TOPOLOGICAL ENTROPY AND TOPOLOGICAL PRESSURE OF A HOMEOMORPHISM ON A DYNAMICAL SPACE

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We introduce the notions of topological G-entropy and topological G-pressure on topological G-spaces and present a method for computing this quantity for G-expansive homeomorphisms. Also, we show that these notions are invariant under topological G-conjugacy.

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1. Introduction

The notion of entropy, as a measure of information content, was first introduced in 1948 by Shannon. The roots of this issue can be traced back to statistical mechanics, which is originated in the work of Boltzmann who studied the relation between entropy and probability in physical systems in 1870's. Entropy has also generalized around 1932 to quantum mechanics by von Neumann.

Topological entropy is a nonnegative real number that measures the complexity of systems on topological spaces and it is the greatest type of entropy of a system. Topological entropy is introduced in 1965 by Adler, Konheim and McAndrew [1], and subsequently studied by many researchers, see for instance [7]. For a system given by an iterated function, the topological entropy represents the exponential growth rate of the number of distinguishable orbits of the iterates. To be more precise, let (X, d) be a compact metric space and $T : X \to X$ be a homeomorphism. Let $d_n(x, y) = \max_{0\leq i\leq n-1} d(T^i(x), T^i(y))$ for all $n \in \mathbb{N}$. Each d_n is a metric on X and the d_n's are all equivalent metrics in the sense that they induce the same topology on X. Fix $\epsilon > 0$ and let $n \in \mathbb{N}$. A set F in X is *(n, ϵ)-spanning* if for every point $x \in X$ there exists a point $y \in F$ such that $d_n(x, y) < \epsilon$. By compactness, there are finite (n, ϵ)-spanning sets. Let $r_n(\epsilon, T)$ be the minimum cardinality of the (n, ϵ)-spanning sets. A set $E \subset X$ is *(n, ϵ)-separated* if the d_n-distance between any two distinct points in E is at least ϵ. Let $s_n(\epsilon, T)$ be the maximum cardinality of

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(n, ϵ)-separated sets. Then

$$h_{top}(T) = \lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log s_n(\epsilon, T) = \lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log r_n(\epsilon, T)$$

is called the *topological entropy* of T, see [2].

We intend to study the topological entropy of a dynamical system (X, T) whose underlying space X is a dynamical space, that is, a space on which some topological group G acts continuously. In this case, the triple (X, G, θ) is called a *metric G-space*, in which $\theta : G \times X \to X$ is a continuous action, see [6]. If $Y \subseteq X$, then Y is G-invariant if $gY = Y$ for all $g \in G$. Given a subset A of X, the G-orbit of A is defined by

$$G(A) = GA = \{ga \mid g \in G, a \in A\}.$$

If $x \in X$, then Gx is the G-orbit passing through x. Clearly, each G-orbit is a G-invariant subset of X. The orbit space for the action of G on X is the quotient topological space X/G. In particular, if G is compact, then the quotient map $\pi : x \in X \mapsto Gx \in X/G$ is an open, closed and proper (the inverse image of each compact set is compact) map and X/G is a Hausdorff space. Moreover, X/G equipped with the metric defined by

$$d(Gx, Gy) = \inf\{d(u, v) : \ u \in Gx, v \in Gy\} \ ; \ Gx, Gy \in X/G$$

is a compact metric space, see [4]. The *isotropy subgroup* at a point $x \in X$ is the set $G_x = \{g \in G \mid gx = x\}$, which is a closed subgroup of G. The action of G on X is called

(1) *trivial* when $G_x = G$, or equivalently, $Gx = \{x\}$ for all $x \in X$;
(2) *transitive* provided that $Gx = X$ for all $x \in X$;
(3) *minimal* if $\overline{Gx} = X$ for all $x \in X$.

Given two metric G-spaces X and Y, a map $T : X \to Y$ is called *G-equivariant* if $T(gx) = gT(x)$ for all $x \in X$ and $g \in G$, and it is called *G-pseudo equivariant* if $T(Gx) = G(Tx)$ for all $x \in X$.

Let $T : X \to X$ and $S : Y \to Y$ be two homeomorphisms of compact G-spaces X and Y, respectively. We say that T is *topologically G-conjugate* to S if there exists a G-equivariant homeomorphism $\phi : X \to Y$ such that $\phi \circ T = S \circ \phi$. The homeomorphism ϕ is called a *G-conjugacy.*

2. Topological G-Entropy of a Homeomorphism

Thomas [9, 10] introduced a measure theoretic entropy for transformations of G-spaces. On the other hand, the notion of topological entropy has been extended from different viewpoints, which can be found in [8], where Malziri and Molaei presented the notion of *base* for a dynamical system on a non compact metric space, and used this notion to define a new kind of entropy. Here, we present a new extension of topological entropy using compact G-spaces. Throughout this paper, X denotes a compact metric G-space in which G a compact group and $T : X \to X$ denotes a homeomorphism of X.

Definition 2.1. *Let X be a compact metric G-space. For each positive integer n we define a metric $\bar{d}_n : X/G \times X/G \to [0, \infty)$ by*

$$\bar{d}_n(Gx, Gy) = \inf\{d_n(u, v) : \ u \in Gx, \ v \in Gy\}, \ Gx, Gy \in X/G,$$

where d_n is defined as in section 1.

Since G is compact, $(X/G, \bar{d}_n)$ is a compact metric space. By $B_n(\delta, Gx)$ we mean an open ball with $\bar{d}_n$-diameter less than δ in X/G.

Definition 2.2. *Let $c_n(\epsilon, G, T)$ be the minimum number of coverings of X/G by the sets of $\bar{d}_n$-diameter less than ϵ. Then the limit*

$$h_G(T, \epsilon) = \lim_{n\to\infty} \frac{1}{n} \log c_n(\epsilon, G, T)$$

exists and is monotonically increasing as $\epsilon \to 0$. So $h_G(T) = \lim_{\epsilon\to 0^+} h_G(T, \epsilon)$ is well defined and we call it the topological G-entropy of T.

A subset $\mathcal{F}$ of X/G is said to be (n, ϵ, G)-*spanning* if for each $Gx \in X/G$ there exists $Gy \in \mathcal{F}$ such that $\bar{d}_n(Gx, Gy) < \epsilon$. Since X/G is compact, there exist (n, ϵ, G)-spanning sets with finite cardinality. Let $r_n(\epsilon, G, T)$ be the minimum cardinality of (n, ϵ, G)-spanning sets of T.

A subset $\mathcal{E}$ of X/G is said to be an (n, ϵ, G)-*separated set* if $Gx, Gy \in \mathcal{E}$ and $G(x) \neq G(y)$ implies that $\bar{d}_n(Gx, Gy) \geq \epsilon$. Let $s_n(\epsilon, G, T)$ be the maximum cardinality of (n, ϵ, G)-separated sets for T.

Lemma 2.1. *If X is a compact metric G-space and $T : X \to X$ is a homeomorphism, then the following results hold.*

(1) $c_n(2\epsilon, G, T) \leq r_n(\epsilon, G, T) \leq s_n(\epsilon, G, T) \leq c_n(\frac{\epsilon}{2}, G, T)$;
(2) For any subgroup H of G, $s_n(\epsilon, H, T) \geq s_n(\epsilon, G, T)$ and $r_n(\epsilon, H, T) \geq r_n(\epsilon, G, T)$.

Proof. (1) If $\mathcal{E}$ is an (n, ϵ, G)-separated set of maximum cardinality, then it is an (n, ϵ, G)-spanning set and hence $r_n(\epsilon, G, T) \leq s_n(\epsilon, G, T)$.

Now, suppose that $\mathcal{F}$ is an (n, ϵ, G)-spanning set of minimum cardinality. The family

$$\{B_n(\epsilon, Gx) : \ Gx \in \mathcal{F}\}$$

is an open cover for X/G. By compactness of X/G we can choose $\epsilon' < \epsilon$ so that the family $\{B_n(\epsilon', Gx) : \ Gx \in \mathcal{F}\}$ covers X/G. Since the diameter of this family is less than 2ϵ, we have $c_n(2\epsilon, G, T) \leq r_n(\epsilon, G, T)$.

To prove the last inequality, let $\mathcal{E}$ be an (n, ϵ, G)-separated set with cardinality $s_n(\epsilon, G, T)$ and let $\mathcal{C}$ be an open cover with $\bar{d}_n$-diameter less than $\epsilon/2$. Then no member of $\mathcal{C}$ contains two elements of $\mathcal{E}$. Therefore $s_n(\epsilon, G, T) \leq c_n(\epsilon/2, G, T)$.

(2) This part follows from the fact that $\bar{d}_n(Gx, Gy) \leq \bar{d}_n(Hx, Hy)$ for each $x, y \in X$. □

Proposition 2.1. *Let X be a compact metric G-space and $T : X \to X$ be a homeomorphism. Then the following results hold.*

(1) $h_G(T) = \lim_{\epsilon\to 0^+} \lim_{n\to\infty} \frac{1}{n} \log s_n(\epsilon, G, T)$;
(2) $h_G(T) = \lim_{\epsilon\to 0^+} \lim_{n\to\infty} \frac{1}{n} \log r_n(\epsilon, G, T)$;

(3) $h_H(T) \geq h_G(T)$ *for each subgroup* H *of* G*;*
(4) $0 \leq h_G(T) \leq h_{top}(T)$*;*
(5) If there is a point $x \in X$ *such that* $\overline{Gx} = X$*, then* $h_G(T) = 0$*;*
(6) If the action is minimal, then $h_G(T) = 0$*;*
(7) If the action is transitive, then $h_G(T) = 0$*;*
(8) If the action is trivial, then $h_G(T) = h_{top}(T)$*.*
(9) If G *is a finite group, then* $h_G(T) = h_{top}(T)$*.*

Proof. The parts (1) and (2) follow immediately from Lemma 2.1(1). Also, the part (3) is a direct result of Lemma 2.1(2). The part (4) follows from Lemma 2.1(2) with $H = \{e\}$.

To prove part (5), let $\epsilon > 0$ be given and $n \in \mathbb{N}$. Then there exists $\delta > 0$ such that $d(x, y) < \delta$ implies $d_n(x, y) < \epsilon$. We show that $\mathcal{F} = \{Gx\}$ is an (n, ϵ, G)-spanning set. To end this, notice that if $Gy \in X/G$ and $g_1 \in G$, then there exists $g_2 \in G$ such that $d(g_1 y, g_2 x) < \delta$. Thus $d_n(g_1 y, g_2 x) < \epsilon$, so that $\overline{d}_n(Gx, Gy) < \epsilon$.

The parts (6) and (7) follow from (5), and part (8) is obvious.

(9) Suppose that $G = \{g_1, g_2, \dots, g_k\}$. Choose $\epsilon > 0$ small enough such that $s_0(\epsilon, T) \geq k^2$ and let $n \in \mathbb{N}$. Then, there exists $\delta \in (0, \epsilon)$ such that $d(x, y) < \delta$ implies that $\max_{g\in G} d_n(gx, gy) < \epsilon$. Hence, there exist a number $m \in \mathbb{N}$ and $0 \leq r \leq K^2 - 1$ such that $s_n(\epsilon, T) = mk^2 + r$. We show that $s_n(\delta, G, T) \geq m$. Suppose that $E = \{x_1, x_2, \dots, x_{mk^2+r}\}$ and $s_n(\delta, G, T) = l < m$. Let $\mathcal{E} = \{Gx_1, \dots, Gx_l\}$ be an $(n, G, \delta/2)$-separated set, hence $d_n(Gx_i, Gx_j) \geq \delta/2$ for each $i, j \in \{1, \dots l\}$, and that for each $i \in \{l + 1, \dots, mk^2 + r\}$ there exists $j \in \{1, \dots l\}$ in such a way that $d_n(Gx_i, Gx_j) < \delta/2$. By invoking pigeonhole principle, there exists an index $j_0 \in \{1, \dots, l\}$ such that the number of indeces $i \in \{l + 1, \dots, mk^2 + r\}$ for which $d_n(Gx_i, Gx_{j_0}) < \delta/2$ is at least $\lfloor (mk^2 + r - l)l^{-1} \rfloor$. The total number of possible selections of disjoint pairs in G is $k(k - 1)$ and it is less than $\lfloor (mk^2 + r - l)l^{-1} \rfloor$. Thus there are an index i and an element $g \in G$ such that $d_n(gx_i, gx_{j_0}) < \delta$. Then $d_n(x_i, x_{j_0}) < \epsilon$, which is a contradiction. Therefore

$$\begin{aligned} h_G(T) &= \lim_{\epsilon\to 0} \lim_{n\to\infty} \frac{1}{n} \log s_n(\delta, G, T) \\ &\geq \lim_{\epsilon\to 0} \lim_{n\to\infty} \frac{1}{n} \log \frac{s_n(\epsilon, T) - r}{k^2} = \lim_{\epsilon\to 0} \lim_{n\to\infty} \frac{1}{n} \log s_n(\epsilon, T) = h_{top}(T), \end{aligned}$$

as required. □

3. Some properties of topological G-entropy

Definition 3.1. *A homeomorphism* $T : X \to X$ *on a metric* G*-space* X *is said to be weak* G*-expansive if there exists* $\delta > 0$ *such that for every* $x, y \in X$ *with* $Gx = Gy$*,* $u \in Gx$ *and* $v \in Gy$*, it follows that*

$$d(T^n(u), T^n(v)) > \delta$$

for some $n = n(u, v) \in \mathbb{Z}$*. The constant* δ *is called a weak* G*-expansive constant for* T*.*

Weak G-expansivity is a generalization of both expansivity and G-expansivity, see [5].

Theorem 3.1. *Let T be a pseudo-equivariant homeomorphism on a compact metric G-space (X,d). If T is weak G-expansive with constant δ, then $h_G(T) = h_G(T,\epsilon)$ for any $\epsilon \in (0,\delta)$.*

Proof. Fix γ and ϵ with $0 < 2\gamma < \epsilon < \delta$. It is enough to show that $h_G(T,2\gamma) = h_G(T,\epsilon)$. Since T is weak G-expansive, for two elements x and y not in the same G-orbit, there exists a number $n = n(x,y)$ such that $d(T^n(x), T^n(y)) \geq \delta > \epsilon$. The set $\{(x,y) \ : \ \overline{d}(Gx,Gy) \geq \gamma\}$ is compact and

$$\{(x,y) \ : \ \overline{d}(Gx,Gy) \geq \gamma\} \subset \bigcup_{i\in\mathbb{Z}} \{(x,y) \ : \ d(T^i(x),T^i(y)) > \epsilon\}.$$

Then, by compactness, there is a number $k \in \mathbb{N}$ such that if $\overline{d}(Gx,Gy) > \gamma$ then $d(T^i(x),T^i(y)) > \epsilon$ for some $|i| \leq k$.

Let $\mathcal{E}$ be an (n,G,γ)-separated set for T and $\tilde{\mathcal{E}} = \tilde{T}^{-k}\mathcal{E}$. We show that $\tilde{\mathcal{E}}$ is an $(n+2k,\epsilon,G)$-separated set for T. Suppose that Gx and Gy are distinct G-orbits in $\tilde{\mathcal{E}}$, $u \in Gx$ and $v \in Gy$. Then $GT^k(u)$ and $GT^k(v)$ are distinct G-orbits in $\mathcal{E}$. Therefore, $\overline{d}_n(GT^k(u), GT^k(v)) \geq \gamma$. So, there is a number i with $|i| \leq k$ such that $d_n(T^{i+k}(u), T^{i+k}(v)) > \epsilon$, and this shows that $d_{n+2k}(Gx,Gy) > \epsilon$. Hence, $c_n(2\gamma,g,T) \leq s_n(\gamma,G,T) \leq s_{n+2k}(\epsilon,G,T)$ and so we obtain $h_G(T,2\gamma) \leq h_G(T,\epsilon)$. On the other hand, by monotonicity, we have $h_G(T,2\gamma) \geq h_G(\epsilon,T)$, giving $h_G(T,2\gamma) = h_G(T,\epsilon)$. □

Theorem 3.2. *For each $m \in \mathbb{N}$, $h_G(T^m) = mh_G(T)$.*

Proof. Let $\mathcal{F}$ be an (mn,ϵ,G)-spanning set for T of maximal cardinality. Then for each $Gx \in X/G$ there exists $Gy \in \mathcal{F}$ such that $\bar{d}_{mn}(Gx,Gy) < \epsilon$, which implies that $d_{mn}(gx,g'y) < \epsilon$ for some $g,g' \in G$. Hence, $d(T^i gx, T^i g'y) < \epsilon$ for all $0 \leq i \leq mn-1$ and so

$$d(T^{mi}gx, T^{mi}g'y) < \epsilon, \quad \text{for all } 0 \leq i \leq n-1.$$

Therefore, $\mathcal{F}$ is an (n,ϵ,G)-spanning set for T^m and accordingly

$$r_n(\epsilon,G,T^m) \leq r_{mn}(\epsilon,G,T),$$

which implies that $h_G(T) \leq mh_G(T)$. Since T is uniformly continuous, for each $\epsilon > 0$ there exists a number $\delta > 0$ such that $\max_{0\leq i\leq m-1} d(T^i x, T^i y) < \epsilon$ whenever $d(x,y) < \delta$. So, every (n,ϵ,G)-spanning set for T^m is an (n,ϵ,G)-spanning set for T, too. Thus $r_n(\delta,G,T^m) \geq r_{mn}(\epsilon,G,T)$ and consequently $h_G(T^m) \geq mh_G(T)$. □

Let (X,d) and (Y,ρ) be metric G- and H-spaces, respectively. We define a metric d on $X \times Y$ by $D((x,y),(x',y')) = \max\{d(x,x'), \rho(y,y')\}$. Obviously, $X \times Y$ is a metric $(G \times H)$-space with the action

$$\begin{array}{rcl} (G\times H)\times(X\times Y) & \to & X\times Y \\ ((g,h),(x,y)) & \mapsto & (gx,hy). \end{array}$$

Utilizing the above action, we have the following result.

Theorem 3.3. *Let T and S be homeomorphisms of G-space X and H-space Y, respectively. Then $h_{G\times H}(T\times S) = h_G(T) + h_H(S)$.*

Proof. Let $\mathcal{F}$ be an (n, ϵ, G)-spanning set with minimal cardinality for T and let $\mathcal{F}'$ be an (n, ϵ, H)-spanning set with minimal cardinality for S. Then $\mathcal{F} \times \mathcal{F}'$ is an $(n, \epsilon, G \times H)$-spanning set for $T \times S$. Hence

$$r_n(\epsilon, G \times H, T \times S) \leq r_n(\epsilon, G, T).r_n(\epsilon, H, S)$$

so that $h_{G \times H}(T \times S) \leq h_G(T) + h_H(S)$.

Now, let $\mathcal{E}$ be an (n, ϵ, G)-separated set with maximal cardinality for T and $\mathcal{E}'$ be an (n, ϵ, H)-separated set with maximal cardinality for S. Then $\mathcal{E} \times \mathcal{E}'$ is an $(n, \epsilon, G \times H)$-separated set for $T \times S$. Hence

$$s_n(\epsilon, G, T).s_n(\epsilon, H, S) \leq s_n(\epsilon, G \times H, T \times S)$$

and consequently $h_G(T) + h_H(S) \leq h_{G \times H}(T \times S)$. □

4. **Topological G-entropy and conjugacy**

In this section, we show that the topological G-entropy is an invariant of topological G-conjugacy.

Theorem 4.1. *Let $T : X \to X$ and $S : Y \to Y$ be homeomorphisms of compact metric G-spaces. If T is topologically G-conjugate to S, then $h_G(T) = h_G(S)$.*

Proof. Let ϕ be a topological G-conjugacy satisfying $\phi \circ T = S \circ \phi$ and assume that $\epsilon > 0$. Then there exists a number $\delta > 0$ such that the inequality $d(x, y) < \delta$ implies $d(\phi(x), \phi(y)) < \epsilon$ for all $x, y \in X$. Let $\mathcal{E}$ be an (n, ϵ, G)-separated set with maximal cardinality for S. We show that $\phi^{-1}\mathcal{E} = \{\phi^{-1}(Gx);\ Gx \in \mathcal{E}\}$ is an (n, δ, G)-separated set for T. To end this, let $G\phi^{-1}(x)$ and $G\phi^{-1}(y)$ be distinct points in $\phi^{-1}\mathcal{E}$. If $d_n(G\phi^{-1}(x), G\phi^{-1}(y)) < \delta$, then

$$\max_{0 \leq i \leq n-1} d(\phi^{-1}S^iGx, \phi^{-1}S^iGy) = \max_{0 \leq i \leq n-1} d(T^i\phi^{-1}Gx, T^i\phi^{-1}Gy) < \delta.$$

Therefore,

$$\bar{d}_n(Gx, Gy) = \max_{0 \leq i \leq n-1} d(S^iG(x), S^iG(y)) < \epsilon,$$

which is a contradiction for Gx and Gy are distinct points of $\mathcal{E}$. This contradiction shows that $d_n(G\phi^{-1}(x), G\phi^{-1}(y)) \geq \delta$, that is, $\phi^{-1}\mathcal{E}$ is an (n, δ, G)-separated set. Thus $s_n(\epsilon, G, S) \leq r_n(\delta, G, T)$, giving $h_G(S) \leq h_G(T)$. Similarly, we have $h_G(T) \leq h_G(S)$, hence the result follows. □

5. **Topological G-pressure**

The well-known notion of topological pressure for additive potentials was introduced by Ruelle [3] in 1973 for expansive maps acting on compact metric spaces.

Definition 5.1. *Let X be a compact metric G-space and $T : X \to X$ be a homeomorphism. Denote by $\mathcal{C}(X, \mathbb{R})$ the space of all real valued functions of X. Let $n \in \mathbb{N}$. For each $f \in \mathcal{C}(X, \mathbb{R})$ define $S_nf = \sum_{i=0}^{n-1} f \circ T^i$ and*

$$P_n(\epsilon, G, T, f) = \inf \left\{ \sum_{Gx \in \mathcal{F}} \inf_{u \in Gx} e^{S_nf(u)} |\ \mathcal{F} \textit{ is an } (n, \epsilon, G) - \textit{spanning set} \right\}.$$

Clearly, $P_n(\epsilon, G, T, 0) = r_n(\epsilon, G, T)$ *and*

$$0 \leq P_n(\epsilon, G, T, f) \leq \|e^{S_n f}\| r_n(\epsilon, G, T).$$

Definition 5.2. *Let X be a compact metric G-space and $T : X \to X$ be a homeomorphism. For $f \in \mathcal{C}(X, \mathbb{R})$ we define*

$$P(\epsilon, G, T, f) = \limsup \frac{1}{n} \log P_n(\epsilon, G, T, f).$$

If $\epsilon_1 < \epsilon_2$ then we have $P_n(\epsilon_1, G, T, f) \geq P_n(\epsilon_2, G, T, f)$. Thus $P_n(\epsilon, G, T, f)$ (and hence $P(\epsilon, G, T, f)$) is decressing in terms of ϵ. Therefore, the following limit exists

$$\lim_{\epsilon \to 0^+} P(\epsilon, G, T, f).$$

Definition 5.3. *Let X be a compact metric G-space and let $T : X \to X$ be a homeomorphism. The topological G-pressure of T is the map $P_G(T, .) : \mathcal{C}(X, \mathbb{R}) \to \mathbb{R} \cup \{\infty\}$ defined via*

$$P_G(T, f) = \lim_{\epsilon \to 0} P(\epsilon, G, T, f).$$

Remark 5.1. *Let X be a compact metric G-space and $T : X \to X$ be a homeomorphism. If $f, g \in \mathcal{C}(X, \mathbb{R})$, then*

(1) $P_G(T, 0) = h_G(T)$,

(2) if $f \leq g$ *then* $P_G(T, f) \leq P_G(T, g)$, *and*

(3) If H is a subgroup of G, then $P_G(T, f) \leq P_H(T, f)$.

Theorem 5.1. *Let (X, d) and (Y, ρ) be two G-spaces. If $T : X \to X$ and $S : Y \to Y$ are two countinuous maps and $\phi : X \to Y$ is a G-pseudoequivariant homeomorphism such that $\phi \circ T = S \circ \phi$, then $P_G(S, f) = P_G(T, f \circ \phi)$ for each $f \in \mathcal{C}(Y, \mathbb{R})$.*

Proof. Given $\epsilon > 0$; there exists a number $\delta > 0$ such that $d(\phi(x), \phi(y)) < \epsilon$ for each $x, y \in X$ satisfying $d(x, y) < \delta$. Let $\mathcal{F} = \{Gx_1, \ldots, Gx_k\}$ be an (n, δ, G)-spanning set for T. We show that $\mathcal{F}' = \{G\phi(x_1), \ldots, G\phi(x_k)\}$ is an (n, ϵ, G)-spanning set for S. For each $Gy \in Y/G$ there exists a point $x \in X$ such that $Gy = G\phi(x)$. Since $Gx \in X/G$, there exists $Gx_i \in \mathcal{F}$ with $\bar{d}_n(Gx, Gx_i) < \delta$. Hence, $d_n(gx, gx_i) < \delta$ for some $g, g' \in G$. Thereofre, $\max_{0 \leq i \leq n-1} d(T^i gx, T^i g' x_i) < \delta$ and so

$$\max_{0 \leq i \leq n-1} d(S^i \phi(gx), S^i \phi(g' x_i)) = \max_{0 \leq i \leq n-1} d(\phi(T^i gx), \phi(T^i g' x_i)) < \epsilon.$$

Hence, $\max_{0 \leq i \leq n-1} d(S^i(h\phi(x), S^i(h'\phi(x_i)) < \epsilon$ for some $h, h' \in G$. So

$$\bar{d}_n(Gy, G\phi(x_i)) = \bar{d}_n(G\phi(x), G\phi(x_i)) < \epsilon.$$

Therefore $\mathcal{F}$ is an (n, ϵ, G)-spanning set for S. Also we have

$$\begin{aligned} &\sum_{i=1}^{k} \inf_{u \in Gx_i} e^{f(\phi(u)) + f(\phi(Tu)) + \cdots + f(\phi(T^{n-1}u))} \\ =& \sum_{i=1}^{k} \inf_{u \in Gx_i} e^{f(\phi(u)) + f(S\phi(u)) + \cdots + f(S^{n-1}\phi(u))} \\ =& \sum_{i=1}^{k} \inf_{u \in G\phi(x_i)} e^{f(y) + f(Sy) + \cdots + f(S^{n-1}y)} \end{aligned}$$

Thus $P_n(\epsilon, G, S, f) \leq P_n(\delta, G, T, f \circ \phi)$ so that $P_G(S, f) \leq P_G(T, f \circ \phi)$. Since ϕ is a homeomorphism, we have

$$P_G(T, f \circ \phi) \leq P_G(S, f \circ \phi \circ \phi^{-1}) = P_G(S, f),$$

as required. □

Theorem 5.2. *Let (X, d) and (Y, ρ) be a G-space and a H-space, respective ly. If $T : X \to X$ and $S : Y \to Y$ are two countinuous maps, $f \in \mathcal{C}(X, \mathbb{R})$ and $g \in \mathcal{C}(Y, \mathbb{R})$. Then*

$$P_{G\times H}(T \times S, f \times g) = P_G(T, f) + P_H(S, g).$$

Proof. Let $\mathcal{F}$ be an (n, ϵ, G)-spanning set with minimal cardinality for T and let $\mathcal{F}'$ be an (n, ϵ, H)-spanning set with minimal cardinality for S. Then $\mathcal{F} \times \mathcal{F}'$ is an $(n, \epsilon, G \times H)$-spanning set for $T \times S$. Also, we have

$$\begin{aligned} &\sum_{(Gx,Hy)\in\mathcal{F}\times\mathcal{F}'} \inf_{(u,v)\in(Gx,Gy)} e^{\sum_{i=0}^{n-1}(f\times g)((T\times S)^i(u,v))} \\ &= \sum_{(Gx,Hy)\in\mathcal{F}\times\mathcal{F}'} \inf_{(u,v)\in(Gx,Gy)} e^{\sum_{i=0}^{n-1} f(T^i u)} e^{\sum_{i=0}^{n-1} g(S^i u)} \\ &= \sum_{(Gx,Hy)\in\mathcal{F}\times\mathcal{F}'} \inf_{u\in Gx} e^{\sum_{i=0}^{n-1} f(T^i u)} \inf_{v\in Gy} e^{\sum_{i=0}^{n-1} g(S^i u)} \end{aligned}$$

Therefore,

$$P_n(\epsilon, G \times H, T \times S, f \times g) \leq P_n(\epsilon, G, T, f).P_n(\epsilon, H, S, g),$$

which implies that $P_{G\times H}(T \times S, f \times g) \leq P_G(T, f) + P_H(S, g)$.

Now consider the quotient map $\pi : X \to X/G$ defined by $\pi(x) = Gx$. For any map $T : X \to X$, the induced map $\tilde{T} : X/G \to X/G$ satisfies $\tilde{T} \circ \pi = \pi \circ T$. Let X be a set and $\mathcal{B}(X)$ denote the group of all bijections on X. We have an action of $\mathcal{B}(X)$ on X defined by

$$\begin{array}{rcl} \mathcal{B}(X) \times X & \to & X \\ (g, x) & \mapsto & g(x). \end{array}$$

Let (X, d) be a compact metric space and $Iso(X) \subset \mathcal{B}(X)$ denote the group of isometries of X. Let ρ be a metric on $Iso(X)$ defined via

$$\rho(f, g) = \sup_{x\in X} d(f(x), g(x))$$

. The group operations of multiplication and inversion are continuous with respect to ρ and $Iso(X)$ has the structure of a topological group. The associated action $Iso(X) \times X \to X$ of $Iso(X)$ on X is continuous. If G is a subgroup of $Iso(X)$, then X/G equipped with the metric

$$\tilde{d}(Gx, Gy) = \inf\{d(u, v) : \ u \in Gx, \ v \in Gy\}$$

is a compact metric space. If $f \in \mathcal{C}(X, \mathbb{R})$ is constant on G-orbits ($f(gx) = f(x)$ for $g \in G$ and $x \in X$), then f induces a map $\tilde{f} \in \mathcal{C}(X/G, \mathbb{R})$ satisfying $\tilde{f}(Gx) = f(x)$. □

Theorem 5.3. *If T is a homeomorphism on the metric G-space X and $f \in \mathcal{C}(X, \mathbb{R})$, then $P(\tilde{T}, \tilde{f}) \leq P_G(T, f)$.*

Proof. Given $\epsilon > 0$. There is $\delta > 0$ such that $\tilde{d}(\pi(x), \pi(y)) < \epsilon$ whenever $d(x, y) < \delta$. Suppose that $\mathcal{F}$ is an (n, δ, G)-spanning set for T of maximum cardinality. We show that $\mathcal{F}$ is an (n, ϵ)-spanning set for $\tilde{T}$. If $Gx \in X/G$ then there exists $Gy \in \mathcal{F}$ such that $\bar{d}_n(Gx, Gy) < \epsilon$. Hence, there are elements $g_1, g_2 \in G$ with

$$\max_{0 \leq i \leq n-1} d(T^i(g_1 x), T^i(g_2 y)) < \delta.$$

Thus,

$$\begin{aligned} \tilde{d}_n(Gx, Gy) &= \max_{0 \leq i \leq n-1} \tilde{d}(\tilde{T}^i(Gx), \tilde{T}^i(Gy)) \\ &= \max_{0 \leq i \leq n-1} \tilde{d}(\tilde{T}^i \circ \pi(g_1 x), \tilde{T}^i \circ \pi(g_2 y)) \\ &= \max_{0 \leq i \leq n-1} \tilde{d}(\pi \circ T^i(g_1 x), \pi \circ T^i(g_2 y)) < \epsilon. \end{aligned}$$

Therefore, $\mathcal{F}$ is an (n, ϵ)-spanning set for $\tilde{T}$. On the other hand, we have

$$\begin{aligned} \tilde{S}_n \tilde{f}(Gx) &= \tilde{f}(Gx) + \tilde{f}(\tilde{T}(Gx)) + \cdots + \tilde{f}(\tilde{T}^{n-1}(Gx)) \\ &= \tilde{f}(Gx) + \tilde{f}(GTx) + \cdots + \tilde{f}(GT^{n-1}x) \\ &= f(x) + f(Tx) + \cdots + f(T^{n-1}x) \\ &= S_n f(x). \end{aligned}$$

Hence,

$$\sum_{Gx \in \mathcal{F}} \inf_{u \in Gx} e^{S_n f(u)} = \sum_{Gx \in \mathcal{F}} e^{S_n f(x)} = \sum_{Gx \in \mathcal{F}} e^{\tilde{S}_n \tilde{f}(Gx)}.$$

Therefore, $P_n(\epsilon, \tilde{T}, \tilde{f}) \leq P_n(\epsilon, G, T, f)$, from which the result follows. □

Corollary 5.1. *If T is a homeomorphism on the metric G-space X, then $h_{top}(\tilde{T}) \leq h_G(T)$.*

6. **Examples**

First we compute the topological G-entropy for the most common action in dynamical systems.

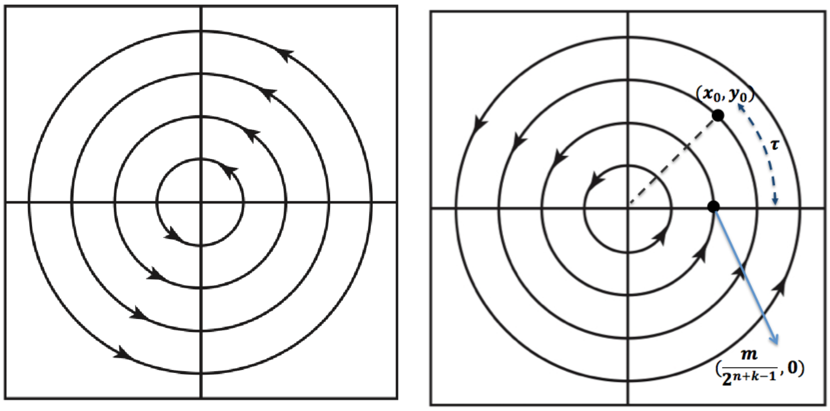
Example 6.1. *Let X be a compact metric space and $T : X \to X$ be a homeomorphism. The group $\mathbb{Z}$ acts continuously on X as follows*

$$\begin{array}{ccc} \mathbb{Z} & \times & X \to X \\ (n, x) & \mapsto & T^n(x). \end{array}$$

Thus X is a compact metric $\mathbb{Z}$-space. In this case, we have $\mathbb{Z}x = \mathcal{O}_T(x)$ and $\bar{d}_n(\mathbb{Z}x, \mathbb{Z}y) = \tilde{d}(\mathbb{Z}x, \mathbb{Z}y)$ for each $n \geq 1$ and $x, y \in X$. Hence, $h_{\mathbb{Z}}(T) = 0$.

Example 6.2. *Consider the linear ordinary differential equation*

$$\dot{X} = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix} X.$$



(A) (B)

Solution to this linear system is given by

$$\begin{cases} x(t) = x_0 \cos(\beta t) - y_0 \sin(\beta t) \\ y(t) = x_0 \sin(\beta t) + y_0 \cos(\beta t) \end{cases},$$

where $(x_0, y_0) = (x(0), y(0))$ *and the trajectories of this system lie on circles as shown in Figure (A). Hence the flow of this linear system is given by*

$$\varphi_t(x_0, y_0) = (x_0 \cos(\beta t) - y_0 \sin(\beta t), x_0 \sin(\beta t) + y_0 \cos(\beta t))\ .$$

Consider the unit disc $\mathcal{D} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ *and define an equivalence relation on* $\mathcal{D}$ *as follows:*

$$X \sim Y \ \Leftrightarrow\ \|X - Y\| \in \mathbb{Z}.$$

Then $X = \mathcal{D}/\sim$ *with the metric*

$$d(X, Y) = \min\{\|X - Y\|, 1 - \|X - Y\|\}$$

is a compact metric space called a quotient metric space [4]*. We know that the group* $G = \mathbb{R}$ *acts continuously on the space* X *as follows:*

$$(t, (x_0, y_0)) \mapsto \varphi_t(x_0, y_0).$$

Now, we compute the topological G*-entropy of the map* $T : (X, d) \to (X, d)$ *with* $T(x_0, y_0) = (2x_0, 2y_0)$*, mod 1 (note that for any* $(x_0, y_0) \in X$ *with* $x_0 \neq 0$*, the point* $T(x_0, y_0)$ *is the intersection of the line* $y = \frac{y_0}{x_0}x$ *and the circle* $x^2 + y^2 = (2\sqrt{x_0^2 + y_0^2} - [2\sqrt{x_0^2 + y_0^2}])^2$ *).*

For each $k \in \mathbb{N}$ *we define*

$$F_k = \left\{(\frac{m}{2^k}, 0) :\ m = 0, 1, \ldots, 2^{k-1}\right\} \subset B.$$

Let $\epsilon > 0$ *and choose* $k \in \mathbb{N}$ *such that* $\frac{1}{2^k} \leq \epsilon < \frac{1}{2^{k-1}}$*. We show that the set* F_{n+k-2} *is an* (n, ϵ, G)*-separated set for each* $n \in \mathbb{N}$*. Suppose that* $(\frac{m_1}{2^{n+k-2}}, 0)$ *and* $(\frac{m_2}{2^{n+k-2}}, 0)$

are two distinct points in F_{n+k-2}. *Then,*

$$\begin{aligned} & d\left(T^{n-1}(\frac{m_1\cos(\beta t)}{2^{n+k-2}}, \frac{m_1\sin(\beta t)}{2^{n+k-2}}), T^{n-1}(\frac{m_2\cos(\beta s)}{2^{n+k-2}}, \frac{m_2\sin(\beta s)}{2^{n+k-2}})\right) \\ =& d\left((\frac{m_1}{2^{k-1}}\cos(\beta t), \frac{m_1}{2^{k-1}}\sin(\beta t)), (\frac{m_2}{2^{k-1}}\cos(\beta s), \frac{m_2}{2^{k-1}}\sin(\beta s))\right) \\ \geq& d\left((\frac{m_1}{2^{k-1}}\cos(\beta t), \frac{m_1}{2^{k-1}}\sin(\beta t)), (\frac{m_2}{2^{k-1}}\cos(\beta t), \frac{m_2}{2^{k-1}}\sin(\beta t))\right) \\ =& \sqrt{\left(\frac{m_1-m_2}{2^{k-1}}\right)^2} = \left|\frac{m_1-m_2}{2^{k-1}}\right| \geq \frac{1}{2^{k-1}} > \epsilon \end{aligned}$$

for all $s,t \in G$. *Hence,* $d_n\left(G(\frac{m_1}{2^{n+k-2}},0), G(\frac{m_2}{2^{n+k-2}},0))\right) > \epsilon$ *so that* $s(n,\epsilon,G) \geq 2^{n+k-2}$ *and hence* $h_G(T) \geq \log(2)$.

Now we show that the set F_{n+k-1} *is an* (n,ϵ,G)*-spanning set. Suppose that* $(x_1,x_2) \in B$. *Then, we can choose* $(\frac{m}{2^{n+k-1}},0) \in F_{n+k-1}$ *such that*

$$\left|\frac{m}{2^{n+k-1}} - \sqrt{x_1^2+y_1^2}\right| < \frac{1}{2^{n+k-1}}.$$

Therefore,

$$\begin{aligned} & d_n\left(G(x_1,y_1), G(\frac{m}{2^{n+k-1}},0)\right) \\ & \leq \inf\{\max_{0\leq i\leq n-1} d\left(T^i(\varphi_t(x_1,y_1)), T^i(\varphi_{t+\tau}(\frac{m}{2^{n+k-1}},0))\right) : \ t \in G\} < \frac{1}{2^k} \leq \epsilon, \end{aligned}$$

where $\tau = \tan^{-1}(\frac{y_1}{x_1})$ *(see Figure (B)). Thus* $r_n(n,\epsilon,G) \leq 2^{n+k-1}$. *This implies that* $h_G(T) \leq \log(2)$. *Therefore,* $h_G(T) = \log(2)$.

7. Conclusion

In this paper, we introduced an extension of the notion of topological entropy. Our approach opens a door to a method for developing a meaningful notion of topological entropy for dynamical systems, which are defined on the solutions of some differential equations. Computation of the topological entropy of such systems will be a topic for future research.

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