by lemma 2.5 we have $lim_{n\to\infty} dist(x_n, \mathcal{F})$ exists. Thus $lim_{n\to\infty} dist(x_n, \mathcal{F}) = 0$. Therefore there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a sequence $\{p_k\}$ in $\mathcal{F}$ such that $\|x_{n_k} - p_k\| < \frac{1}{2^k}$ for all k. Let $\theta_n = (k_n - 1)\|x_n - p\|$ (note that $\sum_{i=1}^{\infty} \theta_n < \infty$). Then we get

$$\begin{aligned}
\|x_{n_{k+1}} - p\| &\leq \|x_{n_{k+1}-1} - p\| + \theta_{n_{k+1}-1} \\
&\leq \|x_{n_{k+1}-2} - p\| + \theta_{n_{k+1}-2} + \theta_{n_{k+1}-1} \\
&\leq \\
&\cdots \\
&\leq \|x_{n_k} - p\| + \sum_{i=1}^{n_{k+1}-n_k-1} \theta_{n_k+i}
\end{aligned}$$

for all $p \in \mathcal{F}$. This implies that

$$\begin{aligned}
\|x_{n_{k+1}} - p\| &\leq \|x_{n_k} - p_k\| + \sum_{i=1}^{n_{k+1}-n_k-1} \theta_{n_k+i} \\
&\leq \frac{1}{2^k} + \sum_{i=1}^{n_{k+1}-n_k-1} \theta_{n_k+i}.
\end{aligned}$$

Now, we show that$\{p_k\}$ is Cauchy sequence in D. Note that

$$\begin{aligned}
\|p_{k+1} - p_k\| &\leq \|p_{k+1} - x_{n_{k+1}}\| + \|x_{n_{k+1}} - p_k\| \\
&< \frac{1}{2^{k+1}} + \frac{1}{2^k} + \sum_{i=1}^{n_{k+1}-n_k-1} \theta_{n_k+i} \\
&< \frac{1}{2^{k-1}} + \sum_{i=1}^{n_{k+1}-n_k-1} \theta_{n_k+i},
\end{aligned}$$

which implies that $\{p_k\}$ is Cauchy sequence in D and hence converges to $q \in D$. Since for each $i = 1, 2, \cdots, m$

$$dist(p_k, T_i(q)) \leq H(T_i(p_k), T_i(q)) \leq \| q - p_k \|$$

and $p_k \to q$ as $k \to \infty$, it follows that $dist(q, T_i(q)) = 0$. Also, by continuity of f_i for $i = 1, 2, ..., m$ we have

$$\|p_k - f_i(p_k)\| \to \|q - f_i(q)\|.$$

Hence $\|q - f_i(q)\| = 0$ which implies that $q \in f_i q$. Therefore $q \in \mathcal{F}$ and $\{x_{n_k}\}$ converges strongly to q. Since $\lim_{n\to\infty} \|x_n - q\|$ exists, we conclude that $\{x_n\}$ converges strongly to q. This complete proof. □

Theorem 3.3. *Suppose that $X, D, f_i, T_i (i = 1, 2, ..., m)$ are as in theorem 3.2 with $\mathcal{F} \neq \emptyset$ and $T_i(p) = \{p\}, (i = 1, 2, \cdots, m)$ for each $p \in \mathcal{F}$. Let $\{x_n\}$ be the iterative process defined by (A), and $a_{n,k}, b_{n,k} \in [a, 1) \subset (0, 1)$ for $k = 0, 1, \cdots, m$. If the mappings $\{T_i, f_i : \quad i = 1, 2, ..., m\}$ satisfy the condition (II), Then $\{x_n\}$ converges strongly to a common fixed point of T_i and f_i for $i = 1, 2, \cdots, m$.*

Proof. By Theorem 3.2, for $i = 1, 2, ..., m$, we have

$$\lim_{n\to\infty} dist(T_i(x_n), x_n) = \lim_{n\to\infty} \|f_i x_n - x_n\| = 0.$$

Since the mappings $\{T_i, f_i; \quad i = 1, 2, ..., m\}$ satisfying the condition (II), we get $\lim_{n\to\infty} dist(x_n, \mathcal{F}) = 0$. The conclusion now follows from Theorem 3.3. □

Theorem 3.4. *Let D be a nonempty compact convex subset of a uniformly convex Banach space X. Let $T_i : D \to CB(D)$, $(i = 1, 2, \cdots, m)$ be a finite family of quasi-nonexpansive multivalued mappings satisfying the condition (E), and $f_i : D \to D$, $(i = 1, 2, \cdots, m)$ be a finite family of asymptotically nonexpansive mappings with sequence $\{k_{n,i}\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, where $k_n = max\{k_{n,i} : 1 \leq i \leq m\}$. Assume that $\mathcal{F} \neq \emptyset$ and $T_i(p) = \{p\}, (i = 1, 2, \cdots, m)$ for each $p \in \mathcal{F}$. Let $\{x_n\}$ be the iterative process defined by (A), and $a_{n,k}, b_{n,k} \in [a, 1) \subset (0, 1)$ for $k = 0, 1, \cdots, m$. Then $\{x_n\}$ converges strongly to a common fixed point of T_i and f_i for $i = 1, 2, \cdots, m$.*

Proof. By Theorem 3.2, we have for $i = 1, 2, ..., m$

$$\lim_{n\to\infty} dist(T_i(x_n), x_n) = \lim_{n\to\infty} \|f_i x_n - x_n\| = 0.$$

Since D is compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim x_{n_k} = w$ for some $w \in D$. By condition (E), there exists $\mu \geq 1$ such that for $i = 1, 2, ..., m$

$$\begin{aligned} dist(w, T_i w) &\leq \|w - x_{n_k}\| + dist(x_{n_k}, T_i w) \\ &\leq \mu\, dist(x_{n_k}, T_i x_{n_k}) + 2\|w - x_{n_k}\| \to 0 \qquad as \quad k \to \infty, \end{aligned}$$

this implies that $w \in F(T_i)$. Also, by continuity of the mappings f_i $(i = 1, 2, ..., m)$ we have

$$\|x_{n_k} - f_i x_{n_k}\| \to \|w - f_i w\| \qquad as \qquad k \to \infty,$$

hence $w \in F(f_i)$. Since $\{x_{n_k}\}$ converges strongly to w and $\lim_{n\to\infty} \|x_n - w\|$ exists (by Theorem 3.2), it follows that $\{x_n\}$ converges strongly to w. □

By similar method as in [[19] , Theorem 3.3], we can prove the following Lemma. However we omit the details of proof.

Lemma 3.1. *Let D be a nonempty closed convex subset of a uniformly convex Banach space X with the Opial property. Suppose that $T : D \to K(D)$ is a multivalued mappings satisfying the condition (E). If $x_n \to v$ weakly and $\lim dist_{n\to\infty}(x_n, Tx_n) = 0$, then $v \in Tv$.*

Theorem 3.5. *Let D be a nonempty closed convex subset of uniformly convex Banach space X with the Opial property. Let $T_i : D \to K(D)$, $(i = 1, 2, \cdots, m)$ be a finite family of quasi-nonexpansive multivalued mappings satisfying the condition (E), and $f_i : D \to D$, $(i = 1, 2, \cdots, m)$ be a finite family of asymptotically nonexpansive mappings with sequence $\{k_{n,i}\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, where $k_n = max\{k_{n,i} : 1 \le i \le m\}$. Assume that $\mathcal{F} \ne \emptyset$ and $T_i(p) = \{p\}, (i = 1, 2, \cdots, m)$ for each $p \in \mathcal{F}$. Let $\{x_n\}$ be the iterative process defined by (A), and $a_{n,k}, b_{n,k} \in [a, 1) \subset (0, 1)$ for $k = 0, 1, \cdots, m$. Then $\{x_n\}$ converges weakly to a common fixed point of T_i, f_i, for $i = 1, 2, \cdots, m$.*

Proof. By Theorem 3.2, $\{x_n\}$ is an approximate fixed point sequence of T_i and f_i for $i = 1, 2, ..., m$, that is

$$\lim_{n\to\infty} dist(x_n, T_i x_n) = \lim_{n\to\infty} \|f_i x_n - x_n\| = 0.$$

Also $\lim_{n\to\infty} \|x_n - w\|$ exists for all $w \in \mathcal{F}$. Thus $\{x_n\}$ is bounded. Since X is uniformly convex, it is reflexive, so that we can assume that $x_n \to q$ weakly as $n \to \infty$ for some $q \in D$. By Lemma 3.6 and 2.7 we have $q \in \mathcal{F}$. Now we prove that $\{x_n\}$ has a unique weak subsequential limit in $\mathcal{F}$. To prove this, let w and v be weak limits of the subsequence $\{x_{n_k}\}$ and $\{x_{n_m}\}$ of $\{x_n\}$, respectively and $v \ne w$. By Lemma 3.6 and 2.7, $w, v \in \mathcal{F}$, and hence by Theorem 3.2, the limits $\lim_{n\to\infty} \|x_n - w\|$ and $\lim_{n\to\infty} \|x_n - v\|$ exists. Then by Opial's property,

$$\begin{aligned}\lim_{n\to\infty} \|x_n - w\| &= \lim_{n_k\to\infty} \|x_{n_k} - w\| \\ &< \lim_{n_k\to\infty} \|x_{n_k} - v\| = \lim_{n\to\infty} \|x_n - v\| \\ &= \lim_{n_m\to\infty} \|x_{n_m} - v\| < \lim_{n_m\to\infty} \|x_{n_m} - w\| \\ &= \lim_{n\to\infty} \|x_n - w\|\end{aligned}$$

which is a contradiction. Therefore $\{x_n\}$ converges weakly to a point in $\mathcal{F}$. □

We now intend to remove the restriction that $T_i(p) = p$ for each $p \in \mathcal{F}$. We define the following iteration process.

(B): Let $T_i : D \to P(D)\, (i = 1, 2, ..., m)$ be a finite family of given multivalued mappings and

$$P_{T_i}(x) = \{y \in T_i(x) :\| x - y \|= dist(x, T_i(x))\}.$$

For fixed $x_0 \in D$ we consider the iterative process defined by:

$$y_n = b_{n,0}x_n + b_{n,1}z_{n,1} + b_{n,2}z_{n,2} + \cdots + b_{n,m}z_{n,m}, \qquad n \ge 0,$$

$$x_{n+1} = a_{n,0}x_n + a_{n,1}f_1^n y_n + a_{n,2}f_2^n y_n + \cdots + a_{n,m}f_m^n y_n, \qquad n \ge 0$$

where $z_{n,i} \in P_{T_i}(x_n)$ and $\{a_{n,k}\}$, $\{b_{n,k}\}$ are sequences of numbers in [0,1] such that for every natural number n

$$\sum_{k=0}^{m} a_{n,k} = \sum_{k=0}^{m} b_{n,k} = 1.$$

We note that if $T : D \to P(D)$ is a multivalued mapping, then $F(T) = F(P_T)$ and for all $p \in F(T)$, $P_T(p) = \{p\}$.

Remark 3.1. *All of the above results holds, if we assume* $T; D \to P(D)$ *is a multivalued mapping, which* P_T *is quasi-nonexpansive.*

Remark 3.2. *A mapping* $T : C \to CB(C)$ *is* $\star$*-nonexpansive* [24] *if for all* $x, y \in C$ *and* $u_x \in Tx$ *with* $d(x, u_x) = \inf\{d(x, z) : z \in Tx\}$*, there exists* $u_y \in Ty$ *with* $d(y, u_y) = \inf\{d(y, w) : w \in Ty\}$ *such that*

$$d(u_x, u_y) \le d(x, y).$$

By the definition of the Hausdorff metric, we obtain that if T *is* $\star$*-nonexpansive, then* P_T *is nonexpansive. It is known that* $\star$*-nonexpansiveness is different from nonexpansiveness for multivalued maps, (see* [25] *for details).*

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