ITERATIVE ALGORITHMS WITH SELF-ADAPTIVE RULE AND KM METHOD FOR SOLVING SPLIT FIXED POINT PROBLEMS

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In this paper, we investigate the split fixed point problem regarding pseudocontractive operators and demicontractive operators in Hilbert spaces. We propose an iterative algorithm with self-adaptive rule and the Krasnoselskii-Mann method for finding a solution of this split problem. The self-adaptive rule does not require the a priori knowledge of the Lipschitz constant of pseudocontractive operators. Under several additional conditions, we prove that the presented algorithm converges strongly to a solution of the considered split problem.

Keywords: split fixed point, pseudocontractive operator, demicontractive operator, self-adaptive rule, Krasnoselskii-Mann method.

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1. **Introduction**

Let H_1 and H_2 be two real Hilbert spaces. Let $(H_1 \supset)C \neq \emptyset$ and $(H_2 \supset)Q \neq \emptyset$ be two closed convex sets. Let $A\colon H_1 \to H_2$ be a linear bounded operator.

(i) SFP: Recall that the split feasibility problem (SFP) is to search an element $u^\dagger \in H_1$ such that

$$u^\dagger \in C \text{ and } Au^\dagger \in Q, \tag{1}$$

which is a model for the intensity modulated radiation therapy ([4]).

(ii) SFFP: Also, the split fixed point problem (SFFP) is to pursuit an element $u^\dagger \in H_1$ such that

$$u \in \mathrm{Fix}(\phi) \text{ and } Au \in \mathrm{Fix}(\psi),$$

where $\mathrm{Fix}(\phi) := \{v \in C : \phi(v) = v\}$ and $\mathrm{Fix}(\psi) := \{\tilde{v} \in Q : \psi(\tilde{v}) = \tilde{v}\}$ in which $\phi\colon C \to H_1$ and $\psi\colon Q \to H_2$ are two nonlinear operators.

A wide variety of problems can be solved by finding a fixed point of a particular operator, and algorithms for reckoning such points play a prominent role in a number of applications [3, 9–11, 15–22, 25, 26, 33]. The SFPP is an extension of the SFP and of the well-known convex feasibility problem, see Youla [30]. The SFPP investigated by Censor and Segal [5] involves a class of directed operators which is an important

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class since it includes the orthogonal projections and the subgradient projectors (see Yu and Yin [31], or Zhan *et al.* [32]). Moudafi [14] further studied the SFPP involved in a class of demicontractive operators which properly includes the class of quasi-nonexpansive mappings and thus that of directed operators. This context is more desirable for example in fixed point methods related to image recovery where, in many cases, it is possible to map the set of images possessing a certain property to the fixed-point set of a nonlinear quasi-nonexpansive operator. The split problems have been studied extensively in the literature, see ([2, 7, 8, 24, 27–29, 35–37]).

The main purpose of this paper is to study the following SFPP of finding an element $u^\dagger \in H_1$ such that

$$u^\dagger \in \mathrm{Fix}(\varphi) \cap \mathrm{Fix}(\phi) \text{ and } Au^\dagger \in \mathrm{Fix}(\psi), \tag{2}$$

where $\phi\colon H_1 \to H_1$ and $\psi\colon H_2 \to H_2$ are two demicontractive operators and $\varphi : H_1 \to H_1$ is a Lipschitz pseudocontractive operator. The solution set of (2) is denoted by Ω, namely,

$$\Omega := \{u^\dagger \in H_1 : u^\dagger \in \mathrm{Fix}(\varphi) \cap \mathrm{Fix}(\phi) \text{ and } Au^\dagger \in \mathrm{Fix}(\psi)\}.$$

Censor and Segal [5] proposed the following algorithm to solve (2):

$$u_0 \in H_1,\ u_{n+1} = \phi(u_n - \eta A^*(I-\psi)Au_n),$$

where ϕ and ψ are two directed operators and $\eta \in (0, 2/\|A\|^2)$. Moudafi further proposed in [14], the following algorithm to solve (2):

$$\begin{cases} u_0 \in H_1, v_n = u_n - \eta A^*(I-\psi)Au_n, \\ u_{n+1} = (1-\lambda_n)v_n + \lambda_n \phi(v_n), \end{cases} \tag{3}$$

where ϕ and ψ are β_1-demicontractiveand β_2-demicontractive, respectively, $\eta \in (0, \frac{1-\beta_2}{\|A\|^2})$ and $\lambda_n \in (\varrho, 1-\beta_n-\varrho)$ for a small enough $\varrho > 0$. It should be pointed out that in Algorithm (3), the Krasnoselskii-Mann method ([12, 13]) was applied.

Motivated by the works in this direction, in this paper we investigate the SFPP (2) involved in a pseudocontractive operator and two demicontractive operators in Hilbert spaces. We propose an iterative algorithm with self-adaptive rule and the Krasnoselskii-Mann method for finding a solution of the SFPP (2). The self-adaptive rule has no need to know a priori the Lipschitz constant of pseudocontractive operators. Under several additional conditions, we prove that the presented algorithm converges strongly to an element in Ω provided $\Omega \neq \emptyset$.

2. **Preliminaries**

Let H be a real Hilbert space. The following equality is well-known in this setting.

$$\|(1-\theta)x + \theta y\|^2 = (1-\theta)\|x\|^2 + \theta\|y\|^2 - \theta(1-\theta)\|x-y\|^2, \forall x, y \in H, \forall \theta \in \mathbb{R}. \tag{4}$$

Let $\Gamma \subset H$ be a nonempty, closed, and convex set. For every point $x \in H$, there exists a unique nearest point in Γ, denoted by $P_\Gamma(x)$. This point satisfies the inequality

$$\|P_\Gamma(x) - x\| \leq \|y - x\|,\ \forall y \in \Gamma.$$

The mapping P_Γ is called the metric projection of H onto Γ. The metric projection P_Γ is characterized by the fact that $P_\Gamma(x) \in \Gamma$ and for all $x \in H$,

$$\langle x - P_\Gamma(x), y - P_\Gamma(x)\rangle \leq 0, \ \forall y \in \Gamma.$$

In the sequel, we use the following marks:

• " $\rightharpoonup$ " and " $\to$ " denote weak convergence and strong convergence, respectively.

• $\omega_w(p_n) := \{\tilde{p} \in H : \exists \{p_{n_k}\}_{k=1}^{\infty} \subset \{p_n\}_{n=1}^{\infty} \subset H \text{ with } p_{n_k} \rightharpoonup \tilde{p}(k \to \infty)\}$.

Let $f \colon H \to H$ be a nonlinear operator. f is said to be

(i) L-Lipschitz if

$$\|f(x) - f(y)\| \leq L\|x - y\|, \ \forall x, y \in H,$$

where $L > 0$.

f is said to be L-contractive provided $0 < L < 1$, and is called nonexpansive provided $L = 1$. It is well-known that the metric projection is nonexpansive.

(ii) ϱ-demicontractive if

$$\langle x - f(x), x - y\rangle \geq \tfrac{1-\varrho}{2}\|f(x) - x\|^2, \ \forall x \in H, y \in \mathrm{Fix}(f),$$

which is the same as

$$\|f(x) - y\|^2 \quad \leq \quad \|x - y\|^2 + \varrho\|f(x) - x\|^2, \ \forall x \in H, y \in \mathrm{Fix}(f),$$

where $\varrho \in [0, 1)$.

(iii) pseudocontractive if

$$\langle x - f(x) - (y - f(y)), x - y\rangle \quad \geq \quad 0, \ \forall x, y \in H,$$

which is equivalent to

$$\|f(x) - f(y)\|^2 \quad \leq \quad \|x - y\|^2 + \|x - y - (f(x) - f(y))\|^2, \ \forall x, y \in H.$$

(iv) demiclosed at the origin if for any sequence $\{p_n\} \subset H$ and $p \in H$, the following relation holds:

$$p_n \rightharpoonup p \text{ and } p_n - f(p_n) \to 0 \Rightarrow p \in \mathrm{Fix}(f).$$

The following issues have to be emphasized.

• Obviously, between the previous concepts there is the next connextion: (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii).

• The class of demicontractive operators is fundamental because many common types of operators arising is optimization belong to this class, see, for example Bauschke and Combettes [1].

• We are interested in the class of pseudocontractive operators because of their relationship with the class of monotone operators ([1, 6]).

Recall that a linear operator $g \colon H \to H$ is said to be τ-strongly positive if

$$\langle x, g(x)\rangle \geq \tau\|x\|^2, \ \forall x \in H,$$

where $\tau > 0$. It is known that

$$\|I - \gamma g\| \leq 1 - \gamma\tau, \text{ when } \gamma \in \left(0, \frac{1}{\tau}\right).$$

The next property characterizes continuous pseudocontractive operators.

Lemma 2.1 ([34]). *Let $h\colon H \to H$ be a continuous pseudocontractive operator. Then, $I - h$ is demiclosed at the origin (thus, a nonexpansive operator is also demiclosed).*

The following lemma on the convergence of sequence holds true.

Lemma 2.2 ([23]). *Suppose the sequences $\{r_n\}$, $\{\gamma_n\}$ and $\{w_n\}$ satisfy the following conditions:*

(i) $\forall n \geq 0,\ r_n \in [0,\infty), \gamma_n \in [0,1], w_n \in \mathbb{R}$;
(ii) $\sum_{n=0}^{\infty} \gamma_n = \infty$ *and* $\limsup_{n\to\infty} w_n \leq 0$;
(iii) $\forall n \geq 0, r_{n+1} \leq (1-\gamma_n)r_n + \gamma_n w_n$.
Then $\lim_{n\to\infty} r_n = 0$.

3. **Main results**

In this section, to solve (2), we first state some necessary assumptions and propose an iterative algorithm. Finally, we demonstrate the strong convergence of the proposed algorithm. Throughout, suppose that the following conditions are satisfied:

(C1): H_1 and H_2 are two real Hilbert spaces;
(C2): $A\colon H_1 \to H_2$ is a nonzero bounded linear operator and $B\colon H_1 \to H_1$ is a τ-strongly positive bounded linear operator;
(C3): $\phi\colon H_1 \to H_1$ is a β_1-demicontractive operator and $\psi : H_2 \to H_2$ is a β_2-demicontractive operator;
(C4): $I - \phi$ and $I - \psi$ are demiclosed at the origin.
(C5): $\rho\colon H_1 \to H_1$ is a ϖ-contractive operator and $\varphi\colon H_1 \to H_1$ be an L-Lipschitz pseudocontractive operator;
(C6): The solution set Ω of (2) is nonempty.
(C7): $\tau \in (0,\infty)$, $\beta_1 \in [0,1)$, $\beta_2 \in [0,1)$, $\varpi \in (0,1)$, $L \in [1,\infty)$, $\lambda \in (0,1]$, $\theta \in \left(0, \frac{1-\beta_1}{2\lambda}\right)$, $\eta \in (0, \frac{1-\beta_2}{2\theta\|A\|^2})$, $\mu \in (0,1)$, $\delta \in (0,1)$, $\epsilon \in \left(0, \frac{1-\delta^2}{2}\right)$ and $\nu \in (0, \tau/\varpi)$;
(C8): $\gamma_n \in (0,1)(\forall n \geq 0)$, $\lim_{n\to\infty} \gamma_n = 0$ and $\sum_{n=0}^{\infty} \gamma_n = \infty$.

Now, we propose the following algorithm for solving (2).

Algorithm 3.1. *Let $u_0 \in H_1$ be an initial point.*

Step 1. Assume the current iterate u_n is given. Compute

$$x_n = (1-\theta)u_n + \theta[(1-\lambda)u_n + \lambda\phi(u_n) - \eta A^*(I-\psi)Au_n]. \tag{5}$$

Step 2. Compute

$$y_n = \left(1 - \frac{\alpha_n}{2}\right) x_n + \frac{\alpha_n}{2}\varphi(z_n), \tag{6}$$

where

$$z_n = (1-\alpha_n)x_n + \alpha_n\varphi(x_n), \tag{7}$$

in which $\alpha_n = \epsilon\mu^k$ and $k = \min\{0,1,2,\cdots\}$ such that

$$\alpha_n\|\varphi(z_n) - \varphi(x_n)\| \leq \delta\|z_n - x_n\|. \tag{8}$$

Step 3. Compute

$$u_{n+1} = \gamma_n \nu \rho(u_n) + (I - \gamma_n B)y_n, \tag{9}$$

and set $n := n+1$ *and return to Step 1.*

Remark 3.1. *By condition* (C7)*, we have the inequality* $\delta^2 < 1 - 2\epsilon$*. It implies that* $1 - 2\alpha_n - \delta^2 > 0$*, for all* n*.*

With respect to the well-posedness of this algorithm, the next property holds.

Lemma 3.1. There exists k such that inequality (8) holds, and

$$\min\left\{\epsilon, \frac{\mu\delta}{L}\right\} \le \alpha_n \le \epsilon, n \ge 0.$$

Proof. In fact, if $z_n = x_n$, we can choose $k = 0$.

Next, we consider the case of $z_n \neq x_n$. In this case, suppose that (8) does not hold for any $k \in \min\{0, 1, 2, \cdots\}$, namely,

$$\epsilon\mu^k \|\varphi(z_n) - \varphi(x_n)\| > \delta\|z_n - x_n\|, \text{ for all } k \ge 0. \tag{10}$$

By (7), we have

$$\|z_n - x_n\| = \alpha_n\|\varphi(x_n) - x_n\| = \epsilon\mu^k\|\varphi(x_n) - x_n\| \tag{11}$$

which together with $z_n \neq x_n$ implies that

$$\|\varphi(x_n) - x_n\| > 0. \tag{12}$$

Combining (10) and (11), we obtain

$$\epsilon\mu^k\|\varphi(z_n) - \varphi(x_n)\| > \delta\|z_n - x_n\| = \delta\epsilon\mu^k\|\varphi(x_n) - x_n\|, \text{ for all } k \ge 0,$$

which yields that

$$\|\varphi(z_n) - \varphi(x_n)\| > \delta\|\varphi(x_n) - x_n\|. \tag{13}$$

Noting that $\mu \in (0,1)$ and φ is L-Lipschitz, we have

$$\lim_{k\to\infty} \|\varphi(x_n) - \varphi(z_n)\| = \lim_{k\to\infty} \|\varphi(x_n) - \varphi(x_n + \epsilon\mu^k(\varphi(x_n) - x_n))\| = 0,$$

which together with (13) implies that $\|\varphi(x_n) - x_n\| \le 0$. This is a contradiction with (12). Hence, there is k such that inequality (8) holds.

Since φ is L-Lipschitz, we have

$$\alpha_n\|\varphi(z_n) - \varphi(x_n)\| \le \alpha_n L\|z_n - x_n\|. \tag{14}$$

At the same time, from the definition of k, it follows that

$$\frac{\alpha_n}{\mu}\|\varphi(z_n) - \varphi(x_n)\| > \delta\|z_n - x_n\|. \tag{15}$$

From (14) and (15), we have

$$\mu\delta\|z_n - x_n\| < \alpha_n L\|z_n - x_n\|.$$

If $z_n = x_n$, then $k = 0$ and $\alpha_n = \epsilon$. If $z_n \neq x_n$, then $\alpha_n > \frac{\mu\delta}{L}$. □

Next, we prove the following main theorem.

Theorem 3.1. *The sequence $\{u_n\}$ generated by Algorithm 3.1 converges strongly to $x^* = P_\Omega(I - B + \nu\rho)x^*$.*

Proof. Select $q^* \in \Omega$. Then, $q^* = \phi(q^*) = \varphi(q^*)$ and $Aq^* = \psi(Aq^*)$. From (9), we have

$$\begin{aligned}\|u_{n+1} - q^*\| &= \|\gamma_n\nu(\rho(u_n) - \rho(q^*)) + (I - \gamma_n B)(y_n - q^*) + \gamma_n(\nu\rho(q^*) - B(q^*))\| \\ &\le \gamma_n\nu\varpi\|u_n - q^*\| + (1 - \tau\gamma_n)\|y_n - q^*\| + \gamma_n\|\nu\rho(q^*) - B(q^*)\|.\end{aligned} \tag{16}$$

Applying equality (4), we obtain

$$\begin{aligned}\|y_n - q^*\|^2 &= \left\|(1 - \frac{\alpha_n}{2})(x_n - q^*) + \frac{\alpha_n}{2}(\varphi(z_n) - q^*)\right\|^2 \\ &= \left(1 - \frac{\alpha_n}{2}\right)\|x_n - q^*\|^2 + \frac{\alpha_n}{2}\|\varphi(z_n) - q^*\|^2 \\ &\quad - \frac{\alpha_n}{2}\left(1 - \frac{\alpha_n}{2}\right)\|x_n - \varphi(z_n)\|^2,\end{aligned} \tag{17}$$

$$\begin{aligned}\|z_n - q^*\|^2 &= \|(1 - \alpha_n)(x_n - q^*) + \alpha_n(\varphi(x_n) - q^*)\|^2 \\ &= (1 - \alpha_n)\|x_n - q^*\|^2 + \alpha_n\|\varphi(x_n) - q^*\|^2 \\ &\quad - (1 - \alpha_n)\alpha_n\|x_n - \varphi(x_n)\|^2,\end{aligned} \tag{18}$$

and

$$\begin{aligned}\|z_n - \varphi(z_n)\|^2 &= \|(1 - \alpha_n)(x_n - \varphi(z_n)) + \alpha_n(\varphi(x_n) - \varphi(z_n))\|^2 \\ &= (1 - \alpha_n)\|x_n - \varphi(z_n)\|^2 + \alpha_n\|\varphi(x_n) - \varphi(z_n)\|^2 \\ &\quad - \alpha_n(1 - \alpha_n)\|x_n - \varphi(x_n)\|^2.\end{aligned} \tag{19}$$

Since φ is pseudocontractive,

$$\|\varphi(x_n) - q^*\|^2 \le \|x_n - q^*\|^2 + \|x_n - \varphi(x_n)\|^2, \tag{20}$$

and

$$\|\varphi(z_n) - q^*\|^2 \le \|z_n - q^*\|^2 + \|z_n - \varphi(z_n)\|^2. \tag{21}$$

Combining (18) and (20), we deduce

$$\begin{aligned}\|z_n - q^*\|^2 &\le (1 - \alpha_n)\|x_n - q^*\|^2 - (1 - \alpha_n)\alpha_n\|x_n - \varphi(x_n)\|^2 \\ &\quad + \alpha_n(\|x_n - q^*\|^2 + \|x_n - \varphi(x_n)\|^2) \\ &= \|x_n - q^*\|^2 + \alpha_n^2\|x_n - \varphi(x_n)\|^2.\end{aligned} \tag{22}$$

We can rewrite (7) as

$$z_n - x_n = \alpha_n(\varphi(x_n) - x_n),$$

which together with (8) implies that

$$\alpha_n\|\varphi(x_n) - \varphi(z_n)\|^2 \le \frac{\delta^2\|x_n - z_n\|^2}{\alpha_n} = \delta^2\alpha_n\|\varphi(x_n) - x_n\|^2.$$

It follows from (19) that

$$\|z_n - \varphi(z_n)\|^2 \le (1 - \alpha_n)\|x_n - \varphi(z_n)\|^2 - \alpha_n(1 - \alpha_n - \delta^2)\|x_n - \varphi(x_n)\|^2. \tag{23}$$

From (21)-(23), we have

$$\begin{aligned}\|\varphi(z_n) - q^*\|^2 &\leq \|x_n - q^*\|^2 + \alpha_n^2\|x_n - \varphi(x_n)\|^2 + \|z_n - \varphi(z_n)\|^2 \\ &\leq \|x_n - q^*\|^2 + (1-\alpha_n)\|x_n - \varphi(z_n)\|^2 \\ &\quad - \alpha_n(1 - 2\alpha_n - \delta^2)\|x_n - \varphi(x_n)\|^2.\end{aligned} \tag{24}$$

On account of (24) and Remark 3.1, it follows that

$$\|\varphi(z_n) - q^*\|^2 \leq \|x_n - q^*\|^2 + (1-\alpha_n)\|x_n - \varphi(z_n)\|^2. \tag{25}$$

For all n, set

$$v_n = (1-\lambda)u_n + \lambda\phi(u_n) - \eta A^*(I-\psi)Au_n,$$

Take into account (5), we obtain

$$\begin{aligned}\|x_n - q^*\|^2 &= \|u_n - q^* - \theta(u_n - v_n)\|^2 \\ &= \|u_n - q^*\|^2 - 2\theta\langle u_n - q^*, u_n - v_n\rangle + \theta^2\|u_n - v_n\|^2.\end{aligned} \tag{26}$$

Owing to the equality

$$u_n - v_n = \lambda(u_n - \phi(u_n)) + \eta A^*(I-\psi)Au_n,$$

we receive

$$\langle u_n - q^*, u_n - v_n\rangle = \lambda\langle u_n - q^*, u_n - \phi(u_n)\rangle + \eta\langle Au_n - Aq^*, (I-\psi)Au_n\rangle, \tag{27}$$

and

$$\begin{aligned}\|u_n - v_n\|^2 &\leq (\lambda\|u_n - \phi(u_n)\| + \eta\|A\|\|(I-\psi)Au_n\|)^2 \\ &\leq 2\lambda^2\|u_n - \phi(u_n)\|^2 + 2\eta^2\|A\|^2\|(I-\psi)Au_n\|^2.\end{aligned} \tag{28}$$

Since ϕ is β_1-demicontractive and $q^* \in \text{Fix}(\phi)$,

$$\langle u_n - q^*, u_n - \phi(u_n)\rangle \geq \frac{1-\beta_1}{2}\|u_n - \phi(u_n)\|^2. \tag{29}$$

Similarly,

$$\langle Au_n - Aq^*, (I-\psi)Au_n\rangle \geq \frac{1-\beta_2}{2}\|(I-\psi)Au_n\|^2, \tag{30}$$

because ψ is β_2-demicontractive and $Aq^* \in \text{Fix}(\psi)$.

Substituting inequality (29) and relation (30) into equality (27), we get

$$\langle u_n - q^*, u_n - v_n\rangle \geq \frac{\lambda(1-\beta_1)}{2}\|u_n - \phi(u_n)\|^2 + \frac{\eta(1-\beta_2)}{2}\|(I-\psi)Au_n\|^2. \tag{31}$$

From (26), (28) and (31), we attain

$$\begin{aligned}\|x_n - q^*\|^2 &\leq \|u_n - q^*\|^2 - \theta\lambda(1-\beta_1)\|u_n - \phi(u_n)\|^2 - \theta\eta(1-\beta_2)\|(I-\psi)Au_n\|^2 \\ &\quad + 2\theta^2\lambda^2\|u_n - \phi(u_n)\|^2 + 2\theta^2\eta^2\|A\|^2\|(I-\psi)Au_n\|^2 \\ &= \|u_n - q^*\|^2 - \theta\lambda(1-\beta_1-2\theta\lambda)\|u_n - \phi(u_n)\|^2 \\ &\quad - \theta\eta(1-\beta_2-2\theta\eta\|A\|^2)\|(I-\psi)Au_n\|^2.\end{aligned} \tag{32}$$

Combining with (17), (25) and (32), we get

$$\begin{aligned}\|y_n - q^*\|^2 &\leq \|x_n - q^*\|^2 - \frac{\alpha_n^2}{4}\|x_n - \varphi(z_n)\|^2 \\ &\leq \|u_n - q^*\|^2 - \theta\lambda(1-\beta_1-2\theta\lambda)\|u_n - \phi(u_n)\|^2 \\ &\quad - \theta\eta(1-\beta_2-2\theta\eta\|A\|^2)\|(I-\psi)Au_n\|^2 - \frac{\alpha_n^2}{4}\|x_n - \varphi(z_n)\|^2.\end{aligned} \tag{33}$$

In view of (C7), $1-\beta_1-2\theta\lambda > 0$ and $1-\beta_2-2\theta\eta\|A\|^2 > 0$. It follows from (33) that

$$\|y_n - q^*\| \leq \|x_n - q^*\| \leq \|u_n - q^*\|.$$

By (16), we have

$$\begin{aligned}\|u_{n+1} - q^*\| &\leq [1-(\tau-\nu\varpi)\gamma_n]\|u_n - q^*\| + \gamma_n\|\nu\rho(q^*) - B(q^*)\| \\ &\leq \max\{\|u_n - q^*\|, \tfrac{\|\nu\rho(q^*)-B(q^*)\|}{\tau-\nu\varpi}\}.\end{aligned}$$

Using the same inequality repeatedly, we obtain, for any n, that

$$\|u_{n+1} - q^*\| \leq \max\{\|u_0 - q^*\|, \tfrac{\|\nu\rho(q^*)-B(q^*)\|}{\tau-\nu\varpi}\}.$$

Then, the sequence $\{u_n\}$ is bounded.
In light of (9), we have

$$\begin{aligned}\|u_{n+1} - q^*\|^2 &= \langle \gamma_n\nu(\rho(u_n) - \rho(q^*)) + (I-\gamma_n B)(y_n - q^*), u_{n+1} - q^*\rangle \\ &\quad + \gamma_n\langle \nu\rho(q^*) - B(q^*), u_{n+1} - q^*\rangle \\ &\leq (\nu\gamma_n\|\rho(u_n) - \rho(q^*)\| + \|I-\gamma_n B\|\|y_n - q^*\|)\|u_{n+1} - q^*\| \\ &\quad + \gamma_n\langle \nu\rho(q^*) - B(q^*), u_{n+1} - q^*\rangle \\ &\leq (\nu\gamma_n\varpi\|u_n - q^*\| + (1-\tau\gamma_n)\|y_n - q^*\|)\|u_{n+1} - q^*\| \\ &\quad + \gamma_n\langle \nu\rho(q^*) - B(q^*), u_{n+1} - q^*\rangle \\ &\leq \frac{\nu\varpi\gamma_n}{2}\|u_n - q^*\|^2 + \frac{1-\tau\gamma_n}{2}\|y_n - q^*\|^2 + \frac{1}{2}\|u_{n+1} - q^*\|^2 \\ &\quad + \gamma_n\langle \nu\rho(q^*) - B(q^*), u_{n+1} - q^*\rangle,\end{aligned}$$

which yields

$$\begin{aligned}\|u_{n+1} - q^*\|^2 &\leq \nu\varpi\gamma_n\|u_n - q^*\|^2 + (1-\tau\gamma_n)\|y_n - q^*\|^2 \\ &\quad + 2\gamma_n\langle \nu\rho(q^*) - B(q^*), u_{n+1} - q^*\rangle.\end{aligned} \tag{34}$$

On account of (33) and (34), we obtain

$$\begin{aligned}\|u_{n+1}-q^*\|^2 \leq{}& [1-(\tau-\nu\varpi)\gamma_n]\|u_n-q^*\|^2+(\tau-\nu\varpi)\gamma_n \\ & \times\big(-(1-\tau\gamma_n)\theta\lambda(1-\beta_1-2\theta\lambda)\frac{\|u_n-\phi(u_n)\|^2}{(\tau-\nu\varpi)\gamma_n} \\ & -(1-\tau\gamma_n)\theta\eta(1-\beta_2-2\theta\eta\|A\|^2)\frac{\|(I-\psi)Au_n\|^2}{(\tau-\nu\varpi)\gamma_n} \\ & +\frac{2}{\tau-\nu\varpi}\langle\nu\rho(q^*)-B(q^*),u_{n+1}-q^*\rangle \\ & -(1-\tau\gamma_n)\frac{\alpha_n^2}{4}\frac{\|x_n-\varphi(z_n)\|^2}{(\tau-\nu\varpi)\gamma_n}\big).\end{aligned} \tag{35}$$

Put $r_n=\|u_n-q^*\|^2$ and

$$\begin{aligned}w_n={}&-(1-\tau\gamma_n)\theta\lambda(1-\beta_1-2\theta\lambda)\frac{\|u_n-\phi(u_n)\|^2}{(\tau-\nu\varpi)\gamma_n} \\ & -(1-\tau\gamma_n)\theta\eta(1-\beta_2-2\theta\eta\|A\|^2)\frac{\|(I-\psi)Au_n\|^2}{(\tau-\nu\varpi)\gamma_n} \\ & +\frac{2}{\tau-\nu\varpi}\langle\nu\rho(q^*)-B(q^*),u_{n+1}-q^*\rangle \\ & -(1-\tau\gamma_n)\frac{\alpha_n^2}{4}\frac{\|x_n-\varphi(z_n)\|^2}{(\tau-\nu\varpi)\gamma_n}.\end{aligned} \tag{36}$$

We can rewrite (35), for all $n\geq 0$, as

$$r_{n+1}\leq[1-(\tau-\nu\varpi)\gamma_n]r_n+(\tau-\nu\varpi)\gamma_n w_n.$$

Next, we prove $\limsup_{n\to\infty}w_n\geq -1$. If $\limsup_{n\to\infty}w_n<-1$, then there is a positive integer m satisfying $w_n<-1$ when $n\geq m$. Then, $r_{n+1}\leq r_n-(\tau-\nu\varpi)\gamma_n$, $n\geq m$, which results in

$$r_{n+1}\leq r_m-\sum_{i=m}^{n}(\tau-\nu\varpi)\gamma_i.$$

Hence, $\limsup_{n\to\infty}r_{n+1}\leq-\infty$ which is impossible. Thus, $\limsup_{n\to\infty}w_n\geq -1$. At the same time, due to (36), we conclude that

$$w_n\leq\frac{2}{\tau-\nu\varpi}\|\nu\rho(q^*)-B(q^*)\|\|u_{n+1}-q^*\|,$$

which yields that $\limsup_{n\to\infty}w_n<\infty$.

Next, we prove $\omega_w(u_n)\subset\Omega$. Pick up $\hat{u}\in\omega_w(u_n)$, which means that there is $\{u_{n_k}\}\subset\{u_n\}$ satisfying $u_{n_k}\rightharpoonup\hat{u}(k\to\infty)$ and $\limsup_{n\to\infty}w_n=\lim_{k\to\infty}w_{n_k}$. Without loss of generality, assume $\lim_{k\to\infty}\langle\nu\rho(q^*)-B(q^*),u_{n_k+1}-q^*\rangle$ exists. Hence, the following limit exists

$$\lim_{k\to\infty}\Big(-(1-\tau\gamma_{n_k})\theta\lambda(1-\beta_1-2\theta\lambda)\frac{\|u_{n_k}-\phi(u_{n_k})\|^2}{(\tau-\nu\varpi)\gamma_{n_k}} - (1-\tau\gamma_{n_k})\theta\eta(1-\beta_2-2\theta\eta\|A\|^2)\frac{\|(I-\psi)Au_{n_k}\|^2}{(\tau-\nu\varpi)\gamma_{n_k}} - (1-\tau\gamma_{n_k})\frac{\alpha_{n_k}^2}{4}\frac{\|x_{n_k}-\varphi(z_{n_k})\|^2}{(\tau-\nu\varpi)\gamma_{n_k}}\Big),$$

which yields that

$$\lim_{k\to\infty}\|u_{n_k}-\phi(u_{n_k})\| = \lim_{k\to\infty}\|(I-\psi)Au_{n_k}\| = \lim_{k\to\infty}\|x_{n_k}-\varphi(z_{n_k})\| = 0. \tag{37}$$

From (5), we have

$$\|x_{n_k}-u_{n_k}\| \le \theta\lambda\|\phi(u_{n_k})-u_{n_k}\| + \theta\eta\|A\|\|(I-\psi)Au_{n_k}\|,$$

which together with (37) implies that

$$\lim_{k\to\infty}\|x_{n_k}-u_{n_k}\| = 0. \tag{38}$$

By (6),

$$\|y_{n_k}-x_{n_k}\| \le \frac{\alpha_{n_k}}{2}\|\varphi(z_{n_k})-x_{n_k}\|.$$

It follows from (37) that

$$\lim_{k\to\infty}\|y_{n_k}-x_{n_k}\| = 0. \tag{39}$$

Taking into account (7) and (8), we have

$$\begin{aligned}\|z_{n_k}-x_{n_k}\| &\le \alpha_{n_k}\|\varphi(x_{n_k})-\varphi(z_{n_k})\| + \alpha_{n_k}\|\varphi(z_{n_k})-x_{n_k}\|\\ &\le \delta\|x_{n_k}-z_{n_k}\| + \alpha_{n_k}\|\varphi(z_{n_k})-x_{n_k}\|.\end{aligned}$$

It follows that $\|z_{n_k}-x_{n_k}\| \le \frac{\alpha_{n_k}}{1-\delta}\|\varphi(z_{n_k})-x_{n_k}\|$ and hence

$$\lim_{k\to\infty}\|z_{n_k}-x_{n_k}\| = 0. \tag{40}$$

Combining (37) and (40), we deduce

$$\lim_{k\to\infty}\|z_{n_k}-\varphi(z_{n_k})\| = 0. \tag{41}$$

As $\gamma_n \to 0$, from relations (9), (38) and (39), the next relations

$$\begin{aligned}\|u_{n+1}-u_n\| &= \|\gamma_n\nu\rho(u_n) + (I-\gamma_n B)y_n - u_n\|\\ &\le \gamma_n(\nu\|\rho(u_n)\| + \|B(y_n)\|) + \|y_n-u_n\|\\ &\le \gamma_n(\nu\|\rho(u_n)\| + \|B(y_n)\|) + \|y_n-x_n\| + \|x_n-u_n\|\end{aligned}$$

lead to

$$\lim_{k\to\infty}\|u_{n_k+1}-u_{n_k}\| = 0. \tag{42}$$

Noticing that $u_{n_k} \rightharpoonup \hat{u}$, we obtain

$$Au_{n_k} \rightharpoonup A\hat{u}, \text{ and } z_{n_k} \rightharpoonup \hat{u}.$$

Therefore,

$$\left.\begin{array}{r} u_{n_k} \rightharpoonup \hat{u}, \|u_{n_k} - \phi(u_{n_k})\| \to 0 \\ I - \phi \text{ is demiclosed at the origin} \end{array}\right\} \Rightarrow \hat{u} \in \text{Fix}(\phi),$$

$$\left.\begin{array}{r} z_{n_k} \rightharpoonup \hat{u}, \|z_{n_k} - \varphi(z_{n_k})\| \to 0 \\ I - \varphi \text{ is demiclosed at the origin} \end{array}\right\} \Rightarrow \hat{u} \in \text{Fix}(\varphi),$$

and

$$\left.\begin{array}{r} Au_{n_k} \rightharpoonup A\hat{u}, \|(I - \psi)Au_{n_k}\| \to 0 \\ I - \psi \text{ is demiclosed at the origin} \end{array}\right\} \Rightarrow A\hat{u} \in \text{Fix}(\psi).$$

So, $\hat{u} \in \Omega$ and $\omega_w(u_n) \subset \Omega$. Since $x^* = P_\Omega(I - B + \nu\rho)x^*$ is equivalent to

$$\langle (I - B + \nu\rho)x^* - x^*, y - x^* \rangle \leq 0 (\forall y \in \Omega),$$

we attain

$$\begin{aligned} \limsup_{n\to\infty} \langle \nu\rho(x^*) - B(x^*), u_{n+1} - x^* \rangle &= \lim_{k\to\infty} \langle \nu\rho(x^*) - B(x^*), u_{n_k+1} - x^* \rangle \\ &= \langle \nu\rho(x^*) - B(x^*), \hat{u} - x^* \rangle \leq 0. \end{aligned} \tag{43}$$

Based on (35), we have

$$\|u_{n+1} - x^*\|^2 \leq [1 - (\tau - \nu\varpi)\gamma_n]\|u_n - x^*\|^2 + 2\gamma_n \langle \nu\rho(x^*) - B(x^*), u_{n+1} - x^* \rangle. \tag{44}$$

Based on (43), (44) and Lemma 2.2, we conclude $u_n \to x^*$. □

By considering A as the null operator, we are led to the following algorithm.

Algorithm 3.2. *Let $u_0 \in H_1$ be an initial point.*
Step 1. Assume the current iterate u_n is given. Compute

$$x_n = (1 - \theta)u_n + \theta[(1 - \lambda)u_n + \lambda\phi(u_n)].$$

Step 2. Compute

$$y_n = (1 - \frac{\alpha_n}{2})x_n + \frac{\alpha_n}{2}\varphi(z_n),$$

where

$$z_n = (1 - \alpha_n)x_n + \alpha_n\varphi(x_n),$$

in which $\alpha_n = \epsilon\mu^k$ and $k = \min\{0, 1, 2, \cdots\}$ such that

$$\alpha_n\|\varphi(z_n) - \varphi(x_n)\| \leq \delta\|z_n - x_n\|.$$

Step 3. Compute $u_{n+1} = \gamma_n\nu\rho(u_n) + (I - \gamma_n B)y_n$. Set $n := n + 1$ and return to Step 1.

Corollary 3.1. *If $\Omega_1 := \text{Fix}(\phi) \cap \text{Fix}(\varphi) \neq \emptyset$, then the sequence $\{u_n\}$ generated by Algorithm 3.2 converges strongly to $y^* = P_{\Omega_1}(I - B + \nu\rho)(y^*)$.*

4. **Conclusion**

In this paper we study the SFPP (2) which is an extension of the SFP, as well as of the well-known convex feasibility problem. Our problem involves in a pseudocontractive operator and two demicontractive operators in Hilbert spaces. To solve this problem, we proposed an iterative algorithm with self-adaptive rule and the Krasnoselskii-Mann method for the computation of a solution of the SFPP (2). The self-adaptive rule has does not rely on an a priori the Lipschitz constant of the involved pseudocontractive operators. Under several adequate conditions, we proved that the presented algorithm converges strongly to an element in Ω.

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