A NEW APPLICATION OF QUASI-F-POWER INCREASING SEQUENCES TO FACTORED INFINITE SERIES

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In [Politehn. Univ.Bucharest Sci. Bull. Ser. A Appl. Math. Phys., 75 (3) (2013), 37-40], we proved a main theorem dealing with an application of quasi-f-power increasing sequences to absolute Cesàro summability methods. In this paper, we generalize this theorem for a general summability method. Some new results have also been deduced.

Keywords: Power increasing sequences, Cesàro mean, sequence spaces, infinite series.
MSC2010: 40D 15, 40F 05, 40G 05, 40 G99 , 46A 45.

1. Introduction

A sequence (λ_n) is said to be of bounded variation, denote by $(\lambda_n) \in BV$, if $\sum_{n=1}^{\infty} |\Delta\lambda_n| = \sum_{n=1}^{\infty} \mid \lambda_n - \lambda_{n+1} \mid < \infty$. A positive sequence $X = (X_n)$ is said to be a quasi-f-power increasing sequence, if there exists a constant $K = K(X, f) \geq 1$ such that $Kf_n X_n \geq f_m X_m$, holds for $n \geq m \geq 1$, where $f = (f_n) = [n^\sigma (\log n)^\gamma, \gamma \geq 0, 0 < \sigma < 1]$ (see [13]). If we take γ=0, then we obtain a quasi-σ-power increasing sequence (see [11]). Let $\sum a_n$ be a given infinite series. We denote by $t_n^{\alpha,\beta}$ the nth Cesàro mean of order (α, β), with $\alpha + \beta > -1$, of the sequence (na_n), that is (see [7])

$$t_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^\beta v a_v, \tag{1}$$

where

$$A_n^{\alpha+\beta} = O(n^{\alpha+\beta}), \quad A_0^{\alpha+\beta} = 1, \quad \text{and} \quad A_{-n}^{\alpha+\beta} = 0 \quad for \quad n > 0. \tag{2}$$

The series $\sum a_n$ is said to be summable $\mid C, \alpha, \beta; \delta \mid_k$, $k \geq 1$ and $\delta \geq 0$, if (see [4])

$$\sum_{n=1}^{\infty} n^{\delta k-1} \mid t_n^{\alpha,\beta} \mid^k < \infty. \tag{3}$$

If we take $\delta = 0$, then $\mid C, \alpha, \beta; \delta \mid_k$ summability reduces to $\mid C, \alpha, \beta \mid_k$ summability (see [8]). Also, if we take $\beta = 0$ and $\delta = 0$, then $\mid C, \alpha, \beta; \delta \mid_k$ summability reduces to $\mid C, \alpha \mid_k$ summability (see [9]). Furthermore, if we take $\beta = 0$, then we get $\mid C, \alpha; \delta \mid_k$ summability (see [10]).

2. Known results.

The following theorems are known dealing with the absolute Cesàro summability factors of infinite series.

Theorem A ([2]). Let $(\lambda_n) \in BV$ and (X_n) be a quasi-f-power increasing sequence for some σ $(0 < \sigma < 1)$. Suppose also that there exist sequences (β_n) and (λ_n), such that

$$\mid \Delta\lambda_n \mid \leq \beta_n \tag{4}$$

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$$\beta_n \to 0 \quad as \quad n \to \infty \tag{5}$$

$$\sum_{n=1}^{\infty} n \mid \Delta\beta_n \mid X_n < \infty \tag{6}$$

$$\mid \lambda_n \mid X_n = O(1) \quad as \quad n \to \infty. \tag{7}$$

If the sequence (u_n^α) defined by (see [12])

$$u_n^\alpha = \begin{cases} |t_n^\alpha|, & \alpha = 1, \\ \max_{1\le v\le n} |t_v^\alpha|, & 0 < \alpha < 1 \end{cases} \tag{8}$$

satisfies the condition

$$\sum_{n=1}^{m} n^{\delta k} \frac{(u_n^\alpha)^k}{n} = O(X_m) \quad as \quad m \to \infty, \tag{9}$$

then the series $\sum a_n\lambda_n$ is summable $\mid C,\alpha;\delta \mid_k$, $k \ge 1$ and $0 \le \delta < \alpha \le 1$.
Theorem B ([6]). Let (X_n) be a quasi-f-power increasing sequence. If conditions from (4) to (7) are satisfied and the sequence (u_n^α) defined by (8) satisfies the condition

$$\sum_{n=1}^{m} n^{\delta k} \frac{(u_n^\alpha)^k}{n\ X_n^{k-1}} = O(X_m) \quad as \quad m \to \infty, \tag{10}$$

then the series $\sum a_n\lambda_n$ is summable $\mid C,\alpha;\delta \mid_k$, $k \ge 1$ and $0 \le \delta < \alpha \le 1$.

3. The main result.

The aim of this paper is to generalize Theorem B for $\mid C,\alpha,\beta;\delta \mid_k$ summability method. Now, we shall prove the following more general theorem.
Theorem. Let (X_n) be a quasi-f-power increasing sequence. If conditions from (4) to (7) are satisfied and the sequence $(u_n^{\alpha,\beta})$ defined by (see [3])

$$u_n^{\alpha,\beta} = \begin{cases} \left|t_n^{\alpha,\beta}\right|, & \alpha = 1, \beta > -1, \\ \max_{1\le v\le n} \left|t_v^{\alpha,\beta}\right|, & 0 < \alpha < 1, \beta > -1. \end{cases} \tag{11}$$

satisfies the condition

$$\sum_{n=1}^{m} n^{\delta k} \frac{(u_n^{\alpha,\beta})^k}{n\ X_n^{k-1}} = O(X_m) \quad as \quad m \to \infty, \tag{12}$$

then the series $\sum a_n\lambda_n$ is summable $\mid C,\alpha,\beta;\delta \mid_k$, $k \ge 1$, $\beta > -1$, $0 \le \delta < \alpha \le 1$, and $(\alpha+\beta-\delta-1)k > 0$.
We need the following lemmas for the proof of our theorem.
Lemma 1([3]). If $0 < \alpha \le 1$, $\beta > -1$, and $1 \le v \le n$, then

$$\mid \sum_{p=0}^{v} A_{n-p}^{\alpha-1} A_p^\beta a_p \mid \le \max_{1\le m\le v} \mid \sum_{p=0}^{m} A_{m-p}^{\alpha-1} A_p^\beta a_p \mid . \tag{13}$$

Lemma 2([5]). Under the conditions on (X_n), (β_n) and (λ_n) as expressed in the statement of the theorem, we have the following ;

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty. \tag{14}$$

$$nX_n\beta_n = O(1), \tag{15}$$

4. Proof of the theorem. Let $(T_n^{\alpha,\beta})$ be the nth (C,α,β) mean of the sequence $(na_n\lambda_n)$. Then, by (1), we have $T_n^{\alpha,\beta}=\frac{1}{A_n^{\alpha+\beta}}\sum_{v=1}^{n}A_{n-v}^{\alpha-1}A_v^{\beta}va_v\lambda_v$. Applying Abel's transformation first and then using Lemma 1, we have that

$$\begin{aligned}
T_n^{\alpha,\beta} &= \frac{1}{A_n^{\alpha+\beta}}\sum_{v=1}^{n-1}\Delta\lambda_v\sum_{p=1}^{v}A_{n-p}^{\alpha-1}A_p^{\beta}pa_p+\frac{\lambda_n}{A_n^{\alpha+\beta}}\sum_{v=1}^{n}A_{n-v}^{\alpha-1}A_v^{\beta}va_v,\\
\mid T_n^{\alpha,\beta}\mid &\leq \frac{1}{A_n^{\alpha+\beta}}\sum_{v=1}^{n-1}\mid\Delta\lambda_v\mid\mid\sum_{p=1}^{v}A_{n-p}^{\alpha-1}A_p^{\beta}pa_p\mid+\frac{\mid\lambda_n\mid}{A_n^{\alpha+\beta}}\mid\sum_{v=1}^{n}A_{n-v}^{\alpha-1}A_v^{\beta}va_v\mid\\
&\leq \frac{1}{A_n^{\alpha+\beta}}\sum_{v=1}^{n-1}A_v^{(\alpha+\beta)}u_v^{\alpha,\beta}\mid\Delta\lambda_v\mid+\mid\lambda_n\mid u_n^{\alpha,\beta}=T_{n,1}^{\alpha,\beta}+T_{n,2}^{\alpha,\beta}.
\end{aligned}$$

To complete the proof of the theorem, Minkowski's inequality, it is enough to show that

$$\sum_{n=1}^{\infty}n^{\delta k-1}\mid T_{n,r}^{\alpha,\beta}\mid^k<\infty\quad for\quad r=1,2. \tag{16}$$

Whenever $k>1$, we can apply Hölder's inequality with indices k and k', where $\frac{1}{k}+\frac{1}{k'}=1$, we get that

$$\begin{aligned}
\sum_{n=2}^{m+1}n^{\delta k-1}\mid T_{n,1}^{\alpha,\beta}\mid^k &\leq \sum_{n=2}^{m+1}n^{\delta k-1}(A_n^{\alpha+\beta})^{-k}\{\sum_{v=1}^{n-1}(A_v^{\alpha+\beta})^k(u_v^{\alpha,\beta})^k\mid\Delta\lambda_v\mid^k\}\\
&\times \{\sum_{v=1}^{n-1}1\}^{k-1}\\
&= O(1)\sum_{n=2}^{m+1}n^{\delta k-2+k-(\alpha+\beta)k}\{\sum_{v=1}^{n-1}v^{(\alpha+\beta)k}(u_v^{\alpha,\beta})^k\beta_v^k\}\\
&= O(1)\sum_{v=1}^{m}v^{(\alpha+\beta)k}(u_v^{\alpha,\beta})^k\beta_v^k\sum_{n=v+1}^{m+1}\frac{1}{n^{2+(\alpha+\beta-\delta-1)k}}\\
&= O(1)\sum_{v=1}^{m}v^{(\alpha+\beta)k}(u_v^{\alpha,\beta})^k\beta_v^k\int_v^{\infty}\frac{dx}{x^{2+(\alpha+\beta-\delta-1)k}}\\
&= O(1)\sum_{v=1}^{m}(u_v^{\alpha,\beta})^k\beta_v\beta_v^{k-1}v^{\delta k+k-1}\\
&= O(1)\sum_{v=1}^{m}(u_v^{\alpha,\beta})^k\beta_v\left(\frac{1}{vX_v}\right)^{k-1}v^{\delta k+k-1}\\
&= O(1)\sum_{v=1}^{m}v\beta_v v^{\delta k}\frac{(u_v^{\alpha,\beta})^k}{vX_v^{k-1}}\\
&= O(1)\sum_{v=1}^{m-1}\Delta(v\beta_v)\sum_{r=1}^{v}r^{\delta k}\frac{(u_r^{\alpha,\beta})^k}{rX_r^{k-1}}+O(1)m\beta_m\sum_{v=1}^{m}v^{\delta k}\frac{(u_v^{\alpha,\beta})^k}{vX_v^{k-1}}\\
&= O(1)\sum_{v=1}^{m-1}\mid\Delta(v\beta_v)\mid X_v+O(1)m\beta_mX_m\\
&= O(1)\sum_{v=1}^{m-1}\mid(v+1)\Delta\beta_v-\beta_v\mid X_v+O(1)m\beta_mX_m
\end{aligned}$$

$$= O(1)\sum_{v=1}^{m-1} v \mid \Delta\beta_v \mid X_v + O(1)\sum_{v=1}^{m-1}\beta_v X_v + O(1)m\beta_m X_m = O(1) \quad as \quad m\to\infty,$$

by the hypotheses of the theorem and Lemma 2. Finally, we have that

$$\begin{aligned}
\sum_{n=1}^{m} n^{\delta k-1} \mid T_{n,2}^{\alpha,\beta} \mid^k &= \sum_{n=1}^{m} \mid \lambda_n \mid^{k-1} \mid \lambda_n \mid n^{\delta k}\frac{(u_n^{\alpha,\beta})^k}{n}\\
&= O(1)\sum_{n=1}^{m} \mid \lambda_n \mid n^{\delta k}\frac{(u_n^{\alpha,\beta})^k}{nX_n^{k-1}}\\
&= O(1)\sum_{n=1}^{m-1} \Delta \mid \lambda_n \mid \sum_{v=1}^{n} v^{\delta k}\frac{(u_v^{\alpha,\beta})^k}{vX_v^{k-1}}\\
&+ O(1) \mid \lambda_m \mid \sum_{n=1}^{m} n^{\delta k}\frac{(u_n^{\alpha,\beta})^k}{nX_n^{k-1}}\\
&= O(1)\sum_{n=1}^{m-1} \mid \Delta\lambda_n \mid X_n + O(1) \mid \lambda_m \mid X_m\\
&= O(1)\sum_{n=1}^{m-1} \beta_n X_n + O(1) \mid \lambda_m \mid X_m = O(1) \quad as \quad m\to\infty,
\end{aligned}$$

by the hypotheses of the theorem and Lemma 2. This completes the proof of the theorem. It should be noted that, if we take $\delta = 0$ (resp. $\alpha = 1$), then we get a new result for $\mid C,\alpha \mid_k$ (resp. $\mid C,1;\delta \mid_k$) summability. If we set $\beta = 0$, then we obtain Theorem B. Also, if we take $\gamma = 0$, then we have another new result.

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