# IMPACT OF SEMI-SYMMETRIC NON-METRIC CONNECTION ON A LORENTZIAN MANIFOLD

Uday Chand DE[1], Abdallah Abdelhameed SYIED[2], Krishnendu DE[3]

*An analysis of a Lorentzian manifold with a non-metric connection of semi-symmetric type is conducted in this work. We illustrate that if a space-time allows a semi-symmetric non-metric connection, then the integral curves of the associated vector are geodesic. Also it is demonstrated that if a twisted space-time admits a semi-symmetric non-metric connection whose Ricci tensor vanishes, then the space-time represents a perfect fluid space-time. Further, we acquire that if a space-time admits a semi-symmetric non-metric connection whose Ricci tensor vanishes and the torsion tensor is pseudo symmetric, then the space-time is a perfect fluid space-time. In conclusion, to demonstrate the existence of a semi-symmetric type non-metric connection on a Lorentzian manifold, we build a non-trivial example. Lastly, we discuss certain applications of such space-time admitting Ricci solitons and generalized Ricci solitons.*

**Keywords:** semi-symmetric non-metric connection, Lorentzian manifolds, perfect fluid space-times, twisted space-times, Ricci soliton.

**MSC2020:** 53C25, 53C50; 53C80; 53B20.

## 1. **Introduction**

Lorentzian Geometry has appeared as a geometric theory in which general relativity (GR) can be mathematically established. It has been a fascinating topic of research with a significant role in differential geometry and GR for decades. This field involves various mathematical techniques like geometric analysis, functional analysis, Lie algebras and Lie groups. Therefore, any study concentred on it greatly interests many mathematicians.

The novel idea of a semi-symmetric metric connection (SSMC) on a smooth manifold was initiated by Friedmann and Schouten in [22]. A linear connection $\bar{\nabla}$ on a Lorentzian manifold $M$ of dimension $n$ is referred to as a semi-symmetric connection when it possesses non-zero torsion and fulfills

$$\bar{T}^h_{ij} = \delta^h_i \omega_j - \delta^h_j \omega_i, \tag{1}$$

where $\omega_i$ being a $1-$form named the associated vector of the connection.

If $\bar{\nabla}_k g_{ij} = 0$, it is established that the connection satisfying the semi-symmetric condition is identified as a SSMC; otherwise, it is classified as non-metric. Yano and Hayden made advancements in this idea and achieved many intriguing outcomes in the realm of Riemannian manifolds [17, 29]. Subsequently, the characteristics of the curvature tensor of a SSMC in a Sasakian manifold were also examined by [20, 21]. Z. Nakao [26] conducted

[1]Department of Pure Mathematics, University of Calcutta, 35, Ballygaunge Circular Road Kolkata 700019, West Bengal, India, e-mail: uc_de@yahoo.com

[2] Department of Mathematics, Faculty of Science, Zagazig University, P.O. Box 44519, Zagazig, Egypt, e-mail: a.a_syied@yahoo.com

[3]Department of Mathematics, Kabi Sukanta Mahavidyalaya, The University of Burdwan. Bhadreswar, P.O.-Angus, Hooghly, Pin 712221, West Bengal, India., e-mail: krishnendu.de@outlook.in

research on the Gauss curvature equation and the Codazzi-Mainardi equation, focusing on a SSMC on both a Riemannian manifold and a submanifold. We may bring up Zengin et al.'s work in this regard [30, 31]. Gozutok and Esin [18] introduced the concept of investigating the tangent bundle of a hypersurface using SSMC. Demirbag, in [12], examined the characteristics of a weakly Ricci-symmetric manifold that possesses a SSMC. This idea further studied in [4, 15, 32].

After a considerable period of time, the examination of a SSMC $\bar{\nabla}$ that satisfies $\bar{\nabla}_k g_{ij} \neq 0$ was initiated by Prvanovic [28] and referred to as a pseudo-metric semi-symmetric connection. Shortly thereafter, Andorie continued this line of study [3]. In 1992, Agashe and Chafle in [1] introduced the concept of a semi-symmetric non-metric connection (SSNMC). This concept was subsequently advanced by multiple authors [2, 14, 16, 27].

SSNMC is defined as

$$T^h_{ij} = \delta^h_i \omega_j - \delta^h_j \omega_i, \tag{2}$$
$$g_{ij,k} = -\omega_j g_{ik} - \omega_i g_{jk}, \tag{3}$$

where $\omega_i$ is the associated vector of the SSNMC.

Let $K^h_{ijk}$ and $R^h_{ijk}$ indicate the curvature tensor of SSNMC and the Livi-Civita connection respectively. Also, $\bar{R}_{ij}$ and $R_{ij}$ stand for the Ricci tensor of SSNMC and the Livi-Civita connection respectively. Then [1]

$$K^h_{ijk} = R^h_{ijk} + \alpha_{ik}\delta^h_j - \alpha_{ij}\delta^h_k,$$
$$\alpha_{ij} = \nabla_i \omega_j - \omega_i \omega_j = \bar{\nabla}_i \omega_j, \tag{4}$$
$$\bar{R}_{ij} = R_{ij} - (n-1)\,\alpha_{ij}, \tag{5}$$
$$\bar{R} = R - (n-1)\,\alpha,$$

where trace$(\alpha_{ij}) = \alpha$, $\bar{R}$ and $R$ denote the scalar curvature of SSNMC and the manifold $M$, respectively.

Without the cosmological term, Einstein's field Equation is expressed as [25]:

$$R_{ij} - \frac{R}{2} g_{ij} = \kappa T_{ij}, \tag{6}$$

where $\kappa$ represents the gravitational constant and $T_{ij}$ is the energy-momentum tensor(EMT).

Space-time is a Lorentzian manifold equipped with a globally time-like vector, providing the framework for the recent physical world's model. The EMT for a perfect fluid space-time (PFS) has the subsequent structure [25]:

$$T_{ij} = (p + \sigma)\,\omega_i \omega_j + p g_{ij}, \tag{7}$$

with $p$ and $\sigma$ being the isotropic pressure and the energy-density, respectively.

The Ricci tensor of PFS has the following shape [10]:

$$R_{ij} = a g_{ij} + b\omega_i \omega_j, \tag{8}$$

in which $a$ and $b$ are scalars.

Using the earlier equations, we determine

$$a = -\frac{\kappa}{2}(p - \sigma) \quad \text{and} \quad b = \kappa(p + \sigma). \tag{9}$$

If the relations $p = 0$, $p = \sigma$, $3p = \sigma$ and $p + \sigma = 0$ hold, then the PF space-time is called as the dust matter, stiff matter, radiation era and dark matter era (DME) of the Universe [8], respectively. It also covers the phantom era in which $\frac{p}{\sigma} < -1$. The quintessence is a speculative kind of dark energy, or more precisely, a scalar field, that physicists have suggested as a potential explanation for the universe's observed accelerating speed of expansion. Additionally, when $-1 < \frac{p}{\sigma} < 0$, the quintessence phase is included in dark energy.

Existence of a unit time-like and torse-forming vector, among other restrictions, characterize Robertson-Walker (RW) space-times as well as generalized Robertson-Walker (GRW) space-time [23]. They demonstrate that the presence of a single, distinct vector may still define Twisted manifolds without any additional restrictions. Due to their inclusion of a scale function which is both space and time dependent, twisted manifolds generalize RW and GRW space-times.

Twisted space-times are defined for this purpose. Compared to warped space-times, which allows periodic changes to the world, twisted space-times are far more general. Chen [9] first proposed the concept of twisted space-time and described as a Lorentzian manifold $M^n$ with the metric (in local form)

$$ds^2 = g_{jk}dz^j dz^k = -(dt)^2 + \phi^2(z,t)g^*_{jk}dz^j dz^k, \tag{10}$$

where $g^*$ denotes the metric tensor of an (n-1) dimensional Riemannian manifold.The twisted space-time becomes the GRW space-time if $\phi$ is solely a function of t.

The general outline of the research is as follows:

We produce the concepts of SSNMC in Introduction Section. In Section 2 we demonstrate tha main results of our paper. Last Section establishes the existence of a SSNMC on a Lorentzian manifold.

## 2. **SSNMC on a Lorentzian manifold**

This part presents important findings regarding a Lorentzian manifold that possesses a SSNMC.

**Theorem 2.1.** *If a space-time allows a SSNMC whose Ricci tensor vanishes, then the integral curves of the associated vector $\omega_i$ are geodesic.*

*Proof.* Suppose $M^4$ admits a SSNMC whose Ricci tensor vanishes. Then from equation (5) we get

$$R_{ij} = 3\alpha_{ij}. \tag{11}$$

Since $R_{ij}$ is symmetric, therefore the previous equations implies that

$$\alpha_{ij} = \alpha_{ji}.$$

Consequently,

$$\nabla_i\omega_j = \nabla_j\omega_i,$$

that is, the associated vector $\omega_i$ is irrotational.

Multiplying with $\omega^i$, it arises

$$\omega^i\left(\nabla_j\omega_i\right) = \omega^i\left(\nabla_i\omega_j\right) = 0,$$

which indicates that the integral curves of the vector $\omega_i$ are geodesic. □

**Theorem 2.2.** *If a PFS admits a SSNMC whose Ricci tensor vanishes, then the state equation is given by*

$$3p + \sigma = \frac{6}{\kappa}.$$

*Proof.* In virtue of Eqs. (6) and (7), one may get

$$R_{ij} - \frac{R}{2}g_{ij} = \kappa\left[(p+\sigma)\,\omega_i\omega_j + pg_{ij}\right]. \tag{12}$$

Multiplying with $\omega^j$, one finds

$$\omega^j R_{ij} = \left[\frac{R}{2} - \kappa\sigma\right]\omega_i. \tag{13}$$

Assume that $M^4$ admits a SSNMC whose Ricci tensor vanishes. Thus, Eq. (4) implies

$$R_{ij} = 3\left[\nabla_i\omega_j - \omega_i\omega_j\right].$$

Contracting with $\omega^j$, we acquire that

$$\omega^j R_{ij} = 3\omega_i. \tag{14}$$

Thus Eqs. (13) and (14) together give

$$\kappa\sigma = \frac{R}{2} - 3. \tag{15}$$

Multiplying Eq. (12) by $g^{ij}$, we infer

$$R = \kappa\sigma - 3\kappa p. \tag{16}$$

Using Eq. (15) in Eq. (16), one sees that

$$\kappa p = -\frac{R}{6} - 1. \tag{17}$$

The combination of Eqs. (15) and (17) leads to

$$3p + \sigma = \frac{6}{\kappa}.$$

□

**Theorem 2.3.** *If a PFS admits a SSNMC whose Ricci tensor vanishes and the associated vector $\omega_i$ is parallel, then the space-time is vacuum. Consequently, the semi-symmetry and Weyl semi-symmetry are equivalent.*

*Proof.* Let the associated vector $\omega_i$ be parallel with respect to SSNMC. Then

$$\bar{\nabla}_i\omega_j = 0. \tag{18}$$

Consequently, we find

$$\nabla_i\omega_j = \omega_i\omega_j. \tag{19}$$

Hence, we have

$$R_{ij} = 0.$$

This indicates that the space-time is vacuum.

Minkowski space-time, which illustrates empty space without a cosmological constant, is a prominent example of a vacuum space-time. In order to describe an empty cosmos with no curvature, E. A. Milne created the Milne model.

In a vacuum space-time $R^h_{ijk} = C^h_{ijk}$ in some region of the space-time. This implies that semi-symmetry and Weyl semi-symmetry are equivalent is such a space-time. □

In [23] it is established that $M$ represents a twisted space-time iff it allows a time-like and unit torse-forming vector, that is, $\nabla_j u_k = \varphi\{g_{jk} + u_j u_k\}$ and $u^j u_j = -1$. In this article, we prove:

**Theorem 2.4.** *If a twisted space-time admits a SSNMC whose Ricci tensor vanishes, then the space-time becomes a PFS.*

*Proof.* Consider $\omega_j$ is a torse-forming vector with respect to SSNMC, that is,

$$\bar{\nabla}_k\omega_j = \varphi\left(g_{jk} + \omega_k\omega_j\right), \tag{20}$$

where $\varphi$ is a scalar.

In view of Eq. (4), we deduce

$$\nabla_k\omega_j = \varphi g_{jk} + (\varphi + 1)\,\omega_k\omega_j. \tag{21}$$

Hence using (11) we infer

$$\begin{aligned} R_{ij} &= 3\alpha_{ij}, \\ &= 3\left[\nabla_i\omega_j - \omega_i\omega_j\right], \\ &= 3\varphi\left[g_{ij} + \omega_i\omega_j\right]. \end{aligned} \tag{22}$$

This equation represents PFS. □

Comparing Eqs. (8) and (22), we obtain

$$\frac{p}{\sigma} = -\frac{1}{3}.$$

Hence, we state:

**Corollary 2.1.** *A twisted space-time admittting a SSNMC whose Ricci tensor vanishes, represents quintessence phase.*

**Theorem 2.5.** *If a space-time admits a SSNMC whose Ricci tensor vanishes and the torsion tensor is pseudo symmetric, then the space-time is a perfect fluid space-time.*

*Proof.* Assume that the torsion tensor $T$ of SSNMC $\bar{\nabla}$ is pseudo symmetric in the sense of Chaki [7]. Then

$$\bar{\nabla}_k T^h_{ij} = 2b_k T^h_{ij} + b_i T^h_{kj} + b_j T^h_{ik} + b^h T_{kij}, \tag{23}$$

where $T_{kij} = g_{lk}T^l_{ij}$.

From (2) it follows that

$$T^h_{hj} = 3\omega_j. \tag{24}$$

In view of Eq. (2), one finds

$$T_{lij} = g_{il}\omega_j - g_{jl}\omega_i. \tag{25}$$

Eq. (23) leads to

$$\bar{\nabla}_k T^h_{hj} = 2b_k T^h_{hj} + b_h T^h_{kj} + b_j T^h_{hk} + b^h T_{khj}. \tag{26}$$

Now, we get

$$\begin{aligned} b^h T_{khj} &= b^h\left(g_{hk}\omega_j - g_{jk}\omega_h\right), \\ &= b_k\omega_j - f g_{jk}, \end{aligned} \tag{27}$$

where $f = b^h\omega_h$.

Utilizing Eqs. (24) and (27) in Eq. (26), it can be conclude that

$$3\bar{\nabla}_k\omega_j = 8b_k\omega_j - 2f g_{jk} + 3b_j\omega_k.$$

Using Eq. (4), we deduce

$$3\nabla_k\omega_j = 8b_k\omega_j - 2f g_{jk} + 3b_j\omega_k + 3\omega_j\omega_k. \tag{28}$$

Let us assume that the Ricci tensor vanishes with respect to SSNMC. Consequently,

$$\nabla_k\omega_j = \nabla_j\omega_k.$$

Therefore, Eq. (28) implies that

$$b_k\omega_j = b_j\omega_k.$$

Multiplying with $\omega^j$, we obtain

$$b_k = -f\omega_k.$$

Thus Eq. (28) becomes

$$\nabla_k\omega_j = \left(1 - \frac{11f}{3}\right)\omega_k\omega_j - \frac{2f}{3}g_{kj}.$$

Since $\bar{R}_{ij} = 0$, thus

$$\begin{aligned} R_{ij} &= \nabla_i\omega_j - \omega_i\omega_j \\ &= \frac{-11f}{3}\omega_i\omega_j - \frac{2f}{3}g_{ij}, \end{aligned} \tag{29}$$

this means that the space-time is PFS. □

Comparing Eqs. (8) and (29), we obtain

$$p = -\frac{7f}{6\kappa} \text{ and } \sigma = -\frac{5f}{2\kappa}. \tag{30}$$

Therefore, we state:

**Corollary 2.2.** *In a space-time admitting a SSNMC whose Ricci tensor vanishes and the torsion tensor is pseudo symmetric, then p and $\sigma$ are given by* (30).

**Remark 2.1.** *From Eq.* (30)*, we can say that for this space-time the state equation is demonstrated by* $\frac{p}{\sigma} = \frac{7}{15} =$ *constant.*

**Remark 2.2.** *Since p and $\sigma$ are not constants, the current result is consistent with the current state of the Cosmos.*

## 3. Example of semisymmetric non-metric connection

Let

$$M = \left\{y^i,\ i = 1,2,3,4 \text{ and } y^4 \neq 0\right\}$$

denote a semi-Riemannian manifold of dimension 4, $(y^1, y^2, y^3, y^4)$ is the standard coordinates of a point in $\mathbb{R}^4$.

Let $v_1 = e^{\frac{-y^4}{2}}\frac{\partial}{\partial y^1}$, $v_2 = e^{\frac{-y^4}{2}}\frac{\partial}{\partial y^2}$, $v_3 = e^{\frac{-y^4}{2}}\frac{\partial}{\partial y^3}$, $v_4 = e^{\frac{-y^4}{2}}\frac{\partial}{\partial y^4}$ be the linearly independent vectors on $M$ and they form a basis. Also, let $v_4$ be the unite time-like vector in $M$.

The Lorentzian metric is

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Thus, $M$ endowed with $g$ may be described as a 4-dimensional Lorentzian manifold. Then the Lie brackets are given by

$$[v_i, v_j] = \begin{cases} \frac{1}{2}v_i & \text{if } j = 4 \text{ and } i = 1,2,3 \\ -\frac{1}{2}v_j & \text{if } i = 4 \text{ and } j = 1,2,3 \\ 0 & \text{otherwise} \end{cases}. \tag{31}$$

Using the foregoing Lie bracket and Koozul's formula for Livi-Civita connection $\nabla$, we obtain

$$\nabla_{v_i}v_j = \begin{cases} \frac{1}{2}v_4 & \text{for } i,j = 1,2,3 \\ \frac{1}{2}v_i & \text{if } j = 4 \text{ and } i = 1,2,3 \\ 0 & \text{otherwise} \end{cases}. \tag{32}$$

The non-vanishing components of the curvature tensor are given by

$$\begin{aligned}
R(v_1,v_2)v_1 &= \frac{-1}{4}v_2, \quad R(v_1,v_3)v_1 = \frac{-1}{4}v_3, \quad R(v_1,v_4)v_1 = \frac{-1}{4}v_4,\\
R(v_1,v_2)v_2 &= \frac{1}{4}v_1, \quad R(v_2,v_3)v_2 = \frac{-1}{4}v_3, \quad R(v_2,v_4)v_2 = \frac{-1}{4}v_4,\\
R(v_1,v_3)v_3 &= \frac{1}{4}v_1, \quad R(v_2,v_3)v_3 = \frac{1}{4}v_2, \quad R(v_3,v_4)v_3 = \frac{-1}{4}v_4,\\
R(v_1,v_4)v_4 &= \frac{-1}{4}v_1, \quad R(v_3,v_4)v_4 = \frac{-1}{4}v_3, \quad R(v_2,v_4)v_4 = \frac{-1}{4}v_2.
\end{aligned}$$

The rest of the component of curvature tensor may be deduced from the symmetric property of the curvature tensor. From the previous expression of the curvature tensors we acquire the non-vanishing component of the Ricci tensor

$$R(v_1,v_1) = R(v_2,v_2) = R(v_3,v_3) = -R(v_4,v_4) = \frac{3}{4},$$

and the scalar curvature

$$R = 3.$$

Then from the equation

$$\bar{\nabla}_x y = \nabla_x y + \omega(y)x, \tag{33}$$

where $\omega$ is a one-form, it follows from Eq. (32) and Eq. (33) that

$$\bar{\nabla}_{v_i} v_j = \nabla_{v_i} v_j + v_j v_i.$$

From the above we conclude that

$$\bar{\nabla}_{v_i} v_j \neq 0.$$

Hence, the linear connection is non-metric on $M^4$.

## 4. Applications to Ricci Solitons and Generalized Ricci Solitons

In this section we investigate Ricci solitons and generalized Ricci solitons using the connection $\bar{\nabla}$.

The concept of Ricci flow was defined by Hamilton [19] as a way to derive a canonical metric on a differentiable manifold. If the manifold fulfills the evolution equation $\frac{\partial}{\partial t}g_{jk}(t) = -2R_{jk}$, then this is called a Ricci flow equation. Also, Triplet $(g,\omega,\lambda)$ is the Ricci soliton, where $g$ is a semi-Riemannian metric, $\omega$ is a smooth vector (also known as the potential vector), and $\lambda$ is a constant that corresponds to

$$\pounds_\omega g_{jk} + 2R_{jk} - 2\lambda g_{jk} = 0, \tag{34}$$

in which the Lie derivative of $g$ along a vector $\omega$ is written as $\pounds_\omega g$. The soliton mentioned above is an Einstein metric if $\omega$ is zero or Killing. Depending on whether $\lambda$ is positive, zero, or negative, the Ricci soliton is referred to as shrinking, steady, or expanding. In contemporary physics, metrics that fulfil (34) are quite useful. In the context of string theory, theoretical physicists have been examining the Ricci soliton condition. A number of intriguing results about Ricci solitons have been examined in [6, 11, 13].

We know that

$$\pounds_\omega g_{jk} = \nabla_j \omega_k + \nabla_k \omega_j. \tag{35}$$

Using Eq. (4), we acquire

$$\pounds_\omega g_{jk} = \bar{\nabla}_j \omega_k + \bar{\nabla}_k \omega_j + 2\omega j \omega_k. \tag{36}$$

Eqs. (34) and (36) jointly reveal

$$\bar{\nabla}_j \omega_k + \bar{\nabla}_k \omega_j + 2R_{jk} - 2\lambda g_{jk} + 2\omega j \omega_k = 0. \tag{37}$$

Multiplying Eq. (37) with $\omega^j\omega^k$ and using the relation $\omega^j(\bar{\nabla}_j\omega_k) = \omega_k$, we infer

$$R_{jk}\omega^j\omega^k = -\lambda. \tag{38}$$

Suppose the space-time admits a SSNMC whose Ricci tensor vanishes. Then using (5), we provide

$$3\alpha_{jk}\omega^j\omega^k = -\lambda. \tag{39}$$

Then using Eq. (4) and the relation $\omega^j(\nabla_j\omega_k) = 0$, we acquire

$$\lambda = 3. \tag{40}$$

**Theorem 4.1.** *If a space-time with a SSNMC whose Ricci tensor vanishes allows a Ricci soliton, then this Ricci soliton is shrinking.*

Now we examine a generalized Ricci soliton (GRS), which is defined as a semi-Riemannian manifold that admits a vector field u that is differentiable and fulfilling

$$\pounds_\omega g_{jk} + 2\alpha_1\omega_j\omega_k + 2\beta_1 R_{jk} - 2\lambda g_{jk} = 0, \tag{41}$$

where $\alpha_1, \beta_1 \in \mathbb{R}$.

GRS equations are the equations described in (41). In particular, Killing's equation is represented in (41), if $\alpha_1 = \beta_1 = \lambda = 0$. If $\alpha_1 = \beta_1 = 0$ and $\lambda \neq 0$, then the homothetic equation is shown. In this case, the Ricci solitons are represented when $\alpha_1 = 0,\ \beta_1 = -1$ and $\lambda \neq 0$. Moreover, the Einstein-Weyl scenario is represented if $\alpha_1 = 1,\ \beta_1 = -\frac{1}{n-2}$ and $\lambda = 0$. Finally, $\alpha_1 = 1,\ \beta_1 = \frac{1}{2}$ and $\lambda \neq 0$ relate to the vacuum near-horizon geometry equation.

Using Eq. (36) in Eq. (41), we obtain

$$\bar{\nabla}_j\omega_k + \bar{\nabla}_k\omega_j + 2\omega j\omega_k + 2\alpha_1\omega_j\omega_k + 2\beta_1 R_{jk} - 2\lambda g_{jk} = 0, \tag{42}$$

Multiplying Eq. (42) with $\omega^j\omega^k$ and using the relation $\omega^j(\bar{\nabla}_j\omega_k) = \omega_k$, we provide

$$\lambda = \alpha_1 - 3\beta_1. \tag{43}$$

**Theorem 4.2.** *If a space-time with a SSNMC whose Ricci tensor vanishes allows a GRS, then this GRS is expanding for $\alpha_1 < 3\beta_1$; steady if $\alpha_1 = 3\beta_1$; shrinking for $\alpha_1 > 3\beta_1$.*

Assume that a GRS is admitted in a twisted space-time with an SSRMC. After that, we can take

$$\bar{\nabla}_k\omega_j = \varphi\left(g_{jk} + \omega_k\omega_j\right). \tag{44}$$

Now using Eq. (44) in Eq. (41), we infer

$$R_{jk} = \frac{1}{\beta_1}\{(\lambda - \varphi)g_{jk} - (\alpha_1 + \varphi + 1)\omega_j\omega_k\}, \tag{45}$$

this means that the space-time is a PFS.

**Theorem 4.3.** *If a twisted space-time with a SSNMC whose Ricci tensor vanishes allows a GRS, then it becomes a PFS.*

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