EXPLICIT SOLUTION OF THE POSITION-DEPENDENT MASS SCHRÖDINGER EQUATION WITH GORA-WILLIAMS KINETIC ENERGY OPERATOR: CONFINED HARMONIC OSCILLATOR MODEL

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Exactly-solvable confined model of the non-relativistic quantum harmonic oscillator is proposed. Free Hamiltonian of the system under study has a form of the Gora-Williams kinetic energy operator. Explicit solution of this confined harmonic oscillator Schrödinger equation in the canonical approach has achieved thanks to effective mass changing with position. Confinement effect also appears as a result of certain behaviour of the position-dependent effective mass depending from confinement parameter a. It is shown that the discrete energy spectrum of the confined harmonic oscillator with position-dependent mass also depends from confinement parameter and has a non-equidistant form. Wavefunctions of the stationary states of the confined oscillator with position-dependent mass are expressed in terms of the Gegenbauer polynomials. At limit $a \to \infty$, both energy spectrum and wavefunctions recover well-known equidistant energy spectrum and wavefunctions of the stationary non-relativistic harmonic oscillator expressed by Hermite polynomials. Position-dependent effective mass also becomes homogeneous under this limit.

Keywords: Position-dependent effective mass; Quantum harmonic oscillator; Confined model; Gora-Williams kinetic energy operator; Explicit polynomial solution.

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1. Introduction

Quantum harmonic oscillator probably is most attracted exactly solvable problem of the non-relativistic quantum theory due to existence of its elegant and simpler mathematical solutions and enormous applications in the different branches of the modern physics and technologies [1]. Usually, when we talk about quantum harmonic oscillator problem, we mean non-relativistic harmonic oscillator in the canonical approach with discrete energy spectrum having equidistant levels and wavefunctions of the stationary states in terms of the Hermite polynomials, obtained through explicit solution of the corresponding Schrödinger equation [2]. Before solving this second order differential equation, one assumes that wavefunctions of the stationary states vanish to zero at positive and negative infinity values of the position as well as effective mass that appears in both kinetic and potential energy parts of the non-relativistic quantum harmonic oscillator Hamiltonian is homogeneous, i.e. it does not depend from position. Of course, we also do all calculations in configuration space that is continuous. What happens, if we drop one or more listed above conditions or special cases leading to harmonic oscillator wavefunctions expressed by the Hermite polynomials? For example, if we drop canonical commutation relation between non-relativistic momentum and

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position operators, then explicit solution of the harmonic oscillator Schrödinger equation corresponding to non-canonical case leads to the wavefunctions of the stationary harmonic oscillator states expressed by the generalized Laguerre polynomials [3]. If we assume that the problem is relativistic, then, an exactly solvable oscillator model also can be constructed and its wavefunctions are going to be expressed by the Meixner-Pollaczek polynomials [4]. Number of exactly solvable harmonic oscillator models in the discrete or hybrid configuration spaces also exist and their wavefunctions are expressed by the Charlier, Krawtchouk, Meixner and Hahn polynomials [5–9]. Discrete harmonic oscillator models, which wavefunctions are expressed by Krawtchouk or Hahn polynomials have one more attractive property, because, these models are also finite, i.e. in fact they are confined on the discrete position or momentum configuration space. It is necessary to note that the study of the explicitly solvable quantum harmonic oscillator model in the finite-continuous configuration space is of great interest due to recent huge development and changes in the field of nanostructures and low dimensional systems [10]. Interesting approach here is construction of the explicitly solvable quantum harmonic oscillator model being confined by two infinite high walls at values of position $x = \pm a$, $a > 0$. This problem is similar to infinite quantum well problem in the non-relativistic approach. The difference is only behaviour of the potential within infinite well. Stationary Schrödinger equation for case $V(x) = 0$ at $-a < x < a$ is exactly solvable. Expressions of both energy spectrum and wavefunctions of the stationary states are well known. Explicit solution of the stationary Schrödinger equation for the case $V(x) = M\omega^2 x^2/2$ at $-a < x < a$ still is open for research despite that first attempts to solve such a problem explicitly can be traced back to 40-es [11, 12]. Up today, enormous number of papers have been devoted to similar solutions of the confined harmonic oscillator model using different approaches for approximate solutions. In fact, this problem can be solved explicitly in terms of the orthogonal polynomials under the approach that effective mass of the quantum system under study is position-dependent rather than homogeneous. Present work is devoted to obtaining such an explicit solution of the confined quantum harmonic oscillator Shrödinger equation with Gora-Williams kinetic energy term. This kinetic energy term is introduced in [19] within the theory of the electronic minority-carrier transport for semiconductors slowly graded in composition, which allowed to discuss the local radiative-recombination lifetime, local density of states as well as some other phenomena specific only for inhomogeneous semiconductors. Further, quantum dynamical systems with Gora-Williams kinetic energy term free Hamiltonian have been developed in various research directions leading to attractive results. For example, possibility of the position-dependent effective mass as well as band parameters concept to the description of the motion of carriers in graded mixed semiconductors is examined in [13]. Effective Hamiltonian describing the motion of electrons in compositionally graded crystals have been constructed in [15]. The dependence of the band-offset ratio of a $GaAs - Al_xGa_{1-x}As$ quantum well in case of Hermitian Hamiltonian with Gora-Williams kinetic energy term is illustrated in [14] and it is found that the Hamiltonian dependence of the band-offset ratio is significant for accurate models of heterojunctions. [16] constructed a class of η-weak-pseudo-Hermitian position-dependent mass Hamiltonians and used some of Scarf II models as examples for subsequent computations. [17] proposes general form of the kinetic energy with position-dependent mass that includes well-known van Roos kinetic energy operator as a special case.

We are going to solve confined quantum harmonic oscillator Shrödinger equation with Gora-Williams kinetic energy term explicitly under the exact position dependence of the effective mass as $M(x) = ma^2/(a^2 - x^2)$. Feature of such a dependence of the mass from position is that at values of position $x = \pm a$, harmonic oscillator potential behaves itself as applied infinite wall as well as position dependence of the effective mass disappears under the limit $a \to \infty$. We are going to discuss all these issues in the rest of present work.

The paper is structured as follows: Section 2 is devoted to the confinement model of the non-relativistic one-dimensional quantum harmonic oscillator, whose wavefunctions of the stationary states and energy spectrum of the model are obtained through the solution of the corresponding Schrödinger equation with Gora-Williams kinetic energy operator under assumption that the mass of the quantum system under confinement varies with position. First, we present basic review of the non-relativistic one-dimensional quantum harmonic oscillator in the canonical approach with vanishing wavefunctions at infinity. Then, we have shown that the wavefunctions of the stationary states are expressed through the Gegenbauer polynomials. We also present explicit expression of the discrete non-equidistant energy spectrum. Discussions and conclusions are given in Section 3.

2. Schrödinger equation with the Gora-Williams kinetic energy operator: confined harmonic oscillator with a position-dependent effective mass

Before going to solve confined quantum harmonic oscillator Shrödinger equation with Gora-Williams kinetic energy operator, we present basic review on the non-relativistic one-dimensional quantum harmonic oscillator in the canonical approach with vanishing wavefunctions at infinity. It is known that the following stationary Schrödinger equation in the position representation with the non-relativistic harmonic oscillator potential

$$\left[\frac{\hat{p}_x^2}{2m}+\frac{m\omega^2x^2}{2}\right]\psi(x)=E\psi(x), \qquad (\mathbf{2}.1)$$

should be solved explicitly for the non-relativistic harmonic oscillator in order to obtain vanishing wavefunctions at infinity. Here m and ω are the position-independent mass and angular frequency of the non-relativistic quantum harmonic oscillator. Definition of the one-dimensional momentum operator as follows

$$\hat{p}_x=-i\hbar\frac{d}{dx}, \qquad (\mathbf{2}.2)$$

means that Schrödinger equation (**2**.1) is written in the canonical approach. Taking into account the definition of the momentum operator (**2**.2) in eq. (**2**.1) one easily obtain the following second order differential equation:

$$\frac{d^2\psi}{dx^2}+\frac{2m}{\hbar^2}\left(E-\frac{m\omega^2x^2}{2}\right)\psi=0. \qquad (\mathbf{2}.3)$$

Its analytical solution to the explicit expression of the discrete equidistant energy spectrum

$$E\equiv E_n=\hbar\omega\left(n+\tfrac{1}{2}\right), \quad n=0,1,2,\dots, \qquad (\mathbf{2}.4)$$

and wavefunctions of the stationary states in the position representation

$$\psi\equiv\psi_n(x)=\frac{1}{\sqrt{2^nn!}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}e^{-\frac{m\omega x^2}{2\hbar}}H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right), \qquad (\mathbf{2}.5)$$

where, $H_n(x)$ are Hermite polynomials defined in terms of ${}_2F_0$ hypergeometric functions as follows [18]:

$$H_n(x)=(2x)^n\,{}_2F_0\left(\begin{matrix}-n/2,-(n-1)/2\\ -\end{matrix};\ -\tfrac{1}{x^2}\right). \qquad (\mathbf{2}.6)$$

Due to known orthogonality relation for the Hermite polynomials [18], the normalized wavefunctions (**2**.5) satisfy similar orthogonality relation:

$$\int_{-\infty}^{\infty} \psi_m^*(x)\psi_n(x)dx = \delta_{mn}. \qquad (\mathbf{2}.7)$$

Gora-Williams kinetic energy operator introduced in [19] for case of effective mass varying with position has a following general form:

$$\hat{H}_0^{GW} = -\frac{\hbar^2}{4}\left\{M^{-1}(x), \frac{d^2}{dx^2}\right\}. \qquad (\mathbf{2}.8)$$

Here, $\{\cdot,\cdot\}$ means anticommutation relation and it appears due to that unlike constant effective mass case, momentum operator $\hat{p}_x$ (**2**.2) and $M^{-1}(x)$ now do not commute. Taking into account that

$$\left\{M^{-1}(x), \frac{d^2}{dx^2}\right\} = \frac{2}{M}\left[\frac{d^2}{dx^2} - \frac{M'}{M}\frac{d}{dx} - \frac{1}{2}\frac{M''}{M} + \left(\frac{M'}{M}\right)^2\right],$$

and introducing confined harmonic oscillator potential as

$$V(x) = \begin{cases} \frac{M(x)\omega^2 x^2}{2}, & -a < x < a, \\ \infty, & x = \pm a, \end{cases} \qquad (\mathbf{2}.9)$$

one can rewrite full Hamiltonian describing non-relativistic harmonic ocsillator with position-dependent effective mass as follows:

$$\hat{H}^{GW} = -\frac{\hbar^2}{2M}\left[\frac{d^2}{dx^2} - \frac{M'}{M}\frac{d}{dx} - \frac{1}{2}\frac{M''}{M} + \left(\frac{M'}{M}\right)^2\right] + \frac{M\omega^2 x^2}{2}. \qquad (\mathbf{2}.10)$$

Now, we have to define position-dependent effective mass thanks to that confinement effect of the harmonic oscillator potential (**2**.9) will be satisfied. It has the following analytical expression:

$$M \equiv M(x) = \frac{a^2 m}{a^2 - x^2}. \qquad (\mathbf{2}.11)$$

One can easily check that position-dependent effective mass $M(x)$ (**2**.11) recovers homogeneous mass m under the simple limit

$$\lim_{a\to\infty} \frac{a^2 m}{a^2 - x^2} = m \qquad (\mathbf{2}.12)$$

as well as quantum harmonic oscillator potential (**2**.9) with position-dependent effective mass $M(x)$ (**2**.11) satisfies the correct boundary conditions

$$V(-a) = V(a) = \infty. \qquad (\mathbf{2}.13)$$

Next, taking into account that

$$\frac{M'}{M} = \frac{2x}{a^2 - x^2}$$

as well as

$$\frac{M''}{M} = \frac{2}{a^2 - x^2} + \frac{8x^2}{(a^2 - x^2)^2},$$

one needs to solve explicitly the following Schrödinger equation:

$$\left[\frac{d^2}{dx^2} - \frac{2x}{a^2 - x^2}\frac{d}{dx} - \frac{1}{a^2 - x^2}\right]\psi + \left(\frac{2ma^2E}{\hbar^2(a^2 - x^2)} - \frac{m^2\omega^2 a^4 x^2}{\hbar^2(a^2 - x^2)^2}\right)\psi = 0. \qquad (\mathbf{2}.14)$$

Introduction of the new dimensionless variable ξ as

$$\xi = \frac{x}{a}, \quad \frac{d}{dx} = \frac{1}{a}\frac{d}{d\xi}, \quad \frac{d^2}{dx^2} = \frac{1}{a^2}\frac{d^2}{d\xi^2}$$

will change eq. (**2**.14) to the following second order differential equation form:

$$\psi'' - \frac{2\xi}{1-\xi^2}\psi' + \frac{c_0 - 1 + (1 - c_0 - c_2)\,\xi^2}{\left(1-\xi^2\right)^2}\psi = 0, \tag{2.15}$$

where

$$c_0 = \frac{2ma^2E}{\hbar^2}, \quad c_2 = \frac{m^2\omega^2a^4}{\hbar^2}.$$

We are going to apply Nikiforov-Uvarov method [20] to solve this equation explicitly. This method is applicable to the second order differential equations of type

$$\psi'' + \frac{\tilde{\tau}}{\sigma}\psi' + \frac{\tilde{\sigma}}{\sigma^2}\psi = 0,$$

where, it is assumed that σ and $\tilde{\sigma}$ are arbitrary polynomials of at most second degree and $\tilde{\tau}$ is an arbitrary polynomial of at most first degree. In our case

$$\tilde{\tau} = -2\xi, \qquad \sigma = 1 - \xi^2, \qquad \tilde{\sigma} = c_0 - 1 + (1 - c_0 - c_2)\,\xi^2,$$

that completely satisfies above listed requirements of the applicability of Nikiforov-Uvarov method to explicit solution of eq. (**2**.15). We look for expression of ψ as

$$\psi = \varphi(\xi)\,y,$$

where, $\varphi(\xi)$ should be defined in terms of σ and another arbitrary polynomial of at most first degree π as follows:

$$\varphi(\xi) = e^{\int \frac{\pi(\xi)}{\sigma(\xi)}d\xi}.$$

Further simple computations lead to the following second order differential equation for y:

$$y'' + \frac{2\pi + \tilde{\tau}}{\sigma}y' + \frac{\tilde{\sigma} + \pi^2 + \pi\left(\tilde{\tau} - \sigma'\right) + \pi'\sigma}{\sigma^2}y = 0. \tag{2.16}$$

Let's do now the following substitutions in (**2**.16):

$$\tau = 2\pi + \tilde{\tau}, \quad \bar{\sigma} = \tilde{\sigma} + \pi^2 + \pi\left(\tilde{\tau} - \sigma'\right) + \pi'\sigma.$$

Then, eq.(**2**.16) will be written in the following more compact form:

$$y'' + \frac{\tau}{\sigma}y' + \frac{\bar{\sigma}}{\sigma^2}y = 0. \tag{2.17}$$

Due to that each term in the definition of $\bar{\sigma}$ is a polynomial at most of second degree, one can require here that $\bar{\sigma} = \lambda\sigma$ with $\lambda = const$. If also to introduce new notation $\mu = \lambda - \pi'$, then, we will need to solve the following quadratic equation for $\pi(\xi)$:

$$\pi^2 + \left(\tilde{\tau} - \sigma'\right)\pi + \tilde{\sigma} - \mu\sigma = 0. \tag{2.18}$$

Now, taking into account that $\sigma' = -2\xi$ and as a consequence of $\tilde{\tau} - \sigma' = 0$, eq.(**2**.18) will be completely simplified as follows:

$$\pi^2 + \tilde{\sigma} - \mu\sigma = 0. \tag{2.19}$$

From eq.(**2**.19) it is easy to observe that

$$\pi = \varepsilon\sqrt{\mu\sigma - \tilde{\sigma}} = \varepsilon\sqrt{\mu - c_0 + 1 + (c_0 + c_2 - \mu - 1)\,\xi^2}, \quad \varepsilon = \pm 1. \tag{2.20}$$

We already defined $\pi(\xi)$ as a polynomial at most of first degree. Now parameter μ should be chosen by such a manner that the expression under the square root will have multiple root, i.e. discriminant of the square root expression should equal to zero as follows:

$$\mathcal{D} = -4\,(c_0 + c_2 - 1 - \mu)\,(\mu - c_0 + 1) = 0. \tag{2.21}$$

Then, we obtain the following possible expressions for $\pi(\xi)$:

$$\pi(\xi) = \begin{cases} \varepsilon\sqrt{c_2}\xi, & \mu = c_0 - 1, \\ \varepsilon\sqrt{c_2}, & \mu = c_0 + c_2 - 1. \end{cases} \tag{2.22}$$

We obtain $\varphi(\xi) = \left(1 - \xi^2\right)^{-\frac{1}{2}\varepsilon\sqrt{c_2}}$ for case $\mu = c_0 - 1$ and $\varphi(\xi) = \left(\frac{1+\xi}{1-\xi}\right)^{\frac{1}{2}\varepsilon\sqrt{c_2}}$ for case $\mu = c_0 + c_2 - 1$. Due to the finiteness property of the wavefunction at points $\xi = \pm 1$ (or $x = \pm a$) $\lim\limits_{\xi\to\pm 1}\psi(\xi) = 0$, one observes the condition $\varepsilon = -1$ should be satisfied, and the case $\mu = c_0 - 1$ should be chosen for $\varphi(\xi)$, which leads to the following final expressions of $\pi(\xi)$ and $\varphi(\xi)$:

$$\pi(\xi) = -\sqrt{c_2}\xi, \quad \varphi(\xi) = \left(1 - \xi^2\right)^{\frac{1}{2}\sqrt{c_2}}. \tag{2.23}$$

Also, as a consequence, we find that

$$\lambda = c_0 - 1 - \sqrt{c_2}, \quad \tau(\xi) = -2\left(\sqrt{c_2} + 1\right)\xi. \tag{2.24}$$

Then, eq.(**2**.17) will have the more compact form as follows:

$$\sigma y'' + \tau y' + \lambda y = 0. \tag{2.25}$$

Function $y(\xi)$ should be finite at values $\xi = \pm 1$. Therefore, we have to find its polynomial solutions. For this reason, we compare it with the following second order differential equation for the Gegenbauer polynomials [18]

$$\left(1 - x^2\right)\bar{y}'' - \left(2\bar{\lambda} + 1\right)x\bar{y}' + n\left(n + 2\bar{\lambda}\right)\bar{y} = 0, \quad \bar{y} = C_n^{(\bar{\lambda})}(x),$$

leads us to the following non-equidistant energy spectrum

$$E \equiv E_n^{GW} = \hbar\omega\left(n + \frac{1}{2}\right) + \frac{\hbar^2}{2ma^2}\left(n^2 + n + 1\right), \tag{2.26}$$

and wavefunctions of the stationary states

$$\psi \equiv \psi_n^{GW}(x) = c_n^{GW}\left(1 - \frac{x^2}{a^2}\right)^{\frac{m\omega a^2}{2\hbar}} C_n^{\left(\frac{m\omega a^2}{\hbar} + \frac{1}{2}\right)}\left(\frac{x}{a}\right), \tag{2.27}$$

where, $C_n^{(\bar{\lambda})}(x)$ are Gegenbauer polynomials defined in terms of the ${}_2F_1$ hypergeometric functions as follows:

$$C_n^{(\bar{\lambda})}(x) = \frac{\left(2\bar{\lambda}\right)_n}{n!}\,{}_2F_1\left(\begin{array}{c} -n, n + 2\bar{\lambda} \\ \bar{\lambda} + 1/2 \end{array}; \frac{1-x}{2}\right), \; \bar{\lambda} \neq 0 \tag{2.28}$$

and the normalization factor c_n^{GW} being equal to

$$c_n^{GW} = 2^{\frac{m\omega a^2}{\hbar}}\Gamma\left(\frac{m\omega a^2}{\hbar} + \frac{1}{2}\right)\sqrt{\frac{\left(n + \frac{m\omega a^2}{\hbar} + \frac{1}{2}\right)n!}{\pi a\Gamma\left(n + \frac{2m\omega a^2}{\hbar} + 1\right)}}, \tag{2.29}$$

is defined from the orthogonality relation for Gegenbauer polynomials $C_n^{(\bar{\lambda})}(x)$ of the following form

$$\int\limits_{-1}^{1} \left(1-x^2\right)^{\bar{\lambda}-\frac{1}{2}} C_m^{(\bar{\lambda})}(x) C_n^{(\bar{\lambda})}(x) dx = \frac{\pi \Gamma\left(n+2\bar{\lambda}\right) 2^{1-2\bar{\lambda}}}{\left\{\Gamma\left(\bar{\lambda}\right)\right\}^2 \left(n+\bar{\lambda}\right) n!} \delta_{mn} \qquad (\mathbf{2}.30)$$

under conditions $\bar{\lambda} > -\frac{1}{2}$ and $\bar{\lambda} \neq 0$. Therefore, wavefunctions of the stationary states in the position representation (**2**.27) are also orthogonal in the finite region $-a < x < a$:

$$\int\limits_{-a}^{a} \left[\psi_m^{GW}(x)\right]^* \psi_n^{GW}(x) dx = \delta_{mn}. \qquad (\mathbf{2}.31)$$

We are going to discuss impact of non-equidistant behaviour of the energy spectrum appeared due to confinement effect as well as possible special cases and correct limit expressions of both (**2**.26) and (**2**.27) in final section.

3. Discussions and Conclusion

Taking into account that explicit expressions of the wavefunctions of the stationary states and discrete energy spectrum are obtained by solving the Schrödinger equation with Gora-Williams kinetic energy operator for the confined harmonic oscillator model (**2**.14), now let's explore what kind of primary differences appear in properties of wavefunctions and energy spectrum of the oscillator under the confinement effect and mass varying with position. First of all, let's note that oscillator model with wavefunctions expressed by Gegenbauer polynomials is not new. We have to note [21] that proposes quantum non-linear oscillator model with wavefunctions expressed by Gegenbauer polynomials and non-equidistant energy spectrum. If one compares energy spectrum (**2**.26) with energy spectrum from [21] and other papers, which developed later proposed non-linear oscillator model, then, one observes that unlike non-linear oscillator energy spectrum, ground state energy level in our case substantially differs from the ground state energy level of the non-relativistic harmonic oscillator $E_0 = \hbar\omega/2$ and it depends from the confinement parameter a.

In Fig.1, we present behaviour of the confined quantum harmonic oscillator potential (**2**.9) and its corresponding non-equidistant energy levels (**2**.26) as well as probability densities $\left|\psi_n^{GW}(x)\right|^2$ computed from wavefunctions of the stationary states (**2**.27) for the ground and excited states for different values of the confinement parameter a ($m = \omega = \hbar = 1$). For simplicity, we depicted probability densities $\left|\psi_n^{GW}(x)\right|^2$ on the corresponding energy level. In each of these four pictures, we observe form of the harmonic oscillator potential that varies due to its indirect dependence from the confinement parameter a. We also observe how non-equidistant energy spectrum discrete levels fill corresponding confined harmonic oscillator potential (**2**.9). Moreover, we observe that in the tendention of confinement parameter a to ∞, non-equidistant energy levels become more equidistant-like.

In Fig.2, we also present dependence of the non-equidistant energy levels (**2**.26) from the confinement parameter a for the ground and 10 excited states ($m = \omega = \hbar = 1$). Main goal of the presentation such a depicting is to exhibit sharp increase energy values, including ground state energy level to infinity upon the reduce of confinement parameter a to zero. It is possible to discuss obtained results more from different aspects, but, we think that main added value of these results to existed ones and their importance is their explicit expressions. Therefore, below we simply want to show briefly, how all these results recover their non-relativistic analogues.

Limit from non-equidistant energy spectrum of the non-relativistic confined oscillator model with position-dependent effective mass to its unbounded analogue is simple. It is obvious that

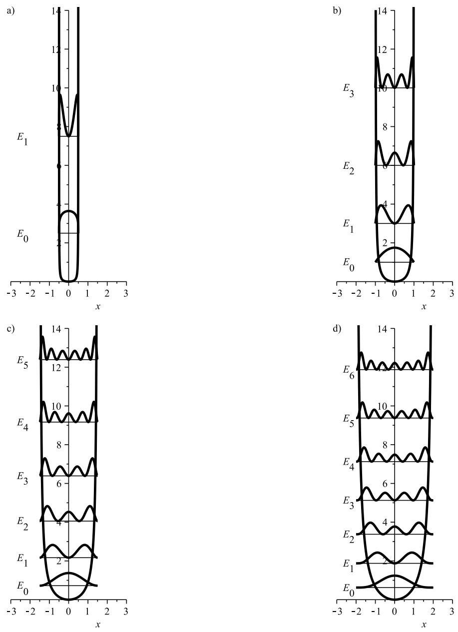

FIGURE 1. Confined quantum harmonic oscillator potential (**2**.9) and behaviour of the corresponding non-equidistant energy levels (**2**.26) and probability densities $\left|\psi_n^{GW}(x)\right|^2$ of the ground and a) 1 excited state for $a=0.5$; b) 3 excited states for $a=1$; c) 5 excited states for $a=1.5$; d) 6 excited states for $a=2$ ($m=\omega=\hbar=1$).

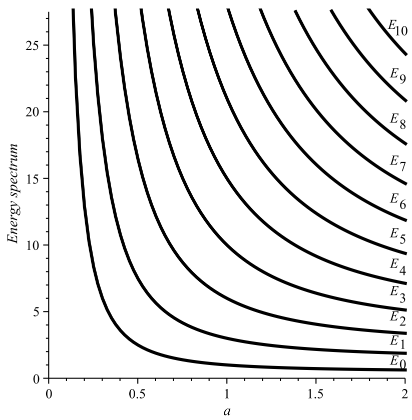

FIGURE 2. Dependence of the non-equidistant energy levels (**2**.26) from the confinement parameter a for the ground and 10 excited states ($m=\omega=\hbar=1$).

$$\lim_{a\to\infty} E_n^{GW} = \hbar\omega\left(n+\tfrac{1}{2}\right) = E_n. \qquad (\mathbf{3}.1)$$

Let's discuss details of the limit between confined and free wavefunctions (**2**.27) and (**2**.5). Due to existence of the following well-known limit relation between Gegenbauer polynomials $C_n^{(\alpha)}(x)$ and Hermite polynomials $H_n(x)$

$$\lim_{\alpha\to\infty} \alpha^{-\frac{1}{2}n} C_n^{\left(\alpha+\frac{1}{2}\right)}\left(\frac{x}{\sqrt{\alpha}}\right) = \frac{H_n(x)}{n!} \qquad (\mathbf{3}.2)$$

that is completely applicable in our case, wavefunctions of the stationary states of the confined quantum harmonic oscillator potential with a position-dependent effective mass $\psi_n^b(x)$ ($\mathbf{2}$.27) reduce to wavefunctions of the stationary states of the unbounded quantum harmonic oscillator potential $\psi_n(x)$ ($\mathbf{2}$.5) under the following limit relation:

$$\lim_{a\to\infty} \psi_n^{GW}(x) = \psi_n(x). \qquad (\mathbf{3}.3)$$

Here, also one needs to use the following special case behaviour for the Gamma function:

$$\Gamma\underset{|z|\to\infty}{(z)} \simeq \sqrt{\frac{2\pi}{z}} e^{z\ln z - z},$$

that leads to the following asymptotics and limit relations $\alpha \to \infty$ ($\alpha = \frac{m\omega a^2}{\hbar}$):

$$\Gamma\underset{\alpha\to\infty}{(\alpha + 1/2)} \simeq \sqrt{2\pi} e^{\alpha\ln\alpha - \alpha},$$

$$\Gamma\underset{\alpha\to\infty}{(n + 2\alpha + 1)} \simeq 2^n \sqrt{4\pi\alpha} e^{(2\alpha+n)\ln\alpha - 2\alpha + 2\alpha\ln 2},$$

$$\lim_{\alpha\to\infty} \alpha^{\frac{n}{2}} c_n^{GW} = \tilde{c}_0 \sqrt{\frac{n!}{2^n}}, \quad \tilde{c}_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4},$$

$$\lim_{\alpha\to\infty} \left(1 - \frac{x^2}{a^2}\right)^{\frac{m\omega a^2}{\hbar}} = e^{-\frac{m\omega x^2}{2\hbar}}.$$

We have to note that present solution applied for confined harmonic oscillator problem can also be used in future for explicit solution of different quantum mechanical problems with position-dependent mass under some confinement effect. We think that number of quantum mechanical problems, for example, free quantum particle, potential box or well problem with position-dependent effective mass can be solved in recent future by using same or at least similar approach and they also will lead to unexpected surprising behaviour of the energy spectrum and wavefunction.

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