APPROXIMATION OF BACKWARD HEAT CONDUCTION PROBLEM USING GAUSSIAN RADIAL BASIS FUNCTIONS

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In this work an efficient numerical method is applied for investigation of the backward heat conduction problem in an unbounded region. The problem is ill-posed, in the sense that the solution if it exists, does not depend continuously on the data. The Gaussian radial basis functions are used for discretization of the problem. The presented method is reducing the problem to an interpolation problem which is more simple than the collocation type method. To regularize the resultant ill-conditioned linear system of equations, we apply successfully both the Tikhonov regularization technique and the L-curve method to obtain a stable numerical approximation to the solution. A new convenient and simply applicable method is derived. The stability and convergence of the proposed method are investigated. Two examples are presented to illustrate efficiency and accuracy of the proposed method.

Keywords: Backward heat conduction problem, ill-posed Problem, Gaussian radial basis function.

1. Introduction

We consider the backward heat conduction problem (BHCP). The BHCP is also referred as a final value problem [1]. As is known, BHCP is severely ill posed; i.e. the solution does not always exist, and when it exists, it does not depend continuously on the given data. So that the numerical simulations are very difficult and some special regularization methods are required. This problem has been considered by several authors in recent decades. Lattes and Lions [2], Showalter [3], Ames et al. [4] and Miller [5] have approximated BHCP by quasi-reversibility method. Tautenhahn and Schroter established an optimal error estimate for a special BHCP [6]. Seidman established an optimal filtering method [7]. Recently Fu et al. used Fourier regularization method [8]. So far there are many papers on the backward heat equation [9, 10, 11], but theoretically the error estimates of most regularization method in the literature are of holder type i.e., the approximate solution v and the exact solution u satisfies $\|u(.,t)-v(.,t)\| \le 2E^{1-\frac{t}{T}}\delta^{\frac{t}{T}}$, where E is a priori bound on $u(x,0)$ and δ is the noise level on final data $u(x,T)$. We note that the right part of above inequality tends to 0 as measurement accuracy is improved ($\delta \longrightarrow 0$), so this

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means that one can achieve arbitrary accuracy in determining the unknown $u\mid_{t\neq 0}$, if one will measure with adequate accuracy. But the error bound at $t=0$ is not more suitable. In this paper we consider the following one dimensional backward heat conduction problem in an unbounded region

$$\begin{cases} u_t = u_{xx}, & -\infty < x < \infty, 0 \le t < T, \\ u(x,T) = h(x), & -\infty < x < \infty, \end{cases} \tag{1}$$

where we want to determine the temperature distribution $u(.,t)$ for $0 \le t < T$ from the final data $h(x)$. The aim of this paper is to provide a new convenient and simply applicable method for obtaining the suitable solution for the problem (1) specially at $t=0$. Therefore we use the Gaussian radial basis functions (GRBFs) for discretization of the problem and the suitable attributes of this functions are used to obtain numerically stable scheme. The unknown coefficients in this method is the unknown coefficient in the interpolation of the given data function h, using RBFs interpolation. The proposed method is reducing the problem to an interpolation problem which is more simple than the collocation type method. Convergence analysis of RBF interpolation has been carried out by several researchers. Results for two numerical examples are presented to demonstrate the efficacy of the method.

2. **The approximate solution**

In this section we try to obtain an approximate solution for (1) based on discretization using Gaussian RBF. No matter how the distinct data points are scattered. In practice we have measured final data $(x_i, h(x_i))|_{(i=1,\ldots,N)}$ at N distinct points. We write the RBF approximation of $u(x,0)$ in the following form:

$$u_N(x,0) = \sum_{i=1}^{N} \lambda_i \phi_i(x), \tag{2}$$

where $\phi_i(x)\mid_{(i=1,\ldots,N)}$ are RBFs and coefficients $\lambda_i\mid_{(i=1,\ldots,N)}$ are unknown to be determined. For solving the problem (1), we have to consider the following initial value problem:

$$\begin{cases} u_t = u_{xx}, & -\infty < x < \infty, 0 \le t < T, \\ u(x,0) = f(x), & -\infty < x < \infty. \end{cases} \tag{3}$$

By following the reference [12], the solution of the above problem exists uniquely

$$u(x,t) = \int_{-\infty}^{+\infty} k(x-s,t) f(s) ds = K_t(f(x)), \tag{4}$$

where K_t is a integral operator with kernel $k(x-s,t)$ and $k(x,t)$ is the fundamental solution of heat equation, $k(x,t) = \frac{1}{\sqrt{4\pi t}} exp(\frac{-x^2}{4t})$. By substituting the final data of (1) into (4), we have

$$u(x,T) = K_T(f(x)) = \int_{-\infty}^{+\infty} k(x-s,T) f(s) ds = h(x), \tag{5}$$

where $u(x,0) = f(x)$ is unknown. We use the Gaussian RBF in the following form

$$\phi_i(x) = exp(-\frac{(x-x_i)^2}{c^2}), \quad (i = 1, \ldots, N), \tag{6}$$

and by operating, K_t in $\phi_i(x)$ we have

$$\Phi_i(x,t) = k_t(\phi_i(x)) = \frac{c}{\sqrt{c^2+4t}}\, exp(-\frac{(x-x_i)^2}{c^2+4t}). \tag{7}$$

Substituting the expansion of $u_N(x,0) = f_N(x)$ of the form (2) into (5),

$$h(x) = \int_{-\infty}^{+\infty} k(x-s,T)\sum_{i=1}^{N}\lambda_i\phi_i(s)ds = \sum_{i=1}^{N}\lambda_i k_T(\phi_i(x)) = \sum_{i=1}^{N}\lambda_i\Phi_i(x,T), \tag{8}$$

where $\phi_i(x)$ and $\Phi_i(x,T)$ are Gaussian RBF of the form (6) and (7), and by substituting collocation points x_i $(i = 1,\dots,N)$ in (8), we have the following system in the matrix form

$$A_N C = b, \tag{9}$$

where

$$A_N = A_{N\times N} = \{a_{ij}\}, a_{ij} = \Phi_i(x_j,T),\ C = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_N \end{pmatrix},\ b = \begin{pmatrix} h(x_1) \\ \vdots \\ h(x_N) \end{pmatrix}. \tag{10}$$

By solving (9), we get the unknown coefficient $\lambda_i \mid_{(i=1,\dots,N)}$. In fact the matrix A is the Gaussian RBF interpolation matrix of the final data function $h(x)$ at distinct points x_i $(i = 1,\dots,N)$, which is symmetric and nonsingular [13], i.e. there is an unique interpolant of the form (8) no matter how the distinct data points are scattered. This is an advantage of the proposed method versus the conventional collocation methods based on radial basis functions. By solving the linear system (9) we approximate $u(x,0) = f(x)$ in following form

$$u_N(x,0) = f_N(x) = \sum_{i=1}^{N}\lambda_i exp(-\frac{(x-x_i)^2}{c^2}), \tag{11}$$

and by using (7) and (4) we have $u_N(x,t)$ for constant t in following form

$$u_N(x,t) = \int_{-\infty}^{+\infty} k(x-s,t)f_N(s)ds = \sum_{i=1}^{N}\lambda_i k_t(exp(-\frac{(x-x_i)^2}{c^2}))$$

$$= \sum_{i=1}^{N}\lambda_i\frac{c}{\sqrt{c^2+4t}}\, exp(-\frac{(x-x_i)^2}{c^2+4t}). \tag{12}$$

Differentiating $\Phi_i(x,t)$ we have

$$\frac{\partial\Phi_i(x,t)}{\partial t} = \Phi_i(x,t)(\frac{-2}{c^2+4t} + \frac{4(x-x_i)^2}{(c^2+4t)^2}),$$

$$\frac{\partial^2\Phi_i(x,t)}{\partial x^2} = \Phi_i(x,t)(\frac{-2}{c^2+4t} + \frac{4(x-x_i)^2}{(c^2+4t)^2}),$$

so by substituting $u_N(x,t)$ defining by (12) in $u_t - u_{xx} = 0$ we can see

$$\frac{\partial u_N(x,t)}{\partial t} - \frac{\partial^2 u_N(x,t)}{\partial x^2} = 0.$$

In fact the problem reduce to a RBF interpolation problem of known data function. The coefficients $\lambda_i \mid_{(i=1,\ldots,N)}$ in (11) are the interpolation coefficients of the known data function h, that obtain by solving the linear system (9).

3. **Convergence**

The superior accuracy of the RBF interpolation is supported by theoretical error estimate. For example, for the case of interpolating a regular function, Madych and Nelson [14] showed that for a class of RBFs including the multiquadric and the Gaussian interpolation error convergence at the exponential rate,

$$\mid f - f_N \mid = O(\gamma^{\frac{1}{\omega}}), \tag{13}$$

where ω is the maximum mesh size, and $0 < \gamma < 1$. For the Gaussian, Wendlandm [15] further refined the error bound as $\mid f - f_N \mid = O(\gamma^{\sqrt{\frac{1}{\omega}}})$.

Lemma 3.1. *Suppose that h and h^δ are exact and measured data such that $\mid h - h^\delta \mid \leq \delta$. Let $u_N(x,0) = f_N$ and $u_N^\delta(x,0) = f_N^\delta$ be solution obtained by proposed method using h and h^δ, respectively. Then we have*

$$\mid f_N - f_N^\delta \mid \leq \frac{\sqrt{c^2+4T}}{c}[2O(\gamma^{\frac{1}{\omega}}) + \delta].$$

Proof. Suppose that h_N and h_N^δ are RBF approximation of h and h^δ. From (13) we have

$$\mid h_N - h_N^\delta \mid = \mid h_N - h + h - h^\delta + h^\delta - h_N^\delta \mid \leq \mid h_N - h \mid + \mid h - h^\delta \mid + \mid h^\delta - h_N^\delta \mid \leq 2O(\gamma^{\frac{1}{\omega}}) + \delta.$$

And we have

$$exp(-\frac{(x-x_i)^2}{c^2}) \leq exp(-\frac{(x-x_i)^2}{c^2+4T}) \Rightarrow \phi_i(x) \leq \frac{\sqrt{c^2+4T}}{c}\Phi_i(x,T), \tag{14}$$

then we have

$$\mid f_N - f_N^\delta \mid = \mid \sum_{i=1}^{N}(\lambda_i - \lambda_i^\delta)\phi_i(x) \mid \leq \frac{\sqrt{c^2+4T}}{c} \mid \sum_{i=1}^{N}(\lambda_i - \lambda_i^\delta)\Phi_i(x,T) \mid =$$
$$\frac{\sqrt{c^2+4T}}{c} \mid h_N - h_N^\delta \mid \leq \frac{\sqrt{c^2+4T}}{c}[2O(\gamma^{\frac{1}{\omega}}) + \delta].$$

From the Lemma (3.1) we see that the approximate solution given by (2) depends continuously on the given data.

Lemma 3.2. *For any regular function we have, $\| K_t(f(x)) \| \leq \| f(x) \|$.*

Proof. Let $\widehat{f}$ be the fourier transform of f then with respect to Parseval's relation and fourier transform of convolution we have

$$\| k_t(f(x)) \| = \| \widehat{k_t(f)} \| = \| \widehat{k}_t \hat{f} \| = \| e^{-ts^2} \hat{f} \| \leq \| \hat{f} \| = \| f(x) \| . \tag{15}$$

Theorem 3.1. *Suppose that h and h^δ are exact and measured data such that $\mid h - h^\delta \mid \leq \delta$. Let $u_N(x,t)$ and $u_N^\delta(x,t)$ be solution obtained by proposed method using h and h^δ, respectively. Then*

$$\mid u(x,t) - u_N^\delta(x,t) \mid \leq C_1 O(\gamma^{\frac{1}{\omega}}) + C_2\delta,$$

where $\gamma < 1$, ω is maximum step size, $C_1 = 1 + 4\frac{\sqrt{c^2+4T}}{c}$ and $C_2 = \frac{\sqrt{c^2+4T}}{c}$.

Proof. If we substitute the approximation of $f(x)$ with interpolated function using Gaussian RBF in Eq. (5) i.e. $\bar{f}_N(x) = \Sigma_{i=1}^{N}\bar{\lambda}_i\phi_i(x)$ then the right part of the equation is exchanged by a new function that we denote it by $\bar{h}$. So $\bar{h}_N(x) = \int_{-\infty}^{+\infty} k(x-s,T)\bar{f}_N(s)ds$, and we obtain the system

$$A_N\bar{C} = \bar{b}, \tag{16}$$

where

$$A_N = A_{N\times N} = \{a_{ij}\}, a_{ij} = \Phi_i(x_j,T), \bar{C} = \begin{pmatrix} \bar{\lambda}_1 \\ \vdots \\ \bar{\lambda}_N \end{pmatrix}, \bar{b} = \begin{pmatrix} \bar{h}(x_1) \\ \vdots \\ \bar{h}(x_N) \end{pmatrix}. \tag{17}$$

Let $u_N(x,0) = f_N(x)$ then

$$\mid f - f_N \mid \leq \mid f - \bar{f}_N \mid + \mid \bar{f}_N - f_N \mid, \tag{18}$$

by relation (13) we have

$$\mid f - \bar{f}_N \mid \leq O(\gamma^{\frac{1}{\omega}}), \tag{19}$$

and using (14) we have

$$\mid \bar{f}_N - f_N \mid = \mid \sum_{i=1}^{N}(\bar{\lambda}_i - \lambda_i)\phi_i(x) \mid \leq \frac{\sqrt{c^2+4T}}{c} \mid \sum_{i=1}^{N}(\bar{\lambda}_i - \lambda_i)\Phi_i(x,T) \mid$$

$$= \frac{\sqrt{c^2+4T}}{c} \mid \bar{h}_N - h_N \mid, \tag{20}$$

where $\lambda_i \mid_{(i=1,\ldots,N)}$ and $\bar{\lambda}_i \mid_{(i=1,\ldots,N)}$ obtained by (9) and (16), respectively. Using Lemma (3.2) and (13) we have

$$\mid \bar{h}_N - h_N \mid \leq \mid \bar{h}_N - h \mid + \mid h - h_N \mid = \mid K_T(\bar{f}_N - f) \mid + \mid h - h_N \mid$$

$$\leq \mid \bar{f}_N - f \mid + \mid h - h_N \mid \leq 2O(\gamma^{\frac{1}{\omega}}). \tag{21}$$

Then by using Eqs. (18)-(21),

$$\mid f - f_N \mid \leq (1 + 2\frac{\sqrt{c^2+4T}}{c})O(\gamma^{\frac{1}{\omega}}),$$

and by using Lemma (3.1) and Lemma (3.2),

$$\mid u(x,t) - u_N^{\delta}(x,t) \mid = \mid \int k(x-s,t)(f(x) - f_N^{\delta}(x)) \mid \leq \mid f - f_N^{\delta} \mid = \mid f - f_N + f_N - f_N^{\delta} \mid$$

$$\leq \mid f - f_N \mid + \mid f_N - f_N^{\delta} \mid < (1 + 2\frac{\sqrt{c^2+4T}}{c})O(\gamma^{\frac{1}{\omega}}) + 2\frac{\sqrt{c^2+4T}}{c}O(\gamma^{\frac{1}{\omega}}) + \frac{\sqrt{c^2+4T}}{c}\delta.$$

The right hand side of estimates in Theorem (3.1) and Lemma (3.1) contain two terms. The first term represents the error due to interpolation and tends to zero as $\omega \to 0$. The second term represents the error due to noise in the given data and tends to zero as $\delta \to 0$. So this means that one can achieve arbitrary accuracy in results, if one will measure with adequate accuracy and use the suitable interpolation of the given data. The method should not produce results more accurate than the level of error in the given data. Many RBF methods contain a free shape parameter that plays an important role for the accuracy of the method. In proposed method the unknown coefficients $\lambda_i \mid_{(i=1,\ldots,N)}$ in (2) are the interpolation coefficients of the given data function h, that obtain by solving the linear system (9). So the shape parameter in (6) is the shape parameter in (7) that used for RBF interpolation of the given data function h. Despite research done by many scientists to develop algorithms for selecting the value of shape parameter which produce the most accurate interpolation [16, 17, 18], the optimal choice of shape parameter is still an open problem. The proposed methods in literature for selecting the optimal shape parameter for RBF interpolation can be use for improving the method. The main difficulty with the method based on radial basis function is that the condition number of the interpolation matrix A can be enormous. In addition in a real-world application, the right-hand side vector of (9) is always contaminated by various types of errors, such as measurement, approximation and rounding errors. Thus the large condition number of the matrix can be disastrous. Standard method may fail to yield satisfactory results due to the combination of the matrix and noise in data. In order to obtain stable and accurate results, more advanced computational method must be applied to solve the matrix equation. Regularization methods are most powerful and efficient methods for ill-posed problems. In our computation we use Tikhonov regularization method [19] to solve the matrix equation arising from RBF interpolation problem. Other regularization method such as truncated singular value decomposition and conjugate gradient methods, could be considered. The Tikhonov's regularization method finds a solution C_α which minimizes a quadratic functional $\|AC - b\|^2 + \alpha\|C\|^2$, where $\alpha > 0$ is a regularization parameter, which controls the degree of smoothing applied to the problem. The minimization of functional (21) produces the solution

$$C_\alpha = (A^t A + \alpha I)^{-1} A^t b \tag{22}$$

where I is the identity matrix. For $\alpha = 0$ the regularized solution (22) coincides with the solution produced by the least-squares method which is unstable. The choice of the regularization parameter α is crucial and various methods have been proposed for this purpose. However, in this study we use the L-curve method of Hansen and O'Leary [20] for the selection of a suitable value of α. This method plots, on a log-log scale, the $l^2 - norm$ of the regularized solution $\|C_\alpha\|$ versus the $l^2 - norm$ of the residual vector $\|AC_\alpha - b\|$, the graph being called the L-curve due to its L-shaped corner, in general. Since $\|AC_\alpha - b\|$, measures the fit to the data, whereas $\|C_\alpha\|$ measures the smoothness of the numerical solution, the solution at the corner has an optimum balance between fit and smoothness and hence it is considered to be a suitable choice.

4. **Numerical example**

In this section, for testing the accuracy and efficiency of described method we solve two test examples. By using (11) and various number of distinct data points N, and also solving the arising system (9) using Tikhonov regularization method, we can obtain the approximate solution. When the input data contain noises, we simulate noisy data as $h^\delta(x_i) = h(x_i) + r_i\delta$, where $h(x_i)$ is the exact one, $r_i \mid_{(i=1,\ldots,N)}$ are pseudo random values drawn from the standard uniform distribution on the open interval (0,1) produced by "rand" function in Matlab and δ is the noise level.

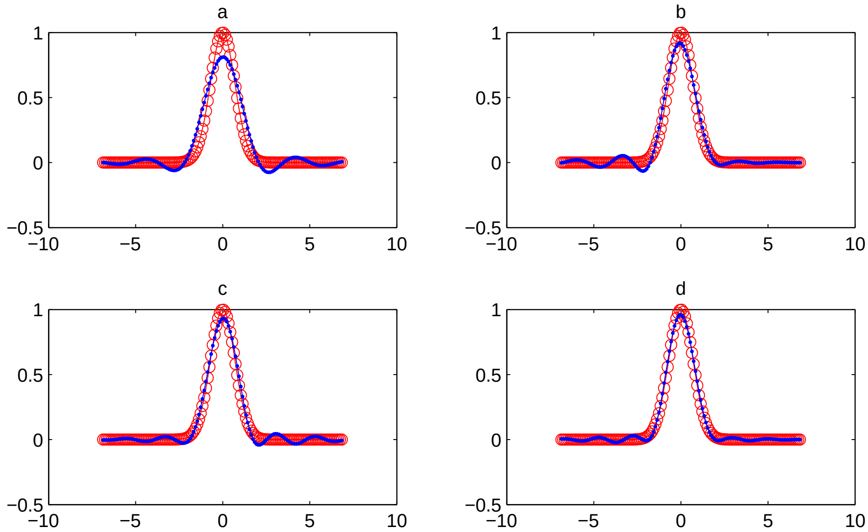

Figure 1. Comparison of approximated solution $u_N^\delta(x,t)$ with the exact one for example 4.1,$T = 1, t = 0, N = 30, c = 1$.(a),(b),(c) and (d) correspond to the results of adding the noise $\delta = 6 \times 10^{-2}, 6 \times 10^{-3}, 6 \times 10^{-4}$ and 6×10^{-5}.

Example 4.1. *Consider the problem (1) with the following final data* $h(x) = \frac{1}{\sqrt{1+4T}} e^{\frac{-x^2}{1+4T}}$ *which has the exact solution* $u(x,t) = \frac{1}{\sqrt{1+4t}} e^{\frac{-x^2}{1+4t}}$.

Example 4.2. *Consider the problem (1) with the following final data* $h(x) = e^{-T} sin(x)$ *which has the exact solution* $u(x,t) = e^{-t} sin(x)$.

We compared our numerical solution with the exact solution in Figures (1-4). These figures show that the approximate solution continuously depend on input data which is consistent with the error estimate in Theorem (3.1) and stability estimate in Lemma (3.1). Our approximate solutions are demonstrate the efficiency of the method computationally.

5. **Conclusions**

In this paper we introduce a convenient and simply applicable method for solving the backward heat conduction problem in unbounded region. We used the Gaussian radial basis functions (GRBFs) for discretization of the problem and the suitable attributes of this functions are used to obtain numerically stable scheme. The stability and convergence of the proposed method are investigated. The introduced method is applicable for approximate the distribution $u(x,t)$ at $t = 0$ which is self-starting and it is an advantage of this method. As for the computational aspect we can easily implement the method. In practice the problem reduced to a RBF interpolation problem. Numerical results show that the method is working well.

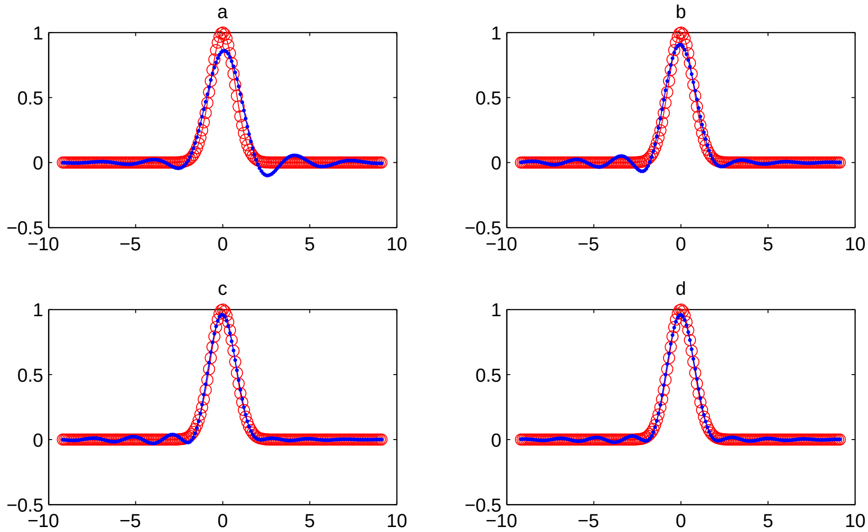

FIGURE 2. Comparison of approximated solution $u_N^\delta(x,t)$ with the exact one for example 4.1, $T = 1, t = 0, N = 50, c = 1$.(a),(b),(c) and (d) correspond to the results of adding the noise $\delta = 6 \times 10^{-2}, 6 \times 10^{-3}, 6 \times 10^{-4}$ and 6×10^{-5}.

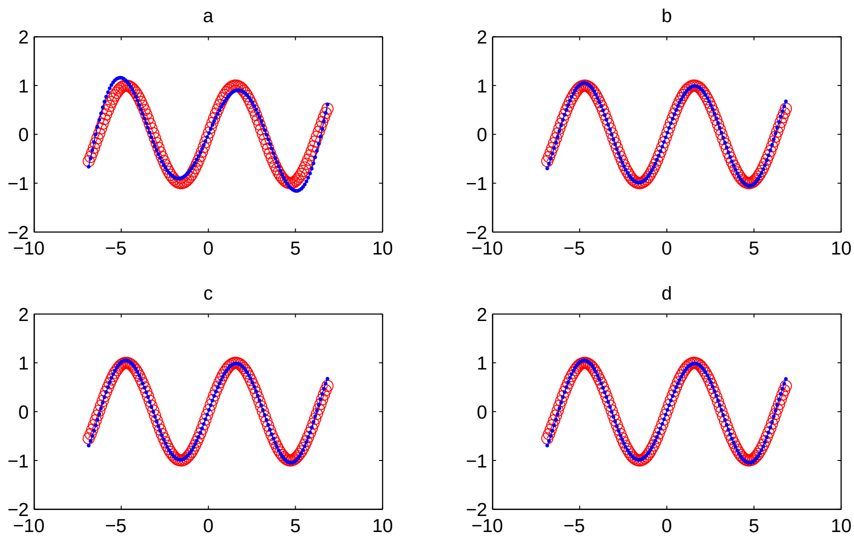

FIGURE 3. Comparison of approximated solution $u_N^\delta(x,t)$ with the exact one for example 4.2, $T = 1, t = 0, N = 30, c = 10$.(a),(b),(c) and (d) correspond to the results of adding the noise $\delta = 6\times 10^{-2}, 6\times 10^{-3}, 6 \times 10^{-4}$ and 6×10^{-5}.

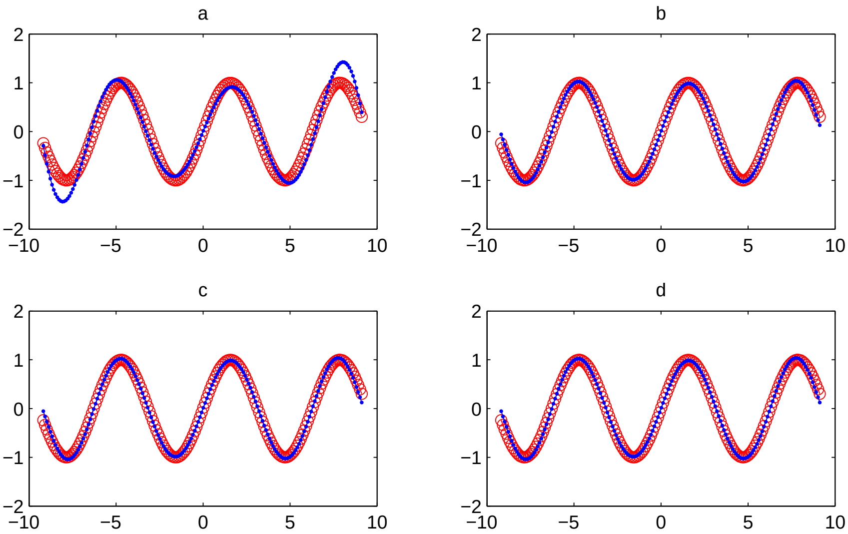

FIGURE 4. Comparison of approximated solution $u_N^\delta(x,t)$ with the exact one for example 4.2, $T = 1, t = 0, N = 50, c = 10$.(a),(b),(c) and (d) correspond to the results of adding the noise $\delta = 6 \times 10^{-2}, 6 \times 10^{-3}, 6 \times 10^{-4}$ and 6×10^{-5}.

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