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FUNDAMENTALS OF Γ-ALGEBRA AND Γ-DIMENSION

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In this paper, we generalize the notion of algebra over a field. A Γ*-algebra is an algebraic structure consisting of a vector space* V*, a groupoid* Γ *together with a map from* $V \times \Gamma \times V$ *to* V*, usually called multiplication. We introduce the notion of* Γ*- dimension and give some examples and prove some properties of* Γ*-algebras. Then, we give some results about* $m \times n$ *real matrices. Also, we study the notion of regular* Γ*-algebra and we obtain some results in this respect. Finally, we define the notions of* T*-functor and* H*-system over a* Γ*-algebra and prove some results. Moreover, we see that there exists a covariant functor between the categories of* Γ*- algebras and algebras. We see that this functor is exact.*

Keywords: Γ-algebra, homomorphism, regular Γ-algebra, H-system, T-functor.
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1. Γ-algebra

In [5], Nobusawa introduced the notion of Γ-ring, as more general than ring. Barnes [2] weakened slightly the conditions in the definition of the Γ-ring in the sense of Nobusawa. After these two papers are published, many mathematicians made good works on Γ-ring in the sense of Barnes and Nobusawa. Luh [4] and Kyuno [3] studied the structure of Γ-rings and obtained various generalization analogous to corresponding parts in ring theory. In [1], Chakraborty and Pau defined an isomorphism, an anti-isomorphism and a Jordan isomorphism in a Γ-ring and developed some important results relating to these concepts, also see [6, 7].

An *algebra* over a field is a vector space equipped with a bilinear vector product. That is to say, it is an algebraic structure consisting of a vector space together with an operation, usually called multiplication, that combines any two vectors to form a third vector; to qualify as an algebra, this multiplication must satisfy certain compatibility axioms with the given vector space structure, such as distributivity. In other words, an algebra over a field is a set together with operations of multiplication, addition, and scalar multiplication by elements of the field. Now, we generalize this notion.

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Definition 1.1. *Let Γ be a groupoid and V be a vector space over a field F. Then, V is called a Γ-algebra over the field F if there exists a mapping $V \times \Gamma \times V \longrightarrow V$ (the image is denoted by $x\alpha y$ for $x, y \in V$ and $\alpha \in \Gamma$) such that the following conditions hold:*

(1) $(x + y)\alpha z = x\alpha z + y\alpha z$, $x\alpha(y + z) = x\alpha y + x\alpha z$,
(2) $x(\alpha + \beta)y = x\alpha y + x\beta y$,
(3) $(cx)\alpha y = c(x\alpha y) = x\alpha(cy)$,
(4) $0\alpha y = y\alpha 0 = 0$,
for all $x, y, z \in V$, $c \in F$ and $\alpha \in \Gamma$.
Moreover, a Γ- algebra is called associative if
(5) $(x\alpha y)\beta z = x\alpha(y\beta z)$,

and *unital* if for every $\alpha \in \Gamma$, there is an element 1_α in V such that $1_\alpha \alpha v = v = v\alpha 1_\alpha$ for all non-zero elements of V.

A non-empty subset V' of a Γ-algebra V is called a Γ-*subalgebra* if it is a subspace of V and for all $x, y \in V'$ and $\alpha \in \Gamma$ we have $x\alpha y \in V'$. A subset I of a Γ-algebra V is called a *left (right) ideal* if it is a Γ-subalgebra of V and for all $a \in I$ and $v \in V$ and $\alpha \in \Gamma$ we have $v\alpha a \in I$ $(a\alpha v \in I)$ and is a *(two-sided) ideal* if it is both a left and right ideal. It easy to see that V and $\{0\}$ are ideals of V. An ideal I such that $\{0\} \subset I \subset V$ is called *proper*.

Let X be a subset of Γ-algebra V. Then, the smallest left (right, two-sided) ideal of V containing X exists and we shall call it the left (right or two-sided) ideal generated by X, and will be denoted by $< X >_l$ ($< X >_r$ or $< X >$). If $X = \{x\}$, then we also write $< x >$ instead of $< \{x\} >$.

Example 1.1. *Let A be a vector space and Γ be a groupoid. For every $x, y \in A$ and $\alpha \in \Gamma$ we define $x\alpha y = 0$. Then, A is a Γ-algebra.*

Example 1.2. *Let F be a field, V and W be two vector spaces and $A = Hom_F(V, W)$, $\Gamma = Hom_F(W, V)$. For every $f, g \in A$ and $\alpha \in \Gamma$ we define $f\alpha g = f \circ \alpha \circ g$, where $\circ$ is the combination operation. Then, A is an associative Γ-algebra.*

Example 1.3. *Let A and Γ be the sets of $n \times m$ and $m \times n$ matrices over the field F, respectively. Then, it is easy to see that A is an associative Γ-algebra.*

Example 1.4. *Consider the pervious example. Let A be the set of 3×2 matrices over the field of real numbers $\mathbb{R}$ and*

$$\Gamma = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \end{pmatrix} \ : \ a, b \in \mathbb{R} \right\}.$$

Then, A is an associative Γ-algebra and

$$B = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \\ 0 & 0 \end{pmatrix} \ : \ x, y \in \mathbb{R} \right\},$$

is a Γ-subalgebra of A.

Let V_1 and V_2 be Γ_1- and Γ_2-algebras respectively, T be a linear transformation from V_1 to V_2, f be a homomorphism from Γ_1 to Γ_2. Then, we say that (T, f) is a (Γ_1, Γ_2)- homomorphism (homomorphism) from (V_1, Γ_1) to (V_2, Γ_2) if $(T, f)(x\alpha y) = T(x\alpha y) = T(x)f(\alpha)T(y)$.

Example 1.5. *Let V_1 be the vector space of $n \times 1$ real matrices generated by $a = (a_{i1})_{n\times 1}$ such that $a_{11} = 1$ and $a_{i1} = 0$ for $i \neq 1$, $\Gamma_1 = \{\begin{pmatrix} r_1 & 0 & \cdots & 0 \end{pmatrix}_{1\times n} : r_1 \in \mathbb{R}\}$,$V_2$ be the vector space of $m \times 1$ real matrices generated by $b = (b_{i1})_{m\times 1}$ such that $b_{11} = 1$ and $b_{i1} = 0$ for $i \neq 1$, $\Gamma_2 = \{\begin{pmatrix} r_2 & 0 & \cdots & 0 \end{pmatrix}_{1\times m} : r_2 \in \mathbb{R}\}$, T be the linear transformation from V_1 to V_2 with the matrix*

$$\begin{pmatrix} k & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

where $0 \neq k \in \mathbb{R}$ and $f : \Gamma_1 \longrightarrow \Gamma_2$ defined by $f(X) = \frac{1}{k} \times X$. Then, (T, f) is a homomorphism from V_1 to V_2.

For non-empty subsets A and B of Γ-algebra V and non-empty subset Γ_1 of Γ. Let

$$A\Gamma_1 B := \{a\gamma b : a \in A, b \in B \text{ and } \gamma \in \Gamma_1\},$$

$$A\Gamma_1^{\sum} B := \Big\{\sum_{i=1}^{n} a_i\gamma_i b_i : a_i \in A, b_i \in B, \gamma_i \in \Gamma_1 \text{ and } n \in \mathbb{N}\Big\},$$

$$\mathbb{Z}X = \Big\{\sum_{i=1}^{n} n_i x_i : n_i \in \mathbb{Z}, x_i \in X\Big\}.$$

If $A = \{a\}$, then we also write $a\Gamma_1 B$ instead of $\{a\}\Gamma_1 B$.

An ideal P is called *prime* if $A\Gamma^{\sum} B \subseteq P$, then $A \subseteq P$ or $B \subseteq P$ and P is called *semiprime* if $A\Gamma^{\sum} A \subseteq P$ then $A \subseteq P$.

Lemma 1.1. *Let V be a Γ-algebra and X be a non-empty subset of V. Then,*

(1) $< X >_r = \mathbb{Z}X + X\Gamma^{\sum} V$,
(2) $< X >_l = \mathbb{Z}X + V\Gamma^{\sum} X$,
(3) $< X > = \mathbb{Z}X + X\Gamma^{\sum} V + V\Gamma^{\sum} X + V\Gamma^{\sum} X\Gamma^{\sum} V$.

Definition 1.2. *Let V be a Γ-algebra. Then, the ordinary dimension of V as a vector space is called the dimension and the dimension of the subspace of V generated by all products of the form $a\alpha b$ is called the Γ-dimension.*

Example 1.6. *Let A be the vector space of 2×3 real matrices with the basis*

$$\left\{\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\},$$

and Γ *be a groupoid of* 3×2 *matrices of the form* $\begin{pmatrix} r & 0 \\ -r & 0 \\ 0 & 0 \end{pmatrix}$, *where* $r\in\mathbb{Z}$. *Then,* A *is a* Γ*-algebra and the dimension of* A *is* 4 *but the* Γ*-dimension is* 0. *Since*

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} r & 0 \\ -r & 0 \\ 0 & 0 \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},\ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} r & 0 \\ -r & 0 \\ 0 & 0 \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$
$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} r & 0 \\ -r & 0 \\ 0 & 0 \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} r & 0 \\ -r & 0 \\ 0 & 0 \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Example 1.7. *Suppose that*

$$A=\left\{\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 0 & b & c \end{pmatrix} : a,b,c\in\mathbb{R}\right\} \text{ and } \Gamma=\left\{\begin{pmatrix} r & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} : r\in\mathbb{R}\right\}.$$

Then, the dimension of A *is 3 and the* Γ*-dimension of* A *is 1.*

2. Results about $m\times n$ matrices

Lemma 2.1. *Let* A *be the vector space of* $m\times n$ *real matrices and* Γ *be a set of* $n\times m$ *real matrices, where the* ij *entire is a real number and the others are zero. Then, the elements of* Γ*-algebra* A *are* $m\times n$*-matrices with dependent rows.*

Proof. The proof is straightforward. □

Proposition 2.1. *Let* A *be the vector space of* $m\times n$ *real matrices and* Γ *is a set of* $n\times m$ *real matrices with* $1\leq k\leq mn$ *non-zero entries. Then, every element of* Γ*-algebra* A *is the sum of* k, $m\times n$ *real matrices with dependent rows.*

Proof. The proof obtains by Lemma 2.1 and the following relation,

$$a(\alpha_1+\alpha_2+\cdots+\alpha_k)b=a\alpha_1 b+\alpha_2 b+,\cdots+a\alpha_k b,$$

where $a,b\in A$ and $\alpha_i\in\Gamma$. □

Proposition 2.2. *Let* A *be the vector space of* $m\times n$ *real matrices and* Γ *is a groupoid of* $n\times m$ *real matrices with at least one non-zero entire. Then, the dimension and* Γ*-dimension of* A *are equals.*

Proof. With out loss of generality, suppose that $\alpha_{n\times m}=\begin{pmatrix} r & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$ is an arbitrary element of Γ. Then, the basis element E_{ij} obtained from the product $A_{m\times n}\alpha_{n\times m}B_{m\times n}$, where $A_{m\times n}=(a_{i'j'})$, $B_{m\times n}=(b_{i'j'})$

$$a_{i'j'}=\begin{cases} \frac{1}{r} & i'=i,\ j'=1 \\ 0 & o.w \end{cases}$$

$$b_{i^{'}j^{'}} = \begin{cases} 1 & i^{'} = 1,\ j^{'} = j \\ 0 & o.w \end{cases}$$

This completes the proof. □

3. Regular Γ-algebra

A Γ-algebra V is *regular* if for every $x \in V$, there exists $y \in V$ and $\alpha, \beta \in \Gamma$ such that

$$x = x\alpha y\beta x.$$

In this case x is called an (α,β)*-regular element*. An ideal I of a Γ-algebra V is called (α,β)*-regular* if every element of I is (α,β)-regular. An element x of a Γ-algebra V is called α*-idempotent* if $x\alpha x = x$.

Example 3.1. *Let F be a field, $V = F \times F$ and Γ be a sub-groupoid of F. For every $\alpha, \beta \in \Gamma$ and $x_1, x_2, x_3, x_4 \in F$, we define*

$$\begin{aligned} (x_1, x_2) \oplus (x_3, x_4) &= (x_1 + x_3, x_2 + x_4), \\ (x_1, x_2)\widehat{(\alpha,\beta)}(x_3, x_4) &= (x_1\alpha x_3, x_2\beta x_4). \end{aligned}$$

Then, V is a regular Γ-algebra.

Notice: Let V be a regular Γ-algebra. Then, $< x >_r = x\Gamma^{\sum}V$. Indeed, since V is regular there exist $\alpha, \beta \in \Gamma$ and $y \in V$ such that $x = x\alpha y\beta x$. Hence, $\mathbb{Z}x = \mathbb{Z}(x\alpha y\beta x) \subseteq x\Gamma^{\sum}V$. This implies that $< x >_r = x\Gamma^{\sum}V$.

Proposition 3.1. *Let V be an associative regular Γ-algebra such that every element is (α,β)-regular. Then, every finitely generated right (left) ideal of V is generated by idempotent elements.*

Proof. Suppose that $x \in V$. Then, there exists $y \in V$ such that $x = x\alpha y\beta x$. We have $(x\alpha y)\beta(x\alpha y) = (x\alpha y\beta x)\alpha y = x\alpha y$. Hence, $x\alpha y$ is a β-idempotent element of V. We see that

$$\begin{aligned} < x\alpha y >_r = (x\alpha y)\Gamma^{\sum}V &= \Big\{ \sum_{i=1}^{n} (x\alpha y)\beta_i v_i : n \in \mathbb{N}, v_i \in V, \beta_i \in \Gamma \Big\} \\ &= \Big\{ \sum_{i=1}^{n} x\alpha(y\beta_i v_i) : n \in \mathbb{N}, v_i \in V, \beta_i \in \Gamma \Big\} \\ &\subseteq x\Gamma^{\sum}V = < x >_r . \end{aligned}$$

On the other hand, since

$$x = x\alpha y\beta x \in (x\alpha y)\Gamma^{\sum}V,$$

$< x >_r \subseteq < x\alpha y >_r$. Therefore, $< x >_r = < x\alpha y >_r$.

Without lose of generality we suppose that $I = < x, y >_r$. Now, $< x >_r = < a >_r$, for some β-idempotent element and since $y - a\beta y \in < x, y >_r$, we have $<$

$x, y >_r = < a, y - a\beta y >_r$, and there exists a β-idempotent element $b \in V$ such that $< b >_r = < y - a\beta y >_r$. Consequently, $a\beta b = 0$ and

$$(b - b\beta a)\beta(b - b\beta a) = b\beta b - (b\beta b)\beta a - (b\beta a)\beta b + (b\beta a)\beta(b\beta a) = b - b\beta a;$$

$$b\beta(b - b\beta a) = b\beta b - b\beta(b\beta a) = b\beta b - (b\beta b)\beta a = b - b\beta a.$$

We conclude that $< b - b\beta a >_r = < b >_r = < y - a\beta y >_r$.

Therefore, $< x, y >_r = < a, b - b\beta a >_r$. This completes the proof. □

Proposition 3.2. *Let V be a Γ-algebra, $x_1 = x - x\alpha y\beta x$ and $x_1 = x_1\alpha a\beta x_1$ for some $a \in V$. Then, $x = x\alpha b\beta x$ for some $b \in V$.*

Proof. We observe that

$$\begin{aligned} x = x_1 + x\alpha y\beta x &= x_1\alpha a\beta x_1 + x\alpha y\beta x \\ &= (x - x\alpha y\beta x)\alpha a\beta(x - x\alpha y\beta x) + x\alpha y\beta x \\ &= x\alpha(a - a\beta x\alpha y - y\beta x\alpha a + y\beta x\alpha a\beta x\alpha y)\beta x. \end{aligned}$$

This implies that $x = x\alpha b\beta x$ for some, $b \in a - a\beta x\alpha y - y\beta x\alpha a + y\beta x\alpha a\beta x\alpha y$. This completes the proof. □

Lemma 3.1. *Let $V_1 \leq V_2$ be ideals in an associative Γ-algebra V. Then, V_2 is (α, β)-regular if and only if V_1 and $[V_2 : V_1]$ are both (α, β)-regular.*

Proof. Suppose that V_2 is (α, β)-regular. Then, obviously $[V_2 : V_1]$ is (α, β)-regular. Let $x \in V_1$. Then, we have $x = x\alpha y\beta x$ for some $y \in V_2$. We set $b = y\beta x\alpha y$. Then, b is an element of V_1 such that

$$x\alpha b\beta x = x\alpha(y\beta x\alpha y)\beta x = (x\alpha y\beta x)\alpha y\beta x = x\alpha y\beta x = x,$$

Then, V_1 is (α, β)-regular.

Conversely, assume that V_1 and $[V_2 : V_1]$ are both (α, β)-regular and $x \in V_1$. Hence, there exist $\widehat{\alpha}, \widehat{\beta} \in \widehat{\Gamma}$ and $y + V_1 \in [V_2 : V_1]$ such that

$$x + V_1 = (x + V_1)\widehat{\alpha}(y + V_1)\widehat{\beta}(x + V_1) = x\alpha y\beta x + V_1,$$

where $\widehat{\Gamma} = \{\widehat{\gamma} : \gamma \in \Gamma\}$. Hence, $x - x\alpha y\beta x \in V_1$ for some $y \in V_2$. Since V_1 is (α, β)-regular,

$$x - x\alpha y\beta x = (x - x\alpha y\beta x)\alpha z\beta(x - x\alpha y\beta x)$$

for some $z \in V_1$, from which we conclude that $x = x\alpha b\beta x$. Therefore, V_2 is (α, β)-regular. □

Proposition 3.3. *Let V be a regular associative Γ-algebra such that every element is (α, α)-regular. Then, $\theta = \{x \in V : x\alpha y = y\alpha x \text{ for all } y \in V, \alpha \in \Gamma : x = x\alpha y\alpha x\}$ is (α, α)-regular.*

Proof. Suppose that $x \in \theta$. There exists $y \in V$ such that $x = x\alpha y\alpha x$. We set $z = y\alpha x\alpha y$. Then, we obtain that

$$x\alpha z\alpha x = x\alpha(y\alpha x\alpha y)\alpha x = (x\alpha y\alpha x)\alpha y\alpha x = x\alpha y\alpha x = x.$$

We have

$$z\alpha v = y\alpha x\alpha y\alpha v = (y\alpha y)\alpha v\alpha x\alpha x = y\alpha x\alpha y\alpha x\alpha v\alpha y = y\alpha x\alpha v\alpha y.$$

In the same way, $v\alpha z = y\alpha v\alpha x\alpha y = y\alpha x\alpha v\alpha y = z\alpha v$, where $v \in V$. Therefore, $z \in \theta$ and θ is (α, α)-regular. □

Let V be a Γ-algebra. An equivalence relation ρ on V is called *regular* if for every a_1, a_2, b_1, b_2, such that (a_1, b_1) and $(a_2, b_2) \in \rho$, then $(a_1 + a_2, b_1 + b_2) \in \rho$ and for all $\alpha \in \Gamma$, $(a_1\alpha a_2, b_1\alpha b_2) \in \rho$ and is called *strong regular* if $(a_1 + a_2, b_1 + b_2) \in \rho$ and $(a_1\alpha a_2, b_1\beta b_2) \in \rho$ for every $\alpha, \beta \in \Gamma$.

Suppose that ρ is a regular relation on a Γ-algebra. We define a binary operations on $[V : \rho]$, the set of all equivalence classes, as follows:

$$\begin{aligned} \rho(a)\widehat{\alpha}\rho(b) &= \rho(a\alpha b), \\ \rho(a) \oplus \rho(b) &= \rho(a + b). \end{aligned}$$

Let $a_1, a_2, b_1, b_2 \in V$ and $\rho(a_1) = \rho(b_1)$ and $\rho(a_2) = \rho(b_2)$. Then,

$$\begin{aligned} (a_1, b_1) \in \rho \text{ and } (a_2, b_2) \in \rho &\Longrightarrow (a_1\alpha a_2, b_1\alpha b_2) \in \rho \\ &\Longrightarrow \rho(a_1)\widehat{\alpha}\rho(a_2) = \rho(b_1)\widehat{\alpha}\rho(b_2) \end{aligned}$$

and $\rho(a_1) \oplus \rho(a_2) = \rho(b_1) \oplus \rho(b_2)$.

It is easy to see that $[V : \rho]$ is a $\widehat{\Gamma}$-algebra. Suppose that ρ is a strong regular relation. Then, for every $\alpha, \beta \in \Gamma$

$$\rho(a)\widehat{\alpha}\rho(b) = \rho(a)\widehat{\beta}\rho(b).$$

Hence, $[V : \rho]$ is an algebra.

Suppose that V is a Γ-algebra and a is an element of V. We say that b is an (α, β)-*inversion* of a if $a\alpha b\beta a = a,\ b\beta a\alpha b = b$.

Example 3.2. *Let $V = \mathbb{R}^3$ and $\Gamma = \{(r, 0, 0) : r \in \mathbb{R}\}$. Then, V is a Γ-algebra with Γ-dimension 1. If $a = (1, 0, 0)$, $b = (3, 0, 0)$, $\alpha = (2, 0, 0)$, $\beta = (\frac{1}{6}, 0, 0)$, then b is an (α, β)-inversion of a.*

Suppose that V is an associative Γ-algebra and a is an (α, β)-regular. Then, there exist $\alpha, \beta \in \Gamma$ and $b \in V$ such that $a = a\alpha b\beta a$. Let $x = b\beta a\alpha b$. Then, we observe that

$$a\alpha x\beta a = a\alpha(b\beta a\alpha b)\beta a = (a\alpha b\beta a)\alpha b\beta a = a\alpha b\beta a = a;$$

$$\begin{aligned} x\beta a\alpha x &= (b\beta a\alpha b)\beta a\alpha(b\beta a\alpha b) = b\beta(a\alpha b\beta a)\alpha(b\beta a\alpha b) \\ &= b\beta a\alpha b\beta a\alpha b = b\beta(a\alpha b\beta a)\alpha b = b\beta a\alpha b = x. \end{aligned}$$

Proposition 3.4. *Let ρ be a regular relation on a regular associative Γ-algebra and $\rho(a)$ be an idempotent in $[V : \rho]$. Then, there exists an idempotent element e in V such that $\rho(a) = \rho(e)$.*

Proof. Suppose that $\rho(a)$ is a γ-idempotent element in $[V:\rho]$. Then, there exists $\gamma \in \Gamma$ such that $\rho(a) = \rho(a)\widehat{\gamma}\rho(a) = \rho(a\gamma a)$. Let x be an (α,β)-inversion of $a\gamma a$. Then,

$$(a\gamma a)\alpha x\beta(a\gamma a) = a\gamma a x\beta(a\gamma a)\alpha x = x.$$

Let $e = a\alpha x\beta a$. Then,

$$e\gamma e = (a\alpha x\beta a)\gamma(a\alpha x\beta a) = a\alpha(x\beta a\gamma a\alpha x)\beta a = a\alpha x\beta a = e.$$

and so e is γ-idempotent. We have

$$(a\alpha x\beta a, (a\gamma a)\alpha x\beta(a\gamma a)) \in \rho,$$

and $(e, a\gamma a) \in \rho$. Therefore, $\rho(e) = \rho(a\gamma a)$. □

Theorem 3.1. *Let V be an associative Γ-algebra such that $\{0\}$ is a semiprime ideal, every family of semiprime ideals has a maximal element and $[V:P]$ is (α,β)-regular for all prime ideal of V. Then, V is a regular algebra.*

Proof. Suppose that V is not regular. Then, there exists $x \in V$ such that $x \notin x\Gamma V\Gamma x$. There exists a semiprime ideal P in V such that it is maximal with respect the property $x \notin x\Gamma V\Gamma x + P$. If $[V:P]$ is regular, then

$$x + P \in (x+P)\widehat{\Gamma}[V:P]\widehat{\Gamma}(x+P).$$

Hence, there exists $y + P \in [V:P]$ such that

$$x + P \in (x+P)\widehat{\alpha}(y+P)\widehat{\beta}(x+P) = x\alpha y\beta x + P.$$

This implies that $x \in x\alpha y\beta x + P \subseteq x\Gamma y\Gamma x + P$, which is a contradiction. Thus, $x \notin x\Gamma V\Gamma x + P$. Then, P is not prime. Hence, there exist ideals A and B such that $A\Gamma^{\Sigma}B \subseteq P$ and $A \nsubseteq P$, $B \nsubseteq P$. Now, suppose that $T_1 = \{v \in V : v\Gamma^{\Sigma}B \subseteq P\}$ and $T_2 = \{v \in V : T_1\Gamma^{\Sigma}v \subseteq P\}$. We see that T_1 and T_2 are semiprime.

Now, let A_1 and A_2 be two ideals such that $A_1\Gamma^{\Sigma}A_1 \subseteq T_1$. Then, $(A_1\Gamma^{\Sigma}A_1)\Gamma^{\Sigma}B$ and $A_1\Gamma^{\Sigma}(A_1\Gamma^{\Sigma}B) \subseteq P$. Since P is prime and $B \nsubseteq P$, implies that $A_1 \subseteq P$. In the same way, one can see that T_2 is a semiprime ideal. On the other hand

$$(T_1 \cap T_2)\Gamma^{\Sigma}(T_1 \cap T_2) \subseteq T_1\Gamma^{\Sigma}T_2 \subseteq P.$$

Hence, $T_1 \cap T_2 \subseteq P$. Since $A \nsubseteq P$ and $B \nsubseteq P$, T_1 and T_2 properly contain P. Because the maximality of P, $[V:T_1]$ and $[V:T_2]$ are regular. Thus, there exist $x_1, x_2 \in V$ such that

$$\begin{aligned} x + P &= (x+P)\widehat{\alpha}(x_1+P)\widehat{\beta}(x+P), \\ x + P &= (x+P)\widehat{\alpha}(x_2+P)\widehat{\beta}(x+P). \end{aligned}$$

Thus, $x - x\alpha x_1\beta x \in T_1$ and $x - x\alpha x_2\beta x \in T_2$. This implies that

$$x - x\alpha(x_1 + x_2 - x_1\beta x\alpha x_2)\beta x = (x - x\alpha x_1\beta x) - (x - x\alpha x_1\beta x)\alpha x\beta x \in T_1$$

and

$$x - x\alpha(x_1 + x_2 - x_1\beta x\alpha x_2)\beta x = (x - x\alpha x_2\beta x) - x\alpha x_1\beta(x - x\alpha x_2\beta x) \in T_2.$$

We conclude that $x \in x\Gamma V\Gamma x + T_1 \cap T_2 \subseteq x\Gamma V\Gamma x + P$, which is a contradiction. Therefore, V must be regular. □

Proposition 3.5. *Let V be an associative unital Γ-algebra and set*

$$\Theta = \left\{ x \in V : V\Gamma^{\sum} x\Gamma^{\sum} V \quad \text{is an } (\alpha,\beta)\text{-regular ideal} \right\}.$$

Then, Θ is an (α,β)-regular ideal and $[V : \Theta]$ has no non-zero (α,β)-regular ideal.

Proof. Suppose that $x, y \in \Theta$. Then, $V\Gamma^{\sum}x\Gamma^{\sum}V$ and $V\Gamma^{\sum}y\Gamma^{\sum}V$ are (α,β)-regular ideals. By Lemma 4.3, $V\Gamma^{\sum}x\Gamma^{\sum}V + V\Gamma^{\sum}x\Gamma^{\sum}V$ is a regular ideal. Since

$$V\Gamma^{\sum}(x+y)\Gamma^{\sum}V \subseteq V\Gamma^{\sum}x\Gamma^{\sum}V + V\Gamma^{\sum}y\Gamma^{\sum}V,$$

$V\Gamma^{\sum}(x+y)\Gamma^{\sum}V$ is regular. In the same way, we can see that $\Theta\Gamma V, V\Gamma\Theta \subseteq \Theta$.

Let J be an (α,β)-regular ideal of V and $x \in J$. Then,,

$$V\Gamma^{\sum}x\Gamma^{\sum}V \subseteq V\Gamma^{\sum}J\Gamma^{\sum}V \subseteq J.$$

Hence, $V\Gamma^{\sum}x\Gamma^{\sum}V$ is (α,β)-regular and $J \subseteq \Theta$. Let $[J : \Theta]$ be an (α,β)-regular ideal of $[V : \Theta]$. Since Θ is (α,β)- regular, J is (α,β)-regular and $J \subseteq \Theta$. This implies that $[V : \Theta]$ has not non-zero (α,β)-regular ideal. □

Proposition 3.6. *Let V be a regular Γ-algebra. Then, the dimension and the Γ-dimension of V are equal.*

Proof. Let $x \in V$. Since V is regular there exist $\alpha, \beta \in \Gamma$ and $y \in V$ such that $x = x\alpha y\beta x$. This completes the proof. □

4. *T*-functor and H-system

The category ΓAL is the category whose objects are Γ- algebras. For Γ_1-algebra V_1 and Γ_2- algebra V_2, $Mor(V_1, V_2)$ is the set of all (Γ_1, Γ_2)- epimorphisms. The composition of morphisms denotes the usual composition of homomorphisms and so satisfies the associative law. $(Id_V, Id_\Gamma) : (V,\Gamma) \longrightarrow (V,\Gamma)$ is the identity map satisfies the required property $(Id_V, Id_\Gamma) \circ (\varphi, f) = (\varphi, f)$ for every $(\varphi, f) \in Mor(V', V)$ and $(\varphi, f) \circ (Id_V, Id_\Gamma) = (\varphi, f)$, for every $(\varphi, f) \in Mor(V, V')$. The category AL is the category whose objects are algebras and $Mor(A_1, A_2)$ is the set of all algebra homomorphisms from A_1 to A_2 and it satisfies the associative law.

Let V be a Γ-algebra and

$$\Delta_V = \left\{ \prod_{i=1}^{n} (x_i, \alpha_i) : \alpha_i \in \Gamma,\ x_i \in V,\ n \in \mathbb{N} \right\}.$$

Then, the relation θ on Δ_V defined by

$$\left(\prod_{i=1}^{n} (x_i, \alpha_i) \right) \theta \left(\prod_{j=1}^{m} (y_j, \beta_j) \right) \text{ if and only if } \sum_{i=1}^{n} x_i\alpha_i x = \sum_{j=1}^{m} y_j\beta_j x,\ \forall x \in V,$$

is an equivalence relation. We denote the equivalence class containing $\prod_{i=1}^{n}(x_i,\alpha_i)$ by $\theta\left(\prod_{i=1}^{n}(x_i,\alpha_i)\right)$. Then, $[\Delta_V:\theta]$ forms a vector space. Now, we define a multiplication on $[\Delta_V:\theta]$ as follows:

$$\theta\left(\prod_{i=1}^{n}(x_i,\alpha_i)\right)\theta\left(\prod_{j=1}^{n}(y_j,\beta_j)\right)=\theta\left(\prod_{i,j}(x_i\alpha_i y_j,\beta_j)\right).$$

We denote this algebra by V_L and is called the *left operator algebra*. In the same way, we can define the *right operator algebra*.

Proposition 4.1. *Let V_1 and V_2 be Γ_1- and Γ_2- algebras, respectively. If $(\varphi,f):(V_1,\Gamma_1)\longrightarrow(V_2,\Gamma_2)$ is an epimorphism, then there exists a unique homomorphism $\overline{(\varphi,f)}:[\Delta_{V_1}:\theta_1]\longrightarrow[\Delta_{V_2}:\theta_2]$ such that the following diagram is commutative:*

$$\begin{array}{ccc}(V_1,\Gamma_1) & \overset{(\varphi,f)}{--\longrightarrow} & (V_2,\Gamma_2)\\ \downarrow & & \downarrow\\ [\Delta_{V_1},\theta_1] & \overset{\overline{(\varphi,f)}}{--\longrightarrow} & [\Delta_{V_2},\theta_2]\end{array}$$

Moreover, if (φ,f) is an isomorphism, then $\overline{(\varphi,f)}$ is an isomorphism.

Proof. We define $\overline{(\varphi,f)}:[\Delta_{V_1},\theta_1]\longrightarrow[\Delta_{V_2},\theta_2]$ by

$$\overline{(\varphi,f)}\left(\theta\left(\prod_{i=1}^{n}(x_i,\alpha_i)\right)\right)=\theta\left(\prod_{i=1}^{n}(\varphi(x_i),f(\alpha_i))\right),$$

for every $\theta\left(\prod_{i=1}^{n}(x_i,\alpha_i)\right)\in[\Delta_{V_1},\theta_1]$. It is easy to see that this function is well-defined and homomorphism. One can see that if (φ,f) is an isomorphism, then induced homomorphism $\overline{(\varphi,f)}$ is an isomorphism. □

Corollary 4.1. *There is a covariant functor between the subcategory of Γ-algebras and the category of algebras.*

Proof. By Proposition 4.1, it is straightforward. □

Let $(\varphi_1,f_1):(V_1,\Gamma_1)\longrightarrow(V_2,\Gamma_2)$ and $(\varphi_2,f_2):(V_1,\Gamma_1)\longrightarrow(V_2,\Gamma_2)$ be homomorphisms. We define

$$S(\varphi_1,\varphi_2)=\left\{\sum_{r=1}^{n}\varphi_i(v_r)f_j(\alpha_r)v : v_r\in V_1,\alpha_r\in\Gamma_1,n\in\mathbb{N},1\leq i,j\leq 2,i\neq j\right\}.$$

This homomorphism is said to be *S-conjugate* if $S(\varphi_1,\varphi_2)=0$.

Let $V_1,V_2,\ldots V_n$ and V be Γ_1-, Γ_2-, …, Γ_n- and $\Gamma=\Gamma_1\times\Gamma_2\ldots\Gamma_n$- algebras, respectively, and suppose that we are given (Γ_i,Γ)- homomorphisms $(\sigma_i,\chi_i):(V_i,\Gamma_i)\longrightarrow(V,\Gamma)$, $(1\leq i\leq n)$ and (Γ,Γ_i)- homomorphism $(\pi_i,\vartheta_i):(V,\Gamma)\longrightarrow(V_i,\Gamma_i)$, $(1\leq i\leq n)$ such that $\pi_j\sigma_i=\delta_{ij}$ and $\sum\sigma_i\pi_i=Id_V$. Then, V is called an H- system.

Proposition 4.2. *Let V be an H-system and $(\varphi_i, f_i) : (V_i, \Gamma_i) \longrightarrow (W, \Gamma)$, $(1 \leq i \leq n)$ are given. Then, there exists a unique homomorphism $(\varphi, f) : (V, \Gamma) \longrightarrow (W, \Gamma)$ such that $(\varphi, f) \circ (\sigma_i, \chi_i) = (\varphi_i, f_i)$. If $(\psi_i, g_i) : (W, \Gamma) \longrightarrow (V_i, \Gamma_i)$, then there exists a unique homomorphism $(\psi, g) : (W, \Gamma) \longrightarrow (V, \Gamma)$ such that $(\psi, \vartheta_i) \circ (\psi, g) = (\psi_i, g_i)$.*

Proof. Suppose that $(\varphi, g) : (W, \Gamma) \longrightarrow (V, \Gamma)$ defined by $\varphi = \sum_{j=1}^{n} \varphi_j \phi_j$. Then,

$$\varphi\sigma_i = \left(\sum_{j=1}^{n} \varphi_j \phi_j \right) \sigma_i = \sum_{j=1}^{n} \varphi_j \phi_j \sigma_i = \sum_{j=1}^{n} \varphi_j \phi_j \delta_{ij} = f_i.$$

It is easy to see that this homomorphism is unique.

Now, we define $\psi : W \longrightarrow V$ by $\psi = \sum_{j=1}^{n} \sigma_j \psi_j$. This is a unique homomorphism such that $\pi_i \psi = \psi_i$. This completes the proof. □

Theorem 4.1. *Let Ω be a subcategory of ΓAL such that for every H-system V of Ω, Δ_V is an H-system in AL. Then, for every morphism φ_1 and φ_2 in Ω, $T(\varphi_1 + \varphi_2) = T(\varphi_1) + T(\varphi_2)$.*

Proof. Suppose that $(\varphi_i, f_i) : (V_i, \Gamma_i) \longrightarrow (W_i, \Gamma_i)$, $(1 \leq i \leq 2)$ are morphisms. Since Δ_{V_1} is an H-system of AL, we have $T(\pi_1)T(\sigma_1 + \sigma_2)$ and $T(\pi_2)T(\sigma_1 + \sigma_2)$ are identity morphisms. Hence,

$$T(\sigma_1 + \sigma_2) = T(\sigma_1)T(\pi_1)T(\sigma_1 + \sigma_2) + T(\sigma_2)T(\phi_2)T(\sigma_1 + \sigma_2).$$

We define $\varphi : W \longrightarrow V_2$ by $\varphi = \varphi_1 \pi_1 + \varphi_2 \pi_2$. Then, $\varphi\sigma_1 = \varphi_1$ and $\varphi\sigma_2 = \varphi_2$. Moreover, $\varphi(\sigma_1 + \sigma_2) = \varphi_1 + \varphi_2$. Hence,

$$T(\varphi_1 + \varphi_2) = T(\varphi_1) + T(\varphi_2).$$

This completes the proof. □

Theorem 4.2. *Let $0 \longrightarrow (V_1, \Gamma_1) \stackrel{(\sigma_1, f_1)}{\longrightarrow} (V, \Gamma) \stackrel{(\pi_2, g_2)}{\longrightarrow} (V_2, \Gamma_2) \longrightarrow 0$ be an exact sequence in ΓAL. Then, the following statements are equivalent:*

1. *There exists (Γ_2, Γ)-homomorphism $(\sigma_2, f_2) : (V_2, \Gamma_2) \longrightarrow (V, \Gamma)$ and (Γ, Γ_1)-homomorphism $(\pi_1, g_1) : (V, \Gamma) \longrightarrow (V_1, \Gamma_1)$ such that V is an H-system.*
2. *There exists Γ-subalgebra of V_1 such that $V = (\sigma_1, f_1)(V_1, \Gamma_1) \oplus V_1$.*

Proof. The proof is straightforward. □

Proposition 4.3. *Let $0 \longrightarrow (V_1, \Gamma_1) \stackrel{(\sigma_1, f_1)}{\longrightarrow} (V, \Gamma) \stackrel{(\pi_2, g_2)}{\longrightarrow} (V_2, \Gamma_2) \longrightarrow 0$ be a split exact sequence in ΓAL. Then, $0 \longrightarrow \Delta_{V_1} \stackrel{T(\sigma_1, f_1)}{\longrightarrow} \Delta_V \stackrel{T(\pi_2, g_2)}{\longrightarrow} \Delta_{V_2} \longrightarrow 0$ is a split exact sequence in AL.*

Proof. The proof is straightforward. □

Proposition 4.4. *Let for every split exact sequence*

$$0 \longrightarrow (V_1, \Gamma_1) \longrightarrow (V, \Gamma) \longrightarrow (V_2, \Gamma_2) \longrightarrow 0$$

implies that $0 \longrightarrow \Delta_{V_1} \longrightarrow \Delta_V \longrightarrow \Delta_{V_2} \longrightarrow 0$ is a split exact sequence. Then, for every homomorphism φ_1,φ_2, we have $T(\varphi_1 + \varphi_2) = T(\varphi_1) + T(\varphi_2)$.

Proof. Suppose that V is an H-system. This implies that

$$0 \longrightarrow (V_1, \Gamma_1) \stackrel{(\sigma_1, f_1)}{\longrightarrow} (V, \Gamma) \stackrel{(\pi_2, g_2)}{\longrightarrow} (V_2, \Gamma_2) \longrightarrow 0$$

is a split exact sequence. By hypothesis

$$0 \longrightarrow \Delta_{V_1} \stackrel{T(\sigma_1, f_1)}{\longrightarrow} \Delta_V \stackrel{T(\pi_2, g_2)}{\longrightarrow} \Delta_{V_2} \longrightarrow 0$$

is a split exact sequence. In the same way

$$0 \longrightarrow \Delta_{V_2} \stackrel{T(\sigma_2, f_2)}{\longrightarrow} \Delta_V \stackrel{T(\pi_1, g_1)}{\longrightarrow} \Delta_{V_1} \longrightarrow 0$$

is a split exact sequence. Hence,

$$T(\pi_2, g_2)T(\sigma_1, f_1) = T((\pi_2, g_2)(\sigma_1, f_1)) = Id,$$

$$T(\pi_1, g_1)T(\sigma_2, f_2) = T((\pi_1, g_1)(\sigma_2, f_2)) = Id.$$

By a routine process, $T(V, \Gamma)$ is an H-system. This completes the proof. □

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